Quantum states of an oscillator with periodic time-dependent frequency under quasiresonant condition: Unperturbed evolution, perturbative effects, and anharmonic effects

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In a recent paper [L. Ferrari, Phys. Rev. A **57**, 2347 (1998)], one of the authors has shown that the mean energy of a quantum oscillator with periodic time-dependent frequency diverges exponentially in time, under certain conditions. In the present paper, we study the explicit form of the evolving state, and compare the results obtained with the general expressions developed by other authors for an arbitrary time-dependent frequency. Then we approach the problem of the anharmonic effects. A first-order calculation is performed in the case of a short-range perturbation. The transition rates between different states, evolving with the unperturbed Hamiltonian, are shown to vanish at long times when the unperturbed oscillator's energy diverges exponentially. A nonperturbative approach must be adopted, in the presence of anharmonic potentials of the form $V(q) \propto q^j$, $j>2$. In the case of weak anharmonicity $(j-2 \ll 1)$, a mean-field procedure can be used to show that the mean energy does actually *saturate* at long times, with the possible exception of periodic peaks, having nonsaturating height, that we call ''special quantum effects.'' [S1050-2947(99)12005-5]

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I. INTRODUCTION

The oscillator with time-dependent frequency

$$\Omega(t) = \Omega_0 \sqrt{1+\delta(t)} \tag{1}$$

is a long-standing problem, originally introduced as a model system for the adiabatic approximation in the case of a frequency varying smoothly between two asymptotic values [1]. The case of a frequency depending periodically or randomly on time, has recently achieved a new impulse, in view of possible applications to the physics of accelerators and to condensed-matter physics. Classically, the problem has been approached as an application of the invariant theory for integrable systems [2], and as a special example of noise-driven motions, in the presence of *random* fluctuations [3]. Starting from a different viewpoint (the analogy existing with a stationary Schrödinger equation in one dimension), one of the authors (Ferrari) has studied the special case in which the classical oscillator's energy increases *exponentially* in time, at the expense of the field producing the frequency fluctuation [4]. In the periodic case, this occurs when the parameters characterizing the frequency fluctuation fall within certain ''bands'' of values. The condition yielding the *maximum* rate of exponential increase is denoted as the ''quasiresonant condition,'' and corresponds to the center of the bands.

The quantum formulation of the problem has a long story, in turn, starting from the early 1950s [5,6]. More advanced methods, extending the invariants' theory to the quantum case, have been used by Dodonov and Man'ko [7], and by Lewis and co-workers [8,9]. Those approaches have a great deal of generality, and provide methods of wide applicability, regardless of the specific model for the fluctuation. In an attempt to approach the quantum version of the case studied in Ref. [4], the author adopted a different method, based on the transfer-matrix formalism. In Ref. [10], the method is applied to the model fluctuation, Fig. 1, that is, to a piecewise constant frequency. It is found that, in agreement with the correspondence principle, even the mean energy of the quantum oscillator increases exponentially in time, under the same conditions discussed above. Furthermore, the Hamiltonian in the Heisenberg representation was shown to be a generalized harmonic form, that is, a linear combination of the (Schrödinger) operators p^2, q^2 and $pq+qp$, with time-dependent coefficients all diverging exponentially in time.

In Sec. II, we study the dynamics of the evolving quantum state $|n,t\rangle$, with initial condition $|n,0\rangle \equiv |n\rangle$, corresponding to the nth excited state of the zero-time Hamiltonian (here and in what follows, the instant $t=0$ marks the onset of the frequency fluctuation). We also make a detailed comparison

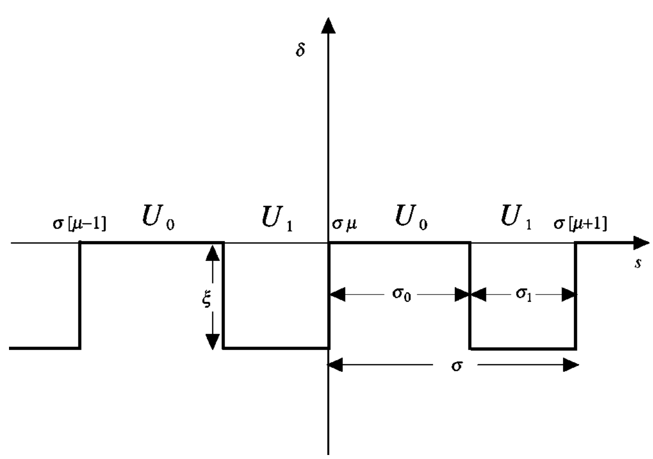

FIG. 1. The square fluctuation $\delta(s/\Omega_0)$ is plotted against the dimensionless time s, between the $(\mu-1)$-th and the $(\mu+1)$-th period.

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between our results and those obtained by other authors, showing that the relevant quantities entering our calculations are special cases of the general expressions developed in Refs. [8,11,12].

In Sec. III, the first-order perturbation theory is applied, by assuming the *evolving* states $|n,t\rangle$ to be unperturbed states, among which a scattering process takes place, due to an additional stationary perturbation $V(q)$. It is shown that the scattering rate between $|n,t\rangle$ and $|m,t\rangle$ vanishes in the long-time limit, if $V(q)$ is short-ranged. This means that the *coherence* of the exponential increase of the mean energy cannot be broken at a perturbative level by short-ranged potentials. Starting from this result, in Sec. IV we discuss the case of the anharmonic terms $V(q)\propto q^j$, $j>2$. We show that such potentials cannot be treated perturbatively *under conditions of exponential increase of the energy*. We then develop a method, reminiscent of the mean-field approximation, showing that a small deviation from harmonicity (measured by the quantity $\epsilon\equiv j-2$) does actually behave as a dissipative channel for the oscillator's energy, and leads to a saturation of the mean energy at long times, if the anharmonic effects tend to increase with the number of elapsed periods of the frequency fluctuation (which is the standard case). A self-consistent equation is produced that makes it possible to calculate the saturation value and the saturation time scale. These are important points in view in the application to realistic systems [4,13].

In Sec. V we discuss an intriguing quantum effect, due to the existence of special instants at which the anharmonic effects (in the coordinate space) tend to *decrease* with the number of elapsed periods of the frequency fluctuation. Accordingly, periodic peaks in the background saturating mean energy are expected to occur at those special instants. It is shown that a saturation of the height of the peaks can be obtained by anharmonic terms in momentum space, proportional to p^j, $j>2$. Hence, besides other possible nonuniversal reasons, the peaks do certainly saturate for relativistic effects, once an expansion of the relativistic kinetic energy in powers of p^2 is accounted for.

