Multipartite pure-state entanglement

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We show that pure states of multipartite quantum systems are *multiseparable* (i.e., give separable density matrices on tracing any party) if and only if they have a generalized Schmidt decomposition. Implications of this result for the quantification of multipartite pure-state entanglement are discussed. Further, as an application of the techniques used here, we show that any purification of a bipartite-bound entangled state having a positive partial transpose is *tri-inseparable*, i.e., has none of its three bipartite partial traces separable. [S1050-2947(99)11805-5]

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I. INTRODUCTION

Quantum entanglement, first noted by Einstein-Podolsky-Rosen (EPR) [1] and Schrödinger [2], is one of the essential features of quantum mechanics. Its famous embodiment, the spin singlet (commonly referred to as the EPR state)

$$|\Psi^{AB}\rangle=\frac{1}{\sqrt{2}}(|\uparrow^{A}\downarrow^{B}\rangle-|\downarrow^{A}\uparrow^{B}\rangle), \tag{1}$$

proposed by Bohm [3], was shown by Bell [4] to have stronger correlations than allowed by any local hidden variable theory. The Greenberger-Horne-Zelinger-Mermin (GHZ) state [5,6]

$$|\Psi^{ABC}\rangle=\frac{1}{\sqrt{2}}(|\uparrow^{A}\uparrow^{B}\uparrow^{C}\rangle+|\downarrow^{A}\downarrow^{B}\downarrow^{C}\rangle) \tag{2}$$

is a canonical three-particle example of quantum entanglement. Contradiction between local hidden variable theories and quantum mechanics occurs even for nonstatistical predictions about the GHZ state [5,6], as opposed to the statistical ones for the EPR singlet. These aspects of quantum mechanics have often been referred to as quantum nonlocality, and form an important aspect of the study of the foundations of quantum mechanics.

Recently it has been realized that quantum resources can be useful in information processing. Quantum entanglement plays a key role in many such applications like quantum teleportation [7], superdense coding [8], quantum error correction [9], quantum key distribution [10], entanglement enhanced classical communication [11], quantum computational speedups [12], quantum distributed computation [13], and entanglement-enhanced communication complexity [14]. In view of its central role [15] in quantum information, it is imperative to have a qualitative as well as quantitative theory of it.

In the last few years much progress has been made in the study of bipartite pure- and mixed-state entanglement. In the rest of this introduction we mention some of these results that are needed. In Sec. II we take a brief look at a recently proposed framework for quantifying tripartite (multipartite) pure-state entanglement [16]. Finally, in Sec. III we present the results: the first result, establishing the equivalence of the set of Schmidt decomposable states and the set of multiseparable states, provides support for the proposed pure-state entanglement measure, and the second result provides a necessary condition for the existence of bound entanglement with positive partial transpose. Now let us look at some basic properties of entanglement.

A. Entanglement basics

Entanglement is a property that only has meaning for a multipartite system, i.e. one whose Hilbert space can be viewed as a product of two or more tensor factors corresponding to physical subsystems of the system. In the EPR example, the two subsystems are the two spin-1/2 particles A and B that form the spin singlet. As a matter of convenience, we think about these subsystems as belonging to different parties: Alice has subsystem A, Bob has subsystem B, and so on. For arbitrary systems, EPR singlets and GHZ states can be made meaningful by labelling any two orthogonal states of each party's subsystem as spin-up and spin-down, respectively.

Operationally, *unentangled* or *separable* states are the ones that can be made by the different parties with (at most) *classically coordinated local operations*, i.e., local operations by the parties, which are coordinated by the exchange of classical information. Here *local operations* include unitary transformations, additions of ancillas, measurements, and throwing away parts of the system, all performed locally by one party on his/her subsystem. Classical communication between parties is included because it allows for the creation of mixed states that are classically correlated but exhibit no quantum correlations.

Thus, mathematically speaking, a pure state $|\Psi^{ABC\cdots}\rangle$ is separable iff it can be written as a tensor product of states belonging to different parties:

$$|\Psi^{ABC\cdots}\rangle=|\phi^{A}\rangle\otimes|\chi^{B}\rangle\otimes|\psi^{C}\rangle\otimes\cdots. \tag{3}$$

A mixed state $\rho^{ABC\cdots}$ is separable if and only if it can be written as a sum of separable pure states [17]:

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$$\rho^{\mathrm{ABC}\cdots}=\sum_i p_i|\phi_i^{\mathrm{A}}\rangle\langle\phi_i^{\mathrm{A}}|\otimes|\chi_i^{\mathrm{B}}\rangle\langle\chi_i^{\mathrm{B}}|\otimes|\psi_i^{\mathrm{C}}\rangle\langle\psi_i^{\mathrm{C}}|\otimes\cdots, \tag{4}$$

where the probabilities $p_i\geqslant 0$ and $\Sigma_i p_i=1$. Finally, states that are not separable are said to be *entangled* or *inseparable*.