II. THE EVOLUTION OF THE QUANTUM STATE IN THE QUASIRESONANT CASE

Given the unperturbed frequency Ω_0 [Eq. (1)] and the mass m of the linear oscillator, we introduce the dimensionless time $s=\Omega_0 t$, momentum $P=p/\sqrt{m\hbar\Omega_0}$, and position $Q=q\sqrt{m\Omega_0/\hbar}$. The model fluctuation Fig. 1, is thereby characterized by a dimensionless period $\sigma=\sigma_0+\sigma_1$, split into a subperiod of duration σ_0, in which the frequency is Ω_0, followed by another subperiod of duration σ_1, in which the frequency is $\Omega_1\equiv\Omega_0\sqrt{1-\xi}$. We stress that ξ will be assumed to be the smallness parameter of the problem. The resulting (dimensionless) Schrödinger Hamiltonian then reads

$$H=\begin{cases}\dfrac{P^2+Q^2}{2}\equiv H_0 & \text{for } s<0 \text{ and } s\in I_0(\mu)\equiv[\mu\sigma,\mu\sigma+\sigma_0[\,,\\[2ex] \dfrac{P^2+(1-\xi)Q^2}{2}\equiv H_1 & \text{for } s\in I_1(\mu)\equiv[\mu\sigma+\sigma_0,(\mu+1)\sigma[\quad(\mu=0,1,2,\ldots).\end{cases}\tag{2}$$

According to Eq. (2) (see also Fig. 1), two unitary operators $U_0(s-\mu\sigma)$ and $U_1(s-\mu\sigma-\sigma_0)$ can be introduced, describing the evolution in the two subperiods, respectively:

$$U_0(s-\mu\sigma)\equiv\exp[-i(s-\mu\sigma)H_0]\quad\text{for } s\in I_0(\mu),$$
$$U_1(s-\mu\sigma-\sigma_0)\equiv\exp[-i(s-\mu\sigma-\sigma_0)H_1]\quad\text{for } s\in I_1(\mu).\tag{3}$$

We study the quantum state evolving in each time interval $I_a(\mu)$ ($a=0,1$) from the initial eigenstate $|n\rangle$ of H_0. They can be obtained from Eq. (3), as $|n,s\rangle_a=\mathcal{U}_a(s)|n\rangle$, where

$$\mathcal{U}_a(s)=U_a(s-(\mu+a)\sigma)[U_1(\sigma_1)U_0(\sigma_0)]^{\mu+a}\quad\text{for } s\in I_a(\mu),\quad a=0,1,\tag{4}$$

is the *total* evolution operator mapping the initial state into the state evolving in the interval $I_a(\mu)$. The Q representation of $|n,s\rangle_a$ in $I_a(\mu)$ follows from the property

$$H_{\text{rev}}^{(a)}(s)=\mathcal{U}_a(s)H_0\mathcal{U}_a^\dagger(s)\Rightarrow H_{\text{rev}}^{(a)}(s)|n,s\rangle_a=E_n^{(0)}|n,s\rangle_a,\tag{5}$$

where $E_n^{(0)}\equiv(n+1/2)$ are the (dimensionless) eigenvalues of H_0. The property, Eq. (5), is useful because the operators $H_{\text{rev}}^{(a)}(s)$ are the "time reversed" form of the Heisenberg Hamiltonians $H^{(a)}(s)=\mathcal{U}_a^\dagger(s)H_a\mathcal{U}_a(s)$. Hence, from the results of Refs. [10,14], one knows that the $H_{\text{rev}}^{(a)}$'s are generalized harmonic Hamiltonians, whose eigenstates in the Q representation can be easily found. The same method used in Ref. [10] to calculate $H^{(a)}(s)$ [therein denoted as $H(t)$] could be applied to $H_{\text{rev}}^{(a)}(s)$ as well, the only difference being a redefinition of the transfer matrices. However, a more efficient method is developed in the Appendix. One of the advantages, with respect to the procedure adopted in Ref. [10], is that the relationship with the *classical* motion equation is made manifest. The resulting expressions for the $H_{\text{rev}}^{(a)}$'s are

$$H_{\text{rev}}^{(a)}(s)=\frac{A_a}{2}P^2+\frac{B_a}{2}Q^2+\frac{C_a}{2}(PQ+QP)\quad(a=0,1).\tag{6}$$

Setting

$$\theta \equiv s - \mu\sigma - \sigma_0, \quad (a-1)\sigma_a \leqslant \theta < a\sigma_a, \tag{7}$$

we get, to first order in ξ [15],

$$\begin{aligned} A_a = &\cosh(2\omega\mu) + \sinh(2\omega\mu)\sin[2\theta(1-\xi)^{a/2} + \sigma_0] \\ &+ a\xi\{\cosh(2\omega\mu)\sin^2(\theta\sqrt{1-\xi}) \\ &+ \sinh(2\omega\mu)\sin(\theta\sqrt{1-\xi})\cos(\theta\sqrt{1-\xi}+\sigma_0)\} \\ &+ aO(\xi^2), \end{aligned} \tag{8a}$$

$$\begin{aligned} B_a = &\cosh(2\omega\mu) - \sinh(2\omega\mu)\sin[2\theta(1-\xi)^{a/2} + \sigma_0] \\ &- a\xi\{\cosh(2\omega\mu)\sin^2(\theta\sqrt{1-\xi}) \\ &- \sinh(2\omega\mu)\sin(\theta\sqrt{1-\xi})\cos(\theta\sqrt{1-\xi}+\sigma_0)\} \\ &+ aO(\xi^2), \end{aligned} \tag{8b}$$

$$\begin{aligned} C_a = &-\sinh(2\omega\mu)\cos[2\theta(1-\xi)^{a/2} + \sigma_0] \\ &- \frac{a\xi}{2}\cosh(2\omega\mu)\sin(2\theta\sqrt{1-\xi}) + aO(\xi^2), \end{aligned} \tag{8c}$$

$$\omega = -\xi\frac{\sin\sigma_0}{2} + O(\xi^2). \tag{8d}$$

In obtaining Eqs. (6)–(8) use has been made of the quasiresonance condition [4,10]:

$$\sigma' \equiv \sigma_0 + \sigma_1\sqrt{1-\xi} = 2\pi l \quad (l \in \mathbb{Z}), \tag{9}$$

which yields the *maximum* rate $|\omega|$ [Eq. (8d)] of exponential increase in the coefficients [16]. The conditions on θ in Eq. (7) ensure that $s \in I_a(\mu)$. It should be noticed that Eqs. (8a)–(8c) are *exact* in the intervals $I_0(\mu)$. The first-order approximation in ξ only affects the quantities in the intervals $I_1(\mu)$ and the value of ω [Eq. (8d)]. This is the reason why the next numerical calculations will be referred to the intervals $I_0(\mu)$.

Given the explicit form, Eq. (6), of $H_{\rm rev}^{(a)}$, we can find its eigenstates $\langle Q|n,s\rangle_a$ in the Q representation under the condition

$$i\frac{d|n,s\rangle_a}{d\theta} = H_a|n,s\rangle_a \quad [s \in I_a(\mu)], \tag{10}$$

that is, the Schrödinger equation in each interval $I_a(\mu)$. Let $\langle Q|n\rangle$ be the Q representation of the nth eigenstate of the Hamiltonian H_0. Then, it can be seen by direct inspection that the *normalized* wave function

$$\begin{aligned} \langle Q|n,s\rangle_a = &\sqrt{y_a(\theta,\mu)}\exp[-i(f_a^{(n)} + Q^2 z_a(\theta,\mu)/2)] \\ &\times\langle y_a(\theta,\mu)Q|n\rangle \end{aligned} \tag{11}$$

is an eigenfunction of $H_{\rm rev}^{(a)}(s)$, with eigenvalue $E_n^{(0)}$, satisfying the condition, Eq. (10), provided that

$$y_a(\theta,\mu) = \frac{1}{\sqrt{A_a(\theta,\mu)}}, \quad z_a(\theta,\mu) = \frac{C_a(\theta,\mu)}{A_a(\theta,\mu)}, \tag{12}$$

$$\frac{df_a^{(n)}}{d\theta} = (n+1/2)y_a^2(\theta,\mu).$$

We notice from Eqs. (8) that $\lim_{\xi\to 0} y_a(\theta,\mu) = 1$ and $\lim_{\xi\to 0} z_a(\theta,\mu) = 0$, so that the standard solution $\langle Q|n,s\rangle_a = \exp(-iE_n^{(0)}s)\langle Q|n\rangle$ is recovered in the limit of vanishing frequency fluctuation. We can thereby conclude that Eqs. (11) and (12) yield the coordinate representation of the evolving quantum states that we are after, in each time interval $I_a(\mu)$. In particular, from Eqs. (2) and (11) it is possible to verify that

$$\begin{aligned} {}_a\langle s,n|H_0|n,s\rangle_a = &E_n^{(0)}[\cosh(2\omega\mu) \\ &+ a\xi\sinh(2\omega\mu)\sin(\theta\sqrt{1-\xi})\cos(\theta\sqrt{1-\xi} \\ &+ \sigma_0)] + aO(\xi^2) \quad (a=0), \end{aligned} \tag{13}$$

which coincides with Eq. (5b) of Ref. [10], when evaluated at $\theta = 0$ (on restoring the dimensioned quantities). A direct inspection reveals the continuity in θ (to first order in ξ) of Eqs. (8) and (12), when passing from $a=0$ to $a=1$ in the μth interval, and from $a=1$ to $a=0$ at the edge between the μth and the $(\mu+1)$-th period. It should be noticed that the evolution of the quantum state is determined by *two* relevant time scales, [see Eqs. (11) and (12)]: one is the ''macroscopic'' (dimensionless) time $\omega\mu$ that enters the hyperbolic functions only, and accounts for the *long-time* effects of the fluctuation δ. These are shown to increase *exponentially* period by period [due to the quasiresonant condition (9)]. In addition, there is the ''microscopic'' time θ (Fig. 2), that enters the trigonometric functions only, and accounts for the details of the evolution within each period of the frequency fluctuation.

By means of Eqs. (11) and (12), one knows the evolution of the basis $\{|n\rangle\}$ in the complementary time intervals $I_a(\mu)$, respectively. So, given any initial state, one can study its complete evolution by projecting it on $\{|n\rangle\}$. In particular, we will take $|n\rangle$ as the initial state.

Taking the square modulus of Eq. (11), the probability density in Q turns out to be $y_a(\theta,\mu)|\langle y_a(\theta,\mu)Q|n\rangle|^2$, which corresponds to the square modulus of the nth eigenstate of a standard harmonic Hamiltonian, with the coordinate rescaled by a factor $y_a(\theta,\mu)$. This factor decreases exponentially with the number $\mu = 0,1,2,\ldots,$ of elapsed periods of the frequency fluctuation [see the first Eq. (12) and Eq. (8a)]. This is true for *almost all* values of the continuous time θ describing the evolution in each subperiod. However, at the isolated values θ_a^* of θ,

$$\begin{aligned} \theta_a^* = \frac{1}{2(1-\xi)^{a/2}}\Bigg[&\frac{\pi}{2} + k\pi - \sigma_0 + \frac{a\xi}{2}\cos\sigma_0\coth(2\omega\mu) \\ &+ aO(\xi^2)\Bigg] \end{aligned} \tag{14}$$

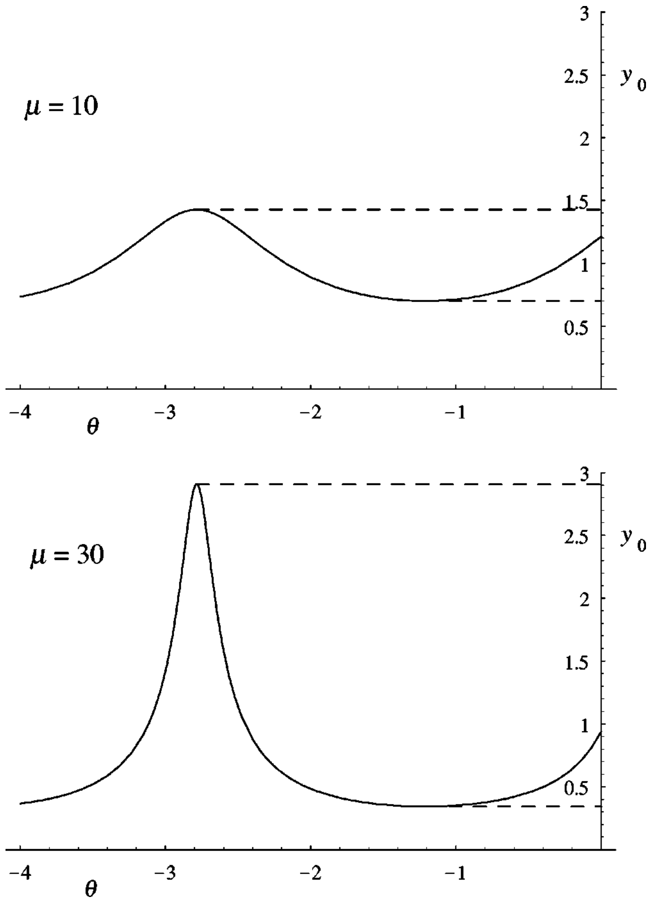

FIG. 2. Plot of $y_0(\theta,\mu)$ [first Eq. (12)] at $\mu=10$ and $\mu=30$, for $-\sigma_0 \leqslant \theta < 0$ and $\sigma_0=4$, $\sigma_1=9$, $l=2$, $\xi=9.40\times 10^{-2}$. Note the increasingly high peak, with decreasingly small width at $\theta=\theta_0^*$.