Because classical communication between parties should not increase their quantum correlations, the expectation of any quantitative measure of entanglement should be nonincreasing under classically coordinated local operations. In addition, any such measure must be invariant under local unitary transformations, because they only correspond to another choice of local bases. Naturally, such a measure must be zero for any separable state. Also, it is natural to require such a measure to be additive for tensor products. To summarize, the four requirements for a good measure of entanglement are [18]: (i) zero for separable states, (ii) invariant under local unitary transformations, (iii) nonincreasing under classically coordinated local operations, and (iv) additive for tensor products. Since bipartite entanglement is the simplest case, let us review it next.

B. Bipartite entanglement

For bipartite pure states it has been shown [19,20] that partial entropy is a good entanglement measure. It is equal both to the state's *entanglement of formation* (the number of singlets asymptotically required to prepare the state, using only classically coordinated local operations) and the state's *distillable entanglement* (the number of singlets asymptotically preparable from the state using only classically coordinated local operations). Here partial entropy is the von Neumann entropy[1] of the reduced density matrix left after tracing out any one of the two parties. Mathematically we write this as

$$E(\Psi^{\mathrm{AB}})=S(\rho^{\mathrm{A}})=S(\rho^{\mathrm{B}}), \tag{5}$$

where $\rho^{\mathrm{A}}=\mathrm{Tr}_{\mathrm{B}}(|\Psi^{\mathrm{AB}}\rangle\langle\Psi^{\mathrm{AB}}|)$, and so on. For mixed states partial entropy is no longer a good measure since it can be nonzero for some separable states like the completely random state. A variety of apparently distinct entanglement measures for bipartite mixed states have been discussed, including entanglement of formation, distillable entanglement [19,21], entanglement of assistance [22], relative entropy entanglement [18], and locally unitarily invariant parameters of the density matrix [23]. However, no measure has been proved to satisfy all the properties required of a good measure.

Qualitatively, the set of inseparable states can be divided into two subsets: *distillable states* — inseparable states that have finite positive distillable entanglement — and *bound entangled states* — inseparable states that have zero distillable entanglement. The *partial transpose* of a density matrix can be used to formulate necessary conditions for separability and distillability, where partial transpose $(\rho^{\mathrm{AB}})^{T_{\mathrm{B}}}$ of a density matrix ρ^{AB} in the basis $|i^{\mathrm{A}}j^{\mathrm{B}}\rangle$ is given by

$$\langle i^{\mathrm{A}}j^{\mathrm{B}}|(\rho^{\mathrm{AB}})^{T_{\mathrm{B}}}|k^{\mathrm{A}}l^{\mathrm{B}}\rangle=\langle i^{\mathrm{A}}l^{\mathrm{B}}|\rho^{\mathrm{AB}}|k^{\mathrm{A}}j^{\mathrm{B}}\rangle. \tag{6}$$

The positivity[2] of the partial transpose[3] of a density matrix, or equivalently the positivity of a density matrix under partial transposition (PPT) is a necessary condition [24] for separability. Similarly, negativity[4] of the density matrix under partial transposition (NPT) is a necessary condition [25] for distillability.

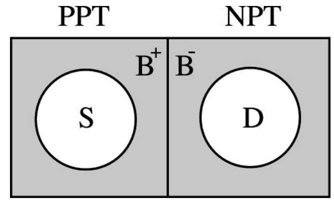

FIG. 1. Types of bipartite entanglement: The left half of the figure represents the set of PPT states, and the right half represents the set of NPT states. S denotes separable states, and D denotes distillable states. $B^+(B^-)$ denotes bound entangled states with P(N)PT. In general, all these sets are known to be nonempty except for B^-, for which no example is known yet.

Thus the set of mixed bipartite states can be divided into four classes, as shown in Fig. 1: the set of separable states (S), the set of distillable states (D), the set of PPT bound entangled states (B^+) and the set of NPT bound entangled states (B^-). Now we are in a position to turn to the concepts of reducibilities and equivalences and their relation to entanglement measures.

II. REDUCIBILITIES, EQUIVALENCES, AND ENTANGLEMENT MEASURES

In this section we will review the concepts of reducibilities and equivalences with respect to classically coordinated local operations [26], which are central to quantifying entanglement. Then we review a suggested way of quantifying tripartite and in general multipartite pure-state entanglement [16].

In what follows we use a quantitative measure of similarity of two states. One such measure is the fidelity [27,28]: the *fidelity* of a mixed state ρ relative to a pure state $|\psi\rangle$ is given by $F(\rho,\psi)=\langle\psi|\rho|\psi\rangle$. It is the probability with which ρ will pass the test for being $|\psi\rangle$, conducted by an observer who knows the state $|\psi\rangle$.

A. Reducibilities and equivalences

We say a pure state $|\Phi\rangle$ is *reducible* ($\leqslant$) to $|\Psi\rangle$ if and only if

$$\exists_{\mathcal{L}} \text{ such that } \mathcal{L}(|\Psi\rangle\langle\Psi|)=|\Phi\rangle\langle\Phi| \tag{7}$$

[1]The von Neumann entropy S of a density matrix ρ is defined to be the Shannon entropy H of its eigenvalues, i.e., $S(\rho)=H(\{\lambda_i\})=-\Sigma_i\lambda_i\log_2(\lambda_i)$, where λ_i are the eigenvalues of ρ.