(with k odd or even according to the positive or negative sign of ω), such that

$$\sin[2\theta_a^*(1-\xi)^{a/2}+\sigma_0] = -\mathrm{sgn}(\omega) + aO(\xi^2) \quad (a=0,1); \tag{15}$$

the scaling factor $y_a(\theta_a^*,\mu)$ *increases* exponentially with μ, as can be seen from Eqs. (12) and (8a) (see also Fig.2). Hence the evolution makes the probability density spread out exponentially with the number of elapsed periods, except at the special instants

$$s_a^*(\mu) = \theta_a^* + \mu\sigma + \sigma_0, \tag{16}$$

for which one has, instead, $\lim_{\mu\to\infty}|\langle Q|n,s_a^*(\mu)\rangle_a|^2 = \delta(Q)$, that is, a spatial probability density shrinking to a δ function. In Secs. IV and V we will see a nontrivial consequence of this result.

At this stage, we recall that the method used here is based on the transfer-matrix formalism (the Appendix). This method is especially useful when the frequency fluctuations are periodic, but the results are (as they must be) special cases of the general expressions developed for any frequency fluctuation. For this comparison we refer to the paper by Lewis and Riesenfeld [8]. One can notice that, by construction, the piecewise defined operator [Eq. (6)]

$$H_{\mathrm{rev}}(s) = \begin{cases} H_{\mathrm{rev}}^{(0)} & \text{for } s \in I_0(\mu), \\ H_{\mathrm{rev}}^{(1)} & \text{for } s \in I_1(\mu) \end{cases} \tag{17}$$

is nothing but the invariant I [Eq. (44) of Ref. [8]] with $I(s=0)=H_0$. In fact, one has, by definition, $H_{\mathrm{rev}}(s) = U(s)H_0U^\dagger(s)$, where $U(s)$ is the evolution operator. However, any operator in the form $F_{\mathrm{rev}} = UF_0U^\dagger$ is a quantum invariant, provided that $\partial F_0/\partial s = 0$. This is because the *Heisenberg* representation $U^\dagger F_{\mathrm{rev}} U$ of F_{rev} is just the time-independent operator F_0. This clarifies the relationship between the transfer-matrix method and the invariant method of Lewis and Riesenfeld [8]. For a more specific comparison, the relevant quantity is the scaling factor $y_a(\theta,\mu)$ [Eqs. (12) and (8a)]. It can be verified by direct inspection that, on setting $\rho_a = y_a^{-1}$, one has

$$\frac{d^2\rho_a}{d\theta^2} + (1-a\xi)\rho_a - \frac{1}{\rho_a^3} = 0 \quad (a=0,1). \tag{18}$$

Hence the piecewise defined function

$$\rho(s) = \begin{cases} y_0^{-1}(\theta,\mu) & \text{for } s \in I_0(\mu), \\ y_1^{-1}(\theta,\mu) & \text{for } s \in I_1(\mu) \end{cases} \tag{19}$$

is just a special case of the function ρ defined by Eq. (45) in Ref. [8]. From the third Eq. (12) and from Eq. (19) it can be seen that the phase

$$f^{(n)}(s) = \begin{cases} f^{(0)} & \text{for } s \in I_0(\mu), \\ f^{(1)} & \text{for } s \in I_1(\mu) \end{cases} \tag{20}$$

is just equivalent to the phase, Eq. (61) of Ref. [8] (a sign apart). In general, our Eq. (11) for the evolving quantum state is a special case of Eq. (27) in Ref. [11] and Eq. (3.6) in Ref. [12]. The only point of caution is that our expressions are piecewise defined, due to the special form of the frequency fluctuations (Fig. 1).

III. FIRST-ORDER PERTURBATION THEORY IN THE QUASIRESONANT CASE

Having found the unperturbed evolving states $|n,s\rangle_a$'s, a perturbation theory can be applied to study the transition probability $P_V(m,n;s)$ between $|n,s\rangle_a$ and $|m,s\rangle_a$, due to a perturbative potential $V(Q)$ that we assume, for the moment, to be *short-ranged*. The first-order approximation yields [17]

$$P_V(m,n;s) = \frac{1}{(\hbar\Omega_0)^2}\left|\int_0^s d\tau_a \langle \tau,m|V|n,\tau\rangle_a\right|^2 \quad (n\neq m), \tag{21}$$

which leads one to study the integral

$$\int_0^s \mathrm{d}\tau\ {}_a\langle\tau,m|V|n,\tau\rangle_a = \underbrace{\sum_{\beta=0}^{\mu-1}\sum_{a=0}^{1}\int_{I_a(\beta)}\mathrm{d}\tau\ldots}_{J(\mu)} + \underbrace{\int_{\mu\sigma}^{s}\mathrm{d}\tau\ldots}_{j(s,\mu)} \quad \text{for } \mu\sigma \le s < (\mu+1)\sigma\ . \tag{22}$$

The first sum $J(\mu)$ in Eq. (22) accounts for the evolution in the μ elapsed periods of the frequency fluctuations, while the second term $j(s,\mu)$ accounts for the ''residual'' evolution in the actual time interval $[\mu\sigma,s]$. By means of Eq. (11), it is possible to write $J(\mu)=\Sigma_{\beta=0}^{\mu-1}\Sigma_{a=0}^{1}j_a(\beta)$, with

$$j_a(\beta)=\int_{(a-1)\sigma_a}^{a\sigma_a} d\theta\, y_a(\theta,\beta)\exp[-i(f_a^{(n)}(\theta,\beta) - f_a^{(m)}(\theta,\beta))]\int_{-\infty}^{\infty} dQ\langle m|y_a(\theta,\beta)Q\rangle V(Q) \times\langle y_a(\theta,\beta)Q|n\rangle. \tag{23}$$

Now we can take Eq. (23) in the limit $\beta \gg 1$, on assuming $0<|\int_{-\infty}^{\infty}dQV(Q)|<\infty$:

$$j_a(\beta)\cong\langle m|0\rangle\langle 0|n\rangle\int_{\infty}^{\infty}dQV(Q)\int_{(a-1)\sigma_a}^{a\sigma_a}d\theta\, y_a(\theta,\beta) \times\exp[-i(f_a^{(n)}(\theta,\beta)-f_a^{(m)}(\theta,\beta))] \quad (\beta\gg 1). \tag{24}$$

Equation (24) follows from the property that the integral of $y_a(\theta,\beta)$ in θ decreases exponentially with β [18]. From the same property, one can conclude that

$$\lim_{\mu\to\infty}|J(\mu)|=|J(\infty)|<\infty\ ,$$

$$\lim_{\mu\to\infty}|j(s,\mu)|=0 \quad \text{uniformly in } s. \tag{25}$$

On using Eq. (25) in Eqs. (21) and (22), one finally gets

$$\lim_{s\to\infty}P_V(m,n,s)=P_V(m,n,\infty)<\infty \Rightarrow \lim_{s\to\infty}\frac{dP_V(m,n,s)}{ds}=0. \tag{26}$$

From Eq. (26) it follows that, for short-ranged potentials, the rate of transition (probability per unit time) between any pair of evolved states vanishes under the quasiresonance condition that leads the mean energy Eq. (13) to diverge exponentially. This result is far from trivial, since in the usual perturbation theory for a *stationary* unperturbed Hamiltonian the integral in Eq. (21) would become proportional to $s\delta(E_n^{(0)}-E_m^{(0)})$ in the limit of large s [19]. A similar expression would be obtained even for our *time-dependent* Hamiltonian, if it were not for the exponential dependence of $y_a(\theta,\mu)$ on μ. In fact, if we took the quantity σ' far enough from the quasiresonant condition Eq. (9), the hyperbolic functions in Eq. (12) would be replaced by *trigonometric* functions [20]. In this case the quantity $j_a(\beta)$ in Eq. (23) would not vanish in the limit $\beta\to\infty$, and the probability of transition, Eq. (21), would be similar to the usual expression. It is indeed possible to explain the formal result, Eq. (26), by a physical argument. The perturbative approach does in general transform the *deterministic* (in the quantum sense) evolution due to the *exact* Hamiltonian (unperturbed plus perturbation) into a *probabilistic* problem. For this to be possible, the perturbative approach must produce finite transition rates. These make it possible to replace (on suitable time scales) the exact evolution with an incoherent sequence of scattering events between the unperturbed states. A measure of the incoherence is given by the broadening of the unperturbed level, which is related to the level's mean life. In the quasiresonant case, there is a basic difference: the unperturbed states themselves evolve in such a way that the mean energy increases exponentially (at the expense of the external field producing the periodic frequency changes) with increasingly large fluctuations. For times short compared to $|\omega|^{-1}$, this *coherent* effect of broadening may be negligible, with respect to the incoherent effect due to the perturbation, but the former becomes overwhelming at long times, since the latter is finite at any time. The coherence reflects itself in the vanishing of the transition rates. As an important consequence, it is impossible, under the quasiresonant condition, to suppress the exponential increase of the oscillator's energy by means of a short-ranged perturbation.

IV. LONG-RANGE ANHARMONIC EFFECTS AND SATURATION UNDER THE QUASIRESONANT CONDITION

So far, the long-range anharmonic effects have been completely ignored. In the standard case of a constant frequency, it is well known that anharmonic potentials of the form

$$V(Q)=\lambda_j Q^j \quad (j=3,4,\ldots) \tag{27}$$

can be treated perturbatively, with the same methods as the short-ranged potentials. In contrast, we show that this is not always possible if the frequency fluctuates. In particular, let the quasiresonant condition, Eq. (9), be satisfied. On inserting the potentials, Eq. (27), in Eq. (23), one gets

$$j_a(\beta)=\lambda_j\langle m|Q^j|n\rangle\int_{(a-1)\sigma_a}^{a\sigma_a}d\theta[y_a(\theta,\beta)]^{-j} \times\exp[-i(f_a^{(n)}(\theta,\beta)-f_a^{(m)}(\theta,\beta))]. \tag{28}$$

Equation (28) shows that the transition rates now diverge in the long-time limit, due to the exponential divergence of $[y_a(\theta,\beta)]^{-j}$. Note that this would *not* be the case for a stationary Hamiltonian, or far enough from the quasiresonant condition, Eq. (9). Once again, it is the exponential behavior in the macroscopic time that marks the difference between a standard case and the present case.

In Refs. [4,13], it was argued that the anharmonic effects lead the oscillator's mean energy to saturate. The stationary levels are therein related to massless bosons (for instance, phonons), and the anharmonic potentials are regarded to as boson-boson interactions producing a finite scattering rate at a perturbative level. Hence, the method looks perturbative at a first glance (which would contrast with the arguments above). However, the anharmonic scattering rate is then *assumed* to act like an ''annihilation'' rate for the bosons, contrasting with the ''creation'' rate, Eq. (8d). This ansatz is the nonperturbative ingredient of the approach just outlined. It is only thanks to it that the saturation of the energy (number of bosons) can be obtained from the detailed balance between the creation and annihilation rate. Our present aim is to show that the saturation does actually follow from anharmonicity, in a more formal way. The approach adopted is totally nonperturbative, and starts from the following physical arguments. As mentioned in Sec. II, the quasiresonant condition, Eq. (9), is of special interest because it yields the *maximum* rate $|\omega|$ of exponential increase of the mean energy. However, there is a band of possible values of $\sigma'\equiv\sigma_0+\sigma_1\sqrt{1-\xi}$ for which an exponential increase is produced, with smaller and smaller rates, vanishing at the band edges [20]. The physical origin of the exponential increase of the classical amplitude (in the absence of anharmonic effects) is that when σ' falls in the band, the moving ''walls'' of the harmonic potential are in constructive phase with the amplitude, and always impart a positive amount of energy to the particle, in each period of the frequency fluctuation. The evidence that this effect goes on without limit, and is not transient (as it would be for a free particle rebounding elastically between two periodically moving walls), is a special feature of the *quadratic* shape of the potential that yields an amplitude-independent period. The anharmonicity does break this ideal condition, and is thus expected to dephase the motion more and more with increasing amplitude, driving the system out of the band where the constructive interference is possible. In an attempt to describe this effect more formally, let us write the Hamiltonian with an anharmonic potential [Eq. (27)], in the dimensionless coordinate, momentum, and time

$$H_{\mathrm{anh}}=\frac{P^2+\varpi^2[1+\xi(s)/\varpi^2]Q^2}{2}\approx\frac{P^2+\varpi^2[1+\xi(s)]Q^2}{2},$$

$$\varpi^2\equiv 1+2\lambda_j Q^{j-2}, \tag{29}$$

where $\xi(s)\equiv\delta(s/\Omega_0)$ is the frequency fluctuation in the dimensionless time (described, for example, by Fig. 1). The factorization in the right-hand side (rhs) of Eq. (29) means that we are trying to include the anharmonic term into a new effective frequency ϖ. The second approximate expression stems from assuming that the relevant effect of the potential Eq. (27) is to be small to order ξ, at most (this will follow self-consistently, in the case of interest). The quantum state evolving with a Hamiltonian in the form of Eq. (29) could be easily found if ϖ were a smooth function of the dimensionless time s, varying on time scales large compared to the period σ of the fluctuation. In this case the state would change *adiabatically* in ϖ, and nonadiabatically in $\xi(s)$. The solution would be still in the form Eq. (11), once $\varpi\Omega_0$ is taken as a new quasistationary frequency. Time, coordinate, and momentum can all be rescaled accordingly, in order to get the same form, Eq. (2), of the Hamiltonian. The rescaling of time, however, yields the effect that we are after; that is, a shifting υ of the *rescaled* σ', from the quasiresonant value of Eq. (9):

$$\sigma'=2\pi l\rightarrow\sigma'\varpi=2\pi l+\upsilon\Rightarrow\upsilon=2\pi l(\varpi-1). \tag{30}$$