[2]We say a matrix A is positive if and only if all its eigenvalues are non-negative. This definition coincides with that of non-negative matrices in mathematical literature.

[3]Clearly, the partial transpose of a density matrix is basis dependent, but its eigenvalues are not.

[4]Here negativity of a Hermitian matrix means that at least one of its eigenvalues is negative.

where $\mathcal{L}$ is a multilocally implementable trace-preserving superoperator [29,30] (a mathematical representation of classically coordinated local operations). Intuitively this means that by doing classically coordinated local operations the parties can make $|\Phi\rangle$ starting from $|\Psi\rangle$. Necessary and sufficient conditions for reducibility of bipartite pure states were found in Ref. [31].

Two states $|\Phi\rangle$ and $|\Psi\rangle$ are said to be *equivalent* ($\equiv$) if $|\Phi\rangle \leq |\Psi\rangle$ and $|\Psi\rangle \leq |\Phi\rangle$. Intuitively this means that the two states are interconvertible by classically coordinated local operations. Here the principle of nonincrease of entanglement implies that equivalent states must have the same entanglement [21]. Obviously, states related by local unitary transformations are equivalent and so are all separable states.

We say that $|\Psi\rangle^{\otimes x}$ and $|\Phi\rangle^{\otimes y}$, with x and y non-negative real numbers,[5] are *asymptotically equivalent* ($\approx$) if $|\Phi\rangle^{\otimes y}$ is *asymptotically reducible* ($\lesssim$) to $|\Psi\rangle^{\otimes x}$, and vice versa, where

$$|\Phi\rangle^{\otimes y} \lesssim |\Psi\rangle^{\otimes x} \text{iff} \quad \forall_{\delta>0,\epsilon>0}; \ \exists_{m,n,\mathcal{L}}$$

$$\text{such that } \frac{|m-n|}{m} < \delta$$

$$\text{and} \quad F(\mathcal{L}(|\Psi\rangle\langle\Psi|^{\otimes(mx)}), |\Phi\rangle^{\otimes(ny)}) \geq 1-\epsilon. \tag{8}$$

Here $\mathcal{L}$ is a multilocally implementable trace preserving superoperator that converts mx copies of $|\Psi\rangle$ into a high-fidelity approximation to ny copies of $|\Phi\rangle$, where m and n are nonnegative integers. These definitions extend the concepts of reducibility and equivalence to encompass the situation of asymptotic interconversion between states. Again the principle of nonincrease of entanglement requires that asymptotically equivalent states must have the same entanglement.

As an example of the usefulness of these concepts let us re-express the bipartite pure-state entanglement result [19,20], mentioned in Sec. I, in terms of asymptotic equivalence. In this new language, any bipartite pure state $|\Psi^{\mathrm{AB}}\rangle$ is asymptotically equivalent to $E(\Psi^{\mathrm{AB}})$ EPR singlets:

$$|\Psi^{\mathrm{AB}}\rangle \approx |\mathrm{EPR}^{\mathrm{AB}}\rangle^{\otimes E(\Psi^{\mathrm{AB}})}. \tag{9}$$

Thus if we take the EPR singlet to be the unit of entanglement (ebit), the partial entropy $E(\Psi^{\mathrm{AB}})$ specifies the EPR singlets that can be obtained from and are required to prepare $|\Psi^{\mathrm{AB}}\rangle$ by classically coordinated local operations.

In proving this result, the concepts of entanglement concentration and dilution [19] are central. The process of asymptotically reducing a given bipartite pure state to EPR singlet form is *entanglement dilution*, and that of reducing EPR singlets to an arbitrary bipartite pure state is *entanglement concentration*. Then the above result means that entanglement concentration and dilution, are reversible in the sense of asymptotic equivalence, i.e., in the sense of approaching unit efficiency and fidelity in the limit of large number of copies n. The crucial requirement for these methods to work is the existence of the Schmidt (normal or polar) decomposition for bipartite pure states [32], in which any pure state say $|\Psi^{\mathrm{AB}}\rangle$ can be written in the form

$$|\Psi^{\mathrm{AB}}\rangle = \sum_i a_i |i^{\mathrm{A}}\rangle \otimes |i^{\mathrm{B}}\rangle, \tag{10}$$

where $|i^{\mathrm{A}}\rangle$ and $|i^{\mathrm{B}}\rangle$ form orthonormal bases in Alice and Bob's Hilbert space, respectively. Notice that, by change of phases of local bases, each of the *Schmidt coefficients* a_i can be made real and non-negative.

B. Schmidt decomposability, multiseparability, and pure-state entanglement

Let Alice, Bob, Charlie, ... , Nancy be the n parties who have one subsystem each of an n-part system, generally in a joint state.

We say an n-party state $|\Psi^{\mathrm{ABC}\cdots\mathrm{N}}\rangle$ is (n-)*Schmidt decomposable* if it has an nth-order Schmidt decomposition, i.e. it can be written in the form

$$|\Psi^{\mathrm{ABC}\dots\mathrm{N}}\rangle = \sum_i a_i |i^{\mathrm{A}}\rangle|i^{\mathrm{B}}\rangle|i^{\mathrm{C}}\rangle\cdots|i^{\mathrm{N}}\rangle, \tag{11}$$

where $|i^{\mathrm{A}}\rangle$, $|i^{\mathrm{B}}\rangle$, $|i^{\mathrm{C}}\rangle, \dots , |i^{\mathrm{N}}\rangle$ form orthonormal bases in Alice, Bob, Charlie, ... , and Nancy's Hilbert space respectively. Again, by change of phases of local bases, each of the *Schmidt coefficients* a_i can be made real and non-negative.

A useful property of Schmidt decomposable states is that the density matrices obtained by tracing out any party are separable. We call this property *n-separability* or *multiseparability*.[6] Further, these density matrices obtained by tracing one party are *eigenseparable*, i.e. they have separable eigenvectors. We call this property *n-eigenseparability* or *multieigenseparability*.

Let us now look at tripartite states for concreteness. It has been noted [34] that arbitrary tripartite pure states are not Schmidt decomposable. A different way of seeing this is by using the facts that any bipartite mixed state can be purified[7] [32] to a tripartite pure state and Schmidt decomposable states are trieigenseparable: Then if this were true, all bipartite states would be eigenseparable, which is false.

The absence of Schmidt decomposition for a general tripartite pure state means that, on the one hand, the techniques developed for bipartite pure states cannot be generalized in a straightforward manner to tripartite pure states, but on the other hand it implies that there are interesting new properties to be discovered about states that are not Schmidt decomposable.

[5]States with non-negative real exponents are defined in an asymptotic sense by $(|\Psi\rangle^{\otimes x})^{\otimes n} = |\Psi\rangle^{\otimes \lfloor n\,x \rfloor}$, where $|\Psi\rangle^{\otimes 0} = |0^{\mathrm{A}}0^{\mathrm{B}}0^{\mathrm{C}}\dots\rangle$. Here $\lfloor x \rfloor$ called the floor of x, is defined as the greatest integer less than or equal to nlx.

[6]In the terminology of Ref. [33], multiseparability is equivalent to all possible partial semiseparability.

[7]A purification of a mixed state ρ^{AB} is a pure state $|\Psi^{\mathrm{ABC}}\rangle$ such that $\rho^{\mathrm{AB}} = \mathrm{Tr}_{\mathrm{C}}(|\Psi^{\mathrm{ABC}}\rangle\langle\Psi^{\mathrm{ABC}}|)$.

Now we are in a position to review the framework used [16] to quantify tripartite (multipartite) entanglement along the lines of the bipartite case. It is based on finding sets of states (analogous to the EPR singlet in the bipartite case) to which every pure tripartite (multipartite) state is asymptotically equivalent. Such sets are called reversibe entanglement generating sets. More precisely, set $\mathcal{G}=\{|\psi_1\rangle,|\psi_2\rangle,\ldots,|\psi_n\rangle\}$ is a *reversible entanglement generating set* (REGS) if and only if for any state $|\Psi\rangle \quad \exists_{x_1,x_2,\ldots,x_n\geqslant 0}$, such that

$$|\Psi\rangle \approx |\psi_1\rangle^{\otimes x_1}\otimes|\psi_2\rangle^{\otimes x_2}\otimes\cdots\otimes|\psi_n\rangle^{\otimes x_n}. \tag{12}$$

The tensor powers $x_1,x_2,\ldots,x_n$ are known as the entanglement measure (or entanglement coefficients) induced by the REGS $\mathcal{G}$.

Of course one would like to know the fewest states needed to make any general pure state. This leads to the definition of a *minimal reversible entanglement generating set* (MREGS) to be a REGS of least cardinality. The set $\mathcal{G}_2=\{|\mathrm{EPR}\rangle\}$ is an example of a MREGS for bipartite entanglement which induces the entanglement measure given by the partial entropy in bits.

As mentioned earlier, bipartite entanglement concentration and dilution protocols depend crucially on the existence of a Schmidt decomposition. Not surprisingly, the bipartite protocols for entanglement concentration and dilution can be generalized [16] to work for tripartite (multipartite) Schmidt decomposable states and used to prove that they are asymptotically equivalent to GHZ (generalized[8] GHZ) states, with the one-party partial entropy as the induced entanglement measure. That is, if $|\Psi^{\mathrm{ABC}}\rangle$ is Schmidt decomposable,

$$|\Psi^{\mathrm{ABC}}\rangle \approx |\mathrm{GHZ}\rangle^{\otimes S(\rho^{\mathrm{A}})}. \tag{13}$$

We note here that for any multipartite Schmidt decomposable state, one-party partial von Neumann entropies are equal to the Shannon entropy of the square of the Schmidt coefficients. Now we are in a position to motivate and present the main results of the paper.