If we wish the rescaled value $\sigma'\varpi$ to fall inside the band of values yielding an exponential increase, υ must be of order ξ, at most, since the bandwidth is of the same order [4,10]. The shifting υ now produces a *smaller* rate of exponential increase $|\omega_{\mathrm{eff}}|$ [20]. In fact, from Eq. (4) of Ref. [4], it is not difficult to see that $\omega_{\mathrm{eff}}=\omega\sqrt{1-\upsilon^2/\omega^2}$, so that, from Eq. (30), one has

$$\omega_{\mathrm{eff}}{}^2\approx\omega^2\left[1-\left(\frac{2\pi l(\varpi-1)}{\omega}\right)^2\right],\quad \mathrm{sgn}(\omega_{\mathrm{eff}})=\mathrm{sgn}(\omega). \tag{31}$$

The crucial point is that $\varpi(Q)$, defined in Eq. (29), is a *Q-dependent operator,* not a smooth function of s alone. A standard procedure for replacing operators with suitable c numbers is the mean-field approximation. One assumes a given form of the state, resulting from a solvable Hamiltonian, in terms of unknown quantities. Then one replaces the ''unsolvable'' operators with their mean values on the state itself. This usually leads to *self-consistent* equations for the unknown quantities. The main advantage of this method is that one may produce nonperturbative solutions. The main disadvantage is that the degree of confidence of the approximation is difficult to control. Hence the validity of the method rests on the physical insights leading one to guess the form of the solution. In our present case, the guess on the state is nonperturbative, since we take the form, Eq. (11), with an unknown $\omega_{\mathrm{eff}}(s)$ replacing ω in the supplementary equations (12). The underlying insight is that the main effect of anharmonicity is to dephase the motion progressively, driving the system off from the ideal constructive interference between the frequency fluctuation and the oscillator's amplitude. It is clear that the deviation from anharmonicity is given by [Eq. (29)]

$$\varpi-1\approx\lambda_j Q^{\epsilon},\quad \epsilon=j-2>0. \tag{32}$$

The mean-field approximation results in assuming that $\omega_{\mathrm{eff}}(s)$ follows from the quantum average of the rhs member of Eq. (31), on account of Eq. (32). Since the quantum state equation (11) now contains $\omega_{\mathrm{eff}}(s)$ in place of ω [Eq. (12)], the procedure indicated yields the following self-consistent equation:

$$\begin{aligned}2\omega_{\mathrm{eff}}\mu&\equiv x\\&=2\omega\mu\left\{1-\left(\frac{2\pi l\lambda_j}{\omega}\right)^2\langle n||Q|^{2\epsilon}|n\rangle\right.\\&\qquad\left.\times[\cosh x+\sin(2\theta+\sigma_0)\sinh x]^{\epsilon}\right\}^{1/2},\end{aligned} \tag{33}$$

for $s \in I_0(\mu)$ [a similar equation can be obtained for $s \in I_1(\mu)$]. In writing down Eq. (33), the rescaling $\sigma_0 \to \varpi \sigma_0$ has been neglected, since the correction is of order ξ at most. We notice that the variable of interest x is now a function both of the macroscopic time $\omega\mu$ and of the microscopic time $\theta \in [-\sigma_0, 0[$. This reflects on the mean quantum energy, which follows from Eq. (13):

$$\langle H_{\text{anh}} \rangle_n = (n+1/2)\cosh[x(\mu,\theta)], \quad s \in I_0(\mu), \tag{34}$$

starting from the nth stationary eigenstate. For Eq. (33) to make sense, it is clear that the initial value $\omega_{\text{eff}}(s=0) = \omega_{\text{eff}}(\mu=0)$ must be real. This means that

$$\left(\frac{2\pi l \lambda_j}{\omega}\right)^2 \langle n||Q|^{2\epsilon}|n\rangle < 1. \tag{35}$$

Equation (35) is an *a priori* condition, in order that the initial state is still available to absorb energy from the frequency fluctuations, despite the presence of an anharmonic term.

In contrast to the harmonic case, the anharmonic mean energy now depends explicitly on θ, which yields nontrivial effects. In Fig. 3 a plot of Eq. (34) is shown, obtained from a numerical solution of Eq. (33). The plot exhibits a saturating background, as expected [Fig. 3(a)], but also periodic *nonsaturating* peaks of decreasingly small width [Fig. 3(b)]. We refer to those peaks as ''special'' quantum effects, to be discussed in Sec. V.

In concluding the present section, we stress that the *true* smallness parameter of the mean-field approximation leading to Eqs. (33) and (34) is $\epsilon = j-2$; that is, the deviation from the quadratic power of the anharmonic potential [Eq. (32)]. The smaller the ϵ, the more reliable our approximation of treating $|Q|^\epsilon$ like a c number and the harmonic factor Q^2 like an operator. Hence it is clear that applying Eqs. (33) and (34), as they stand, to the familiar cases $\epsilon = 1,2,\ldots$, is definitely arbitrary. Those equations should be taken only in the limiting (and certainly unrealistic) case $\epsilon \ll 1$. Notice, in passing, that the unrealistically large saturation values reported in Fig. 3 are just due to the smallness of $\epsilon = 0.1$. In conclusion, the present approach to the anharmonic problem is of little use, in practical cases, but it does certainly help to elucidate why anharmonicity (under the quasiresonant condition) is expected to produce energy saturation as the main effect.

V. SPECIAL QUANTUM EFFECTS

As mentioned above, the mean quantum energy, Eq. (34), exhibits nonsaturating peaks of decreasing width, centered around periodically recurring instants. It is an easy matter to show that those instants are given by Eq. (16), for the special values θ_0^* of θ, such that

$$1+\text{sgn}(\omega)\sin(2\theta_0^*+\sigma_0) = 1+\text{sgn}(x)\sin(2\theta_0^*+\sigma_0) = 0.$$

In this case, in fact, the combination of hyperbolic functions in the rhs of Eq. (33) sums to $\exp[-|x(\mu,\theta_0^*)|]$, so that $x(\mu,\theta_0^*) \to 2\omega\mu$ with diverging μ, and the corresponding mean energy, Eq. (34), tends to the value, Eq. (13), as if the anharmonic terms were ''switched off'' at the special instants, Eq. (16). Similar special points are obviously present

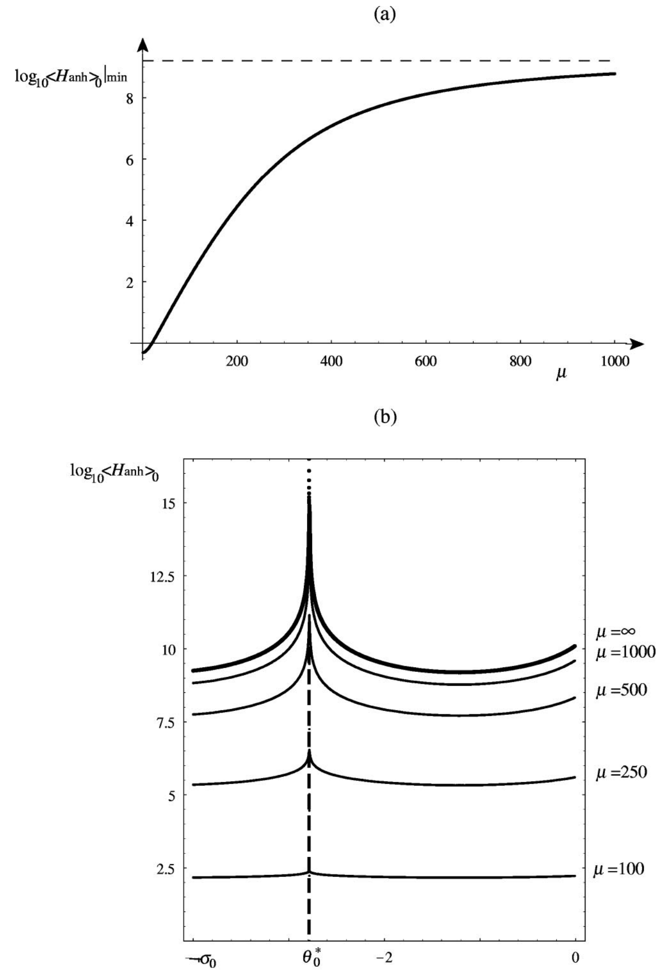

FIG. 3. Saturation of the mean energy due to the anharmonic effects [Eqs. (33) and (34)]. (a) Background saturation of the *minimum* values in each interval $I_0(\mu)$, as a function of the number μ of elapsed periods. (b) Detailed evolution of the mean energy within the intervals $I_0(\mu)$, as a function of the microscopic time θ. Note the peaks at $\theta=\theta_0^*$, with nonsaturating height (special quantum effects). Selected values are $\sigma_0=4$, $\sigma_1=9$, $l=2$, $\xi=9.40\times10^{-2}$, $\lambda_j=10^{-3}$, and $\epsilon=10^{-1}$.

in the intervals $I_1(\mu)$ too [Eq. (15)]. The origin of this effect is quite clear. In Sec. II, we have shown that at the instants, Eq. (16), the evolving probability density in Q shrinks to a δ function with diverging μ, so it is obvious that the influence of any potential of the form, Eq. (27), becomes vanishingly small after many frequency fluctuations have elapsed. The fact that the mean energy remains an exponentially increasing function of μ even at the instants, Eq. (16), simply reveals the genuine *quantum* origin of the effect. A classical oscillator with vanishing amplitude would yield a vanishing energy. It is the Heisenberg principle that makes the mean energy diverge when the wave function becomes more and more localized.

The first-principle origin discussed above for the special quantum effects suggests that at the instants, Eq. (16), the mean energy is actually less sensitive to the anharmonic effects, independently of the formal treatment of the problem. Peaks should then be expected, even from a more rigorous approach. All the way, there is a further mechanism of saturation of the peaks in Fig. 3, ignored so far, that works even in the present scheme of approximation. Since the peaks are due to divergently large fluctuations of the momentum, the simplest way to achieve saturation is to introduce more-than-quadratic terms in momentum space. A universal effect of this kind is certainly produced by the *relativistic* corrections to the kinetic energy, represented by increasing powers of P^2. Should the nonsaturating peaks ''survive'' even to more rigorous calculations, the relativistic corrections would provide their ultimate mechanism of saturation.

VI. CONCLUSIONS

As a continuation of Ref. [10], the present paper deals with the dynamics of a quantum oscillator, whose frequency is forced to fluctuate periodically in time by some external field. Reference [10] was mainly concerned with the exponential increase of the mean energy under the quasiresonant condition. Here we have studied the full evolution of the quantum state.

In the absence of any perturbation (Sec. II), the evolving state turns out to be (a phase factor apart) a standard eigenfunction of the harmonic oscillator [Eq. (11)], whose coordinate scales with a factor $y_a(\theta,\mu)$ [first Eq. (12)]. This scaling factor depends on the number μ of elapsed periods of the frequency fluctuation, and on the continuous time θ, describing the ''residual'' evolution within the actual period considered. For almost all values of θ, the probability density spreads out in space, since $y_a(\theta,\mu)$ decreases exponentially with μ. However, at certain special values θ_a^* [Eq. (15)], $y_a(\theta_a^*,\mu)$ becomes an exponentially *increasing* function of μ (Fig. 2), and the probability density tends to shrink to a δ function with increasing μ. An interesting aspect, which we leave to discussion for future works, is that the phase factor of the wave function is a (quadratic) function of the coordinate, which makes the state carry a (localized) probability current.

For periodic frequency fluctuations, the transfer-matrix method adopted here (the Appendix), is most convenient. In the final part of Sec. II, we have shown that the transfer-matrix method is nothing but an alternative way to calculate the explicit form of the quantum invariant of interest. This provides the relationship between the present approach and the invariant theory [8]. Hence, our piecewise defined quantities, Eqs. (17), (19), and (20) are special cases of the general expressions found elsewhere, with the invariant method or with other related approaches [8,11,12].

In the presence of an additional anharmonic potential $V(Q)$, one is faced with another intriguing result: under the quasiresonant condition, the system is *underperturbed* if $V(Q)$ is short-ranged (Sec. III). This means that $V(Q)$ produces transition rates between the evolving unperturbed states, which vanish in the long-time limit. This follows from the coherent broadening of the energy due to the unperturbed evolution itself, which always overcomes the incoherent broadening due to short-ranged perturbation, at sufficiently long times. In contrast, if $V(Q)$ is a more-than-quadratic power of the coordinate [Eq. (27)], the system becomes *overperturbed*. Namely, the transition rates *diverge* at long times (Sec. IV). The conclusion is that, under the quasiresonance condition (or any other condition of exponential increase of the energy [20]), the perturbation theory is either ''irrelevant,'' or inapplicable. Between these two cases, a border class of potentials (for instance, $V(Q)\propto\ln[|Q|+|c|]$; $[|Q|+|c|]^{-1}$), could actually exist, yielding finite transition rates. We leave these cases to future investigations.