III. RESULTS

For simplicity let us restrict ourselves to the case of tripartite systems. The asymptotic equivalence of Schmidt decomposable states to GHZ states gives a way of quantifying their entanglement. When we look at states that do not lend themselves to the dilution and concentration scheme for Schmidt decomposable states, we notice that it is their bipartite entanglement left after tracing out a party, that somehow "gets in the way" of using these protocols. This fits in with the fact we discussed earlier in Sec. II B that the existence of non-Schmidt decomposable states is intimately connected to the existence of bipartite entangled states. Thus it is natural

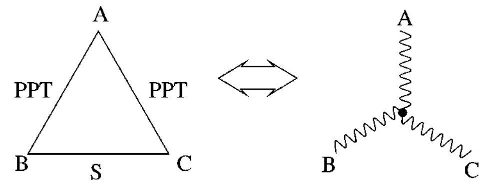

FIG. 2. Equivalence of triseparability and Schmidt decomposability: Here, (A)lice, (B)ob, and (C)harlie represent the three parties. The sides $^{\mathrm{AB}}$, $^{\mathrm{BC}}$, and $^{\mathrm{AC}}$ of the triangle represent the density matrices $\rho^{\mathrm{AB}},\rho^{\mathrm{BC}}$, and ρ^{AC}, respectively. The wiggly lines represent "essential" tripartite entanglement embodied by Schmidt decomposable states.

to expect that any triseparable state is Schmidt decomposable.

Here we prove this claim and in general prove that any multiseparable state is Schmidt decomposable. Let us turn to that next.

A. Equivalence of multiseparability and Schmidt decomposability

The result is trivial for one party. For the bipartite case it states that all pure states have a Schmidt decomposition which, as we mentioned earlier, is known to be true. So we prove the result first for the tripartite case and then extend it to the multipartite case by induction.

Consider a triseparable pure state $|\Psi^{\mathrm{ABC}}\rangle$. By definition, ρ^{AB}, ρ^{BC}, and ρ^{AC} are separable.

Now we show that any triseparable state is Schmidt decomposable. Since PPT is a necessary (but in general not sufficient [35]) condition for separability, we prove a stronger result, namely, *If a tripartite pure state* $|\Psi^{\mathrm{ABC}}\rangle$ *is such that* ρ^{BC} *is separable and* ρ^{AC} *and* ρ^{AB} *are PPT, then it is Schmidt decomposable*.

This result is illustrated in Fig. 2. To prove this we first write $|\Psi^{\mathrm{ABC}}\rangle$ in its Schmidt decomposition [32]

$$|\Psi^{\mathrm{ABC}}\rangle = \sum_{i=1}^{n} \sqrt{\lambda_i}|\lambda_i^{\mathrm{A}}\rangle\otimes|\lambda_i^{\mathrm{BC}}\rangle, \tag{14}$$

where $|\lambda_i^{\mathrm{A}}\rangle$ are eigenvectors of ρ^{A} and $|\lambda_i^{\mathrm{BC}}\rangle$ are eigenvectors of ρ^{BC} corresponding to the nonzero (positive) eigenvalues λ_i.

Since ρ^{BC} is separable it can be written as an ensemble of pure product states. Let $\mathcal{E}=\{p_i,|\psi_i^{\mathrm{B}}\phi_i^{\mathrm{C}}\rangle|i=1,\ldots,m\}$ be such an ensemble with the fewest members (here $m\geqslant n$). Then probabilities $p_i>0,\forall_{i=1,\ldots,m}$ and states $|\psi_i^{\mathrm{B}}\phi_i^{\mathrm{C}}\rangle$ are pairwise linearly independent. Here $|\psi_i^{\mathrm{B}}\phi_i^{\mathrm{C}}\rangle$ is a short way of writing $|\psi_i^{\mathrm{B}}\rangle\otimes|\phi_i^{\mathrm{C}}\rangle$. Now suppose Alice does the following local operations:

(1) She appends an ancilla and performs a local unitary transformation on $|\Psi^{\mathrm{ABC}}\rangle$, resulting in

$$|\tilde{\Psi}^{\mathrm{ABC}}\rangle = \sum_{i=1}^{m} \sqrt{p_i}|i^{\mathrm{A}}\psi_i^{\mathrm{B}}\phi_i^{\mathrm{C}}\rangle, \tag{15}$$

The Hughston-Jozsa-Wootters result [32] ensures that this is always possible.

[8]An n party generalization of the GHZ state which we call the *n-cat* state is defined to be

$$|n\text{-cat}\rangle = \frac{1}{\sqrt{2}}(|\uparrow^{\mathrm{A}}\uparrow^{\mathrm{B}}\ldots\uparrow^{\mathrm{N}}\rangle+|\downarrow^{\mathrm{A}}\downarrow^{\mathrm{B}}\ldots\downarrow^{\mathrm{N}}\rangle).$$

(2) Now Alice chooses two distinct basis vectors $|i^{\rm A}\rangle$ and $|j^{\rm A}\rangle$, and does an incomplete von Neumann measurement projecting the above state into the subspace spanned by these two vectors and its complement. As a result, with probability $(p_i+p_j)>0$, the joint state becomes

$$|\Psi_{ij}^{\rm ABC}\rangle = q_i|i^{\rm A}\psi_i^{\rm B}\phi_i^{\rm C}\rangle + q_j|j^{\rm A}\psi_j^{\rm B}\phi_j^{\rm C}\rangle, \tag{16}$$

with $q_i=\sqrt{p_i/(p_i+p_j)}$ and $q_j=\sqrt{p_j/(p_i+p_j)}$.

This can be rewritten as

$$|\Psi_{ij}^{\rm ABC}\rangle = q_i|i^{\rm A}i^{\rm B}i^{\rm C}\rangle + q_j|j^{\rm A}\rangle(\alpha|i^{\rm B}\rangle+\beta|j^{\rm B}\rangle)(\gamma|i^{\rm C}\rangle+\delta|j^{\rm C}\rangle), \tag{17}$$

where $\{|i^{\rm B}\rangle,|j^{\rm B}\rangle\}$ are orthonormal basis vectors for the *span* of $\{|\psi_i^{\rm B}\rangle,|\psi_j^{\rm B}\rangle\}$ with $|i^{\rm B}\rangle=|\psi_i^{\rm B}\rangle$, and similarly on Charlie's side. Also $|\alpha|^2+|\beta|^2=1$ and $|\gamma|^2+|\delta|^2=1$ for normalization. In this basis the partial transpose of $\rho_{ij}^{\rm AB}$ is

$$(\rho_{ij}^{\rm AB})^{\rm T_B} = \begin{pmatrix} q_i^2 & 0 & q_iq_j\gamma^*\alpha^* & 0 \\ 0 & 0 & q_iq_j\gamma^*\beta^* & 0 \\ q_iq_j\gamma\alpha & q_iq_j\gamma\beta & q_j^2|\alpha|^2 & q_j^2\alpha^*\beta \\ 0 & 0 & q_j^2\alpha\beta^* & q_j^2|\beta|^2 \end{pmatrix}. \tag{18}$$

Since $\rho^{\rm AB}$ is PPT so is $\rho_{ij}^{\rm AB}$ [25]; this requires [36]

$$\begin{vmatrix} 0 & q_1q_2\gamma^*\beta^* \\ q_1q_2\gamma\beta & q_2^2|\alpha|^2 \end{vmatrix} \geqslant 0,$$

implying

$$\gamma=0 \quad \text{or} \quad \beta=0,$$

i.e.,

$$|\phi_j^{\rm C}\rangle \perp |\phi_i^{\rm C}\rangle \quad \text{or} \quad |\psi_j^{\rm B}\rangle = |\psi_i^{\rm B}\rangle. \tag{19}$$

Repeating the above argument for $\rho^{\rm AC}$, we obtain

$$|\psi_j^{\rm B}\rangle \perp |\psi_i^{\rm B}\rangle \quad \text{or} \quad |\phi_j^{\rm C}\rangle = |\phi_i^{\rm C}\rangle. \tag{20}$$

Since any pair of states in ensemble $\mathcal{E}$ are linearly independent, the only consistent solution for the above relations is

$$|\psi_j^{\rm B}\rangle \perp |\psi_i^{\rm B}\rangle \quad \text{and} \quad |\phi_j^{\rm C}\rangle \perp |\phi_i^{\rm C}\rangle. \tag{21}$$

Since Alice can choose any two distinct $i,j=1,\ldots,m$, Eq. (21) implies that $|\Psi^{\rm ABC}\rangle$ is Schmidt decomposable. This completes the proof.

This result is intuitively very satisfying, because it means that if there are no bipartite correlations among any two parties when the third party is traced out, then the tripartite state is Schmidt decomposable and hence asymptotically equivalent to GHZ states. This result supports the hypothesis that the GHZ and EPR states together form a MREGS, with the EPR singlets representing the bipartite entanglement between the parties, and the GHZ state representing ''essential'' tripartite entanglement.

The generalization of this result to the multipartite case follows by induction from the tripartite case. For convenience we illustrate the induction step for the case of four parties: Alice, Bob, Charlie, and David.

Let $|\Psi^{\rm ABCD}\rangle$ be a 4-separable state of Alice, Bob, Charlie, and David. By definition $\rho^{\rm BCD}$, $\rho^{\rm ACD}$, $\rho^{\rm ABD}$, and $\rho^{\rm ABC}$ are separable. Alice can by local operations as in the paragraph before Eq. (15) make it into

$$|\tilde{\Psi}^{\rm ABCD}\rangle = \sum_{i=1}^{m} \sqrt{p_i}|i^{\rm A}\psi_i^{\rm B}\phi_i^{\rm C}\chi_i^{\rm D}\rangle. \tag{22}$$

Joining Charlie and David together into one party and applying the tripartite result — Eq. (21) — implies that the $|\psi_i^{\rm B}\rangle$ form an orthonormal set which we rename $|i^{\rm B}\rangle$. Thus

$$|\tilde{\Psi}^{\rm ABCD}\rangle = \sum_{i=1}^{m} \sqrt{p_i}|i^{\rm A}i^{\rm B}\phi_i^{\rm C}\chi_i^{\rm D}\rangle. \tag{23}$$

Now joining Alice and Bob together as a composite party $\widetilde{\text{Alice}}$ and applying the tripartite result—Eq.(21)—we have that the $|\phi_i^{\rm C}\rangle$ form an orthonormal set and so do the $|\chi_i^{\rm D}\rangle$. This proves the result. After this we apply the two-dimensional projection technique of this section to prove a new necessary condition for bipartite bound entanglement with PPT.

B. No B$^+$-S theorem

It is well known that any two-party mixed state can be purified into a tripartite pure state. Then a connection between tripartite pure-state entanglement and bipartite mixed-state entanglement seems likely. Already the fact that triseparable states are Schmidt decomposable tells us that a purification $|\Psi^{\rm ABC}\rangle$ of a separable bipartite state $\rho^{\rm AB}$ with inseparable eigenvectors cannot be Schmidt decomposable and hence cannot be triseparable. Thus at least one of $\rho^{\rm BC}$ and $\rho^{\rm AC}$ is entangled.

Here we prove another result of this kind: any purification of a bipartite PPT bound-entangled (B$^+$) state is tri-inseparable. More precisely, *if* $|\Psi^{\rm ABC}\rangle$ *is a purification of a bipartite PPT bound-entangled state* $\rho^{\rm AB}$, *then* $\rho^{\rm BC}$ *and* $\rho^{\rm AC}$ *are inseparable and hence* $|\Psi^{\rm ABC}\rangle$ *is tri-inseparable.*

Before proving this result, note that any purification for $\rho^{\rm AB}$ is related to any other, by addition of an ancilla and/or a local unitary transformation by Charlie [32]. Since inseparability is unaffected by such local operations [21], if we prove the above result for one purification it will hold for any other purification. The proof then follows as a trivial consequence of the following result. *If a tripartite pure state* $|\Psi^{\rm ABC}\rangle$ *is such that* $\rho^{\rm BC}$ *is separable and* $\rho^{\rm AB}$ *has positive partial transpose, then* $\rho^{\rm AB}$ *must be separable.* This result is illustrated in Fig. 3.

The argument is similar to that employed in proving the equivalence of Schmidt decomposability and triseparability. The difference is that here, only $\rho^{\rm AB}$ is given to have positive partial transpose; but all the steps from Eqs. (14) to (19) go through. To prove the result, we show that Eq. (19) implies that $|\tilde{\Psi}^{\rm ABC}\rangle$ in Eq. (15) can be written as

$$|\tilde{\Psi}^{\rm ABC}\rangle = \sum_{i=1}^{s} |\mu_i^{\rm B}\rangle \otimes \sum_{j=1}^{t(i)} \sqrt{p_{ij}}|\chi_{ij}^{\rm A}\rangle \otimes |\nu_{ij}^{\rm C}\rangle, \tag{24}$$

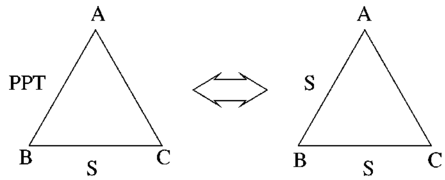

FIG. 3. No B^+-S theorem: Here (A)lice, (B)ob, and (C)harlie represent the three parties. The sides AB, BC, and AC of the triangles represent the density matrices ρ^{AB}, ρ^{BC}, and ρ^{AC}, respectively.

with the kets $|\mu_i^{\mathrm{B}}\rangle$ pairwise linearly independent, $\langle\chi_{ij}^{\mathrm{A}}|\chi_{kl}^{\mathrm{A}}\rangle=\delta_{ik}\delta_{jl}$, and $\langle\nu_{ij}^{\mathrm{C}}|\nu_{kl}^{\mathrm{C}}\rangle=\delta_{ik}\langle\nu_{ij}^{\mathrm{C}}|\nu_{il}^{\mathrm{C}}\rangle$. Here $\Sigma_{i=1}^{s}t(i)=m$ and $p_{ij}>0\forall_{ij}$. Eq. (24) implies

$$\tilde{\rho}^{\mathrm{AB}}=\mathrm{Tr}_{\mathrm{C}}|\tilde{\Psi}^{\mathrm{ABC}}\rangle\langle\tilde{\Psi}^{\mathrm{ABC}}|=\mathrm{Tr}_{\mathrm{C}}\sum_{i=1}^{s}q_i|\mu_i^{\mathrm{B}}\rangle\langle\mu_i^{\mathrm{B}}|\otimes|\chi_i^{\mathrm{AC}}\rangle\langle\chi_i^{\mathrm{AC}}|, \tag{25}$$

with $|\chi_i^{\mathrm{AC}}\rangle=\Sigma_{j=1}^{t(i)}\sqrt{p_{ij}/q_i}|\chi_{ij}^{\mathrm{A}}\rangle\otimes|\nu_{ij}^{\mathrm{C}}\rangle$ and $q_i=\Sigma_{j=1}^{t(i)}p_{ij}>0$; we have used the orthogonality of the $|\nu_{ij}\rangle$'s for different values of subscript i. Performing this trace it is easy to see that $\tilde{\rho}^{\mathrm{AB}}$ is separable. Recall that $\tilde{\rho}^{\mathrm{AB}}$ is obtained from ρ^{AB} by appending an ancilla and/or a unitary rotation by Alice. Since inseparability is preserved under these local operations, separability of $\tilde{\rho}^{\mathrm{AB}}$ implies that ρ^{AB} is also separable.

Now all that remains to be proved is that $|\tilde{\Psi}^{\mathrm{ABC}}\rangle$ has the form shown in Eq. (24) above. For this we use induction on the number of terms in $|\tilde{\Psi}^{\mathrm{ABC}}\rangle$. Obviously the form in Eq. (24) holds in the case when $|\tilde{\Psi}^{\mathrm{ABC}}\rangle$ has just one term. Now assuming that this form holds for $s=r-1$ terms,

$$|\tilde{\Psi}_r^{\mathrm{ABC}}\rangle=\sqrt{1-p_r}|\tilde{\Psi}_{r-1}^{\mathrm{ABC}}\rangle+\sqrt{p_r}|r^{\mathrm{A}}\psi_r^{\mathrm{B}}\phi_r^{\mathrm{C}}\rangle, \tag{26}$$

Where $|\tilde{\Psi}_{r-1}^{\mathrm{ABC}}\rangle$ has the form of Eq. (24) with $\Sigma_{i=1}^{s}t(i)=r-1$. Here $1>p_r>0$. But the condition in Eq. (19) with $j=r$ and $i=1,\ldots,(r-1)$ implies that either $\exists_k$ such that

$$|\psi_r^{\mathrm{B}}\rangle=|\mu_k^{\mathrm{B}}\rangle, \quad \text{with}$$

$$|\phi_r^{\mathrm{C}}\rangle\perp|\nu_{ij}^{\mathrm{C}}\rangle \quad \forall_{i,j;i\neq k}$$

or

$$|\phi_r^{\mathrm{C}}\rangle\perp|\nu_{ij}^{\mathrm{C}}\rangle \quad \forall_{i,j}. \tag{27}$$

Thus $|\tilde{\Psi}_r^{\mathrm{ABC}}\rangle$ can be written in the form given by Eq. (24): in the first case with $s\rightarrow s$ and $t(k)\rightarrow t(k)+1$, and in the second case with $s\rightarrow s+1$ and $t(s+1)=1$. Thus the result is proved.

Given a tripartite pure state $|\Psi^{\mathrm{ABC}}\rangle$, there are many possibilities for the kind of entanglement of the corresponding bipartite states ρ^{AB}, ρ^{BC}, and ρ^{AC}. Figure 4 shows these possibilities and marks the ones ruled out by this result.

IV. CONCLUSIONS AND DISCUSSIONS

We have proved the equivalence of multiseparable and Schmidt decomposable (multipartite pure) states. This result is relevant to the problem of quantifying multipartite pure-state entanglement, because it shows that if there is no $(n-1)$-partite entanglement after tracing any party, then the n-partite state is Schmidt decomposable and hence asymptotically equivalent to the corresponding generalized GHZ state, which represents ''essential'' n-partite entanglement. This result supports the hypothesis that the set of 2-,3-, . . . , n-party generalized GHZ states form a minimal reversible entanglement generating set, with the k-party generalized GHZ states representing ''essential'' k-partite entanglement. Thus this work provides support for the entanglement measure proposed in Ref. [16]. Further work needs to be done to prove that the generalized GHZ states form a minimal reversible entanglement generating set.

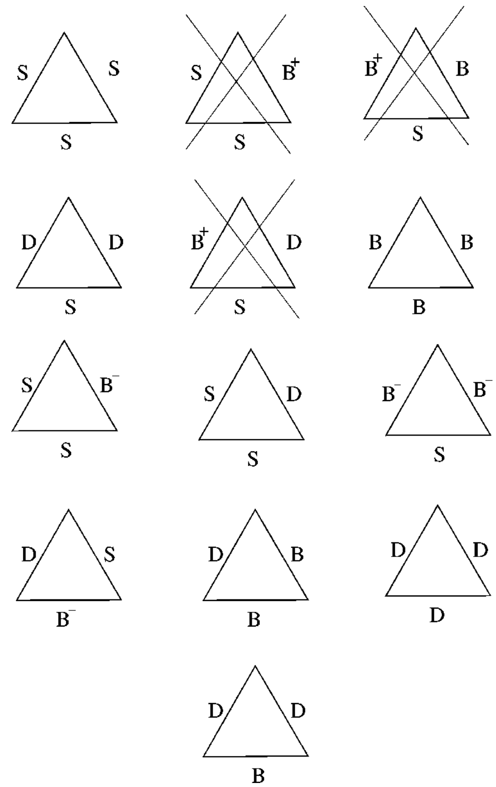

FIG. 4. Here vertices of the triangle represent the three parties, and each of the sides represents the corresponding bipartite density matrix obtained by tracing out the party corresponding to the remaining vertex. The letters near the sides label the kind of bipartite entanglement of the corresponding density matrices: (S)eparable, (D)istillable, B^+,B^-, and B, which stands for both B^+ and B^-.

We have also proved that any purification of a bipartite bound entangled state with positive partial transpose is triinseparable. This provides a new necessary condition for bound entanglement with positive partial transpose. An important question relating to the ''no B^+S'' theorem is whether there exist states like $\mathrm{B}^+-\mathrm{B}^+-\mathrm{B}^+$ and $\mathrm{B}^+-\mathrm{B}^+-\mathrm{D}$. This is related to the question whether *tri-PPT* states are triseparable and whether *bi-PPT* states are biseparable.

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