Since the long-range anharmonic potentials cannot be treated perturbatively, we have proposed a nonperturbative approach in order to support the reasonable guess that the anharmonicity should make the oscillator's mean energy *saturate* at long times (Sec. IV). This approach is based on the idea that anharmonicity does break the ideal constructive interference between the moving ''walls'' of the harmonic well, and the increasingly large amplitude of the oscillator. For this effect to go on indefinitely, it is indeed necessary that the period of the motion be independent of its amplitude. By extracting from the potentials, Eq. (27), the anharmonic factor $|Q|^{j-2}$, and by treating its power $\epsilon\equiv j-2$ as the smallness parameter of a mean-field approximation, we can write down a self-consistent equation for the effective rate of exponential increase of the energy, in the presence of anharmonic terms [Eq. (33)]. This effective rate does actually vanish in the long-time limit, and the corresponding mean energy, Eq. (34), saturates accordingly [Fig. 3(a)]. However, there are special instants, in each period of the frequency fluctuation, at which the saturation does not occur. This results in the presence of peaks [Fig. 3(b)], which we call ''special'' quantum effects. Their genuine quantum origin is discussed in Sec. V. It is argued therein that, besides other possible nonuniversal mechanisms, a saturation process for the peaks is provided by the relativistic corrections to the kinetic energy, which yield anharmonic terms in momentum space.

The future programs of the present research will move along two lines. First, we plan to approach the quasiresonant case in the framework of the invariants' theory [7,8], in an attempt to improve our understanding of the anharmonic effects. Second, we will try to apply the present results to a strongly degenerate system of fermionic oscillators (for example, electrons in a metal under the influence of a fluctuating magnetic field). The effects of the exclusion principle are indeed far from trivial, as anticipated, in a very qualitative way, in Ref. [13].

APPENDIX

We report here the calculations leading to the ''time reversed'' Heisenberg Hamiltonians $H_{\rm rev}^{(a)}(s)=\mathcal{U}_a(s)H_0\mathcal{U}_a^\dagger(s)$ [see Eq. (5)]. First of all, we note that, from the unitarity of $\mathcal{U}_a(s)$ [defined in Eq. (4)], it is sufficient to study the transformation on the Schrödinger operators P and Q. Then, the transformed Hamiltonians $H_{\rm rev}^{(a)}(s)$ are obtained by squaring the resulting transformed operators. Using the factorization of $\mathcal{U}_a(s)$ in terms of the unitary operators U_a, one can take advantage of the formula [14,19]

$$\mathcal{O}_{\rm tr}\equiv e^{iS}\mathcal{O}e^{-iS}=\mathcal{O}-i[\mathcal{O},S]+\frac{(-i)^2}{2}[[\mathcal{O},S],S]+\cdots, \tag{A1}$$

which yields the transform of an arbitrary operator $\mathcal{O}$ under the transformation induced by a Hermitian operator S. In our case $\mathcal{O}$ is simply P or Q and $S\propto H_a$. Now, the key property of the harmonic Hamiltonians is that $P_{\rm tr}$ and $Q_{\rm tr}$ are still *linear* combinations of the initial (Schrödinger) operators P and Q. To be more precise, let us consider the evolution in the subintervals $I_0(\mu)$. The evolution of H_0 in the first subperiod $I_0(0)$ is trivial. So we can conveniently rewrite $H_{\rm rev}^{(0)}(s)$ as

$$\begin{aligned} H_{\rm rev}^{(0)}(s) &= U_0(\theta)[U_0(\sigma_0)U_1(\sigma_1)]^{\mu} \\ &\quad \times H_0[U_1^{\dagger}(\sigma_1)U_0^{\dagger}(\sigma_0)]^{\mu}U_0^{\dagger}(\theta), \\ &\quad \theta \equiv s-\mu\sigma-\sigma_0. \end{aligned} \tag{A2}$$

The μ elapsed periods enter the power $[U_0U_1]^{\mu}$ in Eq. (A2), and can be accounted for by iterating the two basic transformations obtained from Eq. (A1):

$$S=-\sigma_1 H_1 \Rightarrow \begin{pmatrix} P_{\rm tr}^{(1)} \\ Q_{\rm tr}^{(1)} \end{pmatrix} = \begin{pmatrix} c_1 & \sqrt{1-\xi}s_1 \\ -\dfrac{s_1}{\sqrt{1-\xi}} & c_1 \end{pmatrix} \begin{pmatrix} P \\ Q \end{pmatrix},$$

$$S=-\sigma_0 H_0 \Rightarrow \begin{pmatrix} P_{\rm tr}^{(0)} \\ Q_{\rm tr}^{(0)} \end{pmatrix} = \begin{pmatrix} c_0 & s_0 \\ -s_0 & c_0 \end{pmatrix} \begin{pmatrix} P \\ Q \end{pmatrix}, \tag{A3}$$

$$c_0 \equiv \cos\sigma_0, \quad s_0 \equiv \sin\sigma_0,$$

$$c_1 \equiv \cos(\sigma_1\sqrt{1-\xi}), \quad s_1 \equiv \sin(\sigma_1\sqrt{1-\xi}).$$

Keeping track of the order imposed by Eq. (A2), the evolution over one period is ruled by a single transfer matrix:

$$\begin{pmatrix} P_{\rm tr}(\mu=1) \\ Q_{\rm tr}(\mu=1) \end{pmatrix} = \mathsf{T} \begin{pmatrix} P \\ Q \end{pmatrix},$$

$$\mathsf{T} = \begin{pmatrix} c_1c_0-\sqrt{1-\xi}s_1s_0 & c_1s_0+\sqrt{1-\xi}s_1c_0 \\ -\dfrac{s_1c_0}{\sqrt{1-\xi}}-c_1s_0 & c_1c_0-\dfrac{s_1s_0}{\sqrt{1-\xi}} \end{pmatrix}. \tag{A4}$$

At this stage, one observes the strict analogy with the transfer matrix of the *classical* case (see [4] and references quoted therein). The transformed P and Q at $s=\mu$ are obtained through the application of the matrix T^{μ}. It is useful to note that $\det \mathsf{T}=1$, so that the eigenvalues of T can be expressed in the form $\lambda_{\pm}=\exp(\pm\omega)$. Thus, the action of T^{μ} on the basis of eigenvectors of T itself is simply given by $\lambda_{\pm}^{\mu}=\exp(\pm\omega\mu)$. Though these calculations can be made exactly, for our purposes it is sufficient to retain the first-order contribution in ξ. Under the quasiresonant condition, Eq. (9), the maximum value of ω is given by Eq. (8d), and the transformed operators are

$$P_{\rm tr}(\mu) = [\cosh(\omega\mu)+s_0\sinh(\omega\mu)]P - c_0\sinh(\omega\mu)Q,$$

$$Q_{\rm tr}(\mu) = -c_0\sinh(\omega\mu)P + [\cosh(\omega\mu)-s_0\sinh(\omega\mu)]Q. \tag{A5}$$

Now one is left with the evolution in the residual time θ, under the action of U_0 [Eq. (A2)]. The transfer matrix is similar to Eq. (A3), with θ replacing σ_0. The transformed P and Q at time $s\in I_0(\mu)$ are given by the linear combinations

$$\begin{aligned} P_{\rm tr}(s) &= A_{\mu}(\theta)P + B_{\mu}(\theta)Q, \\ Q_{\rm tr}(s) &= C_{\mu}(\theta)P + D_{\mu}(\theta)Q, \end{aligned} \tag{A6}$$

with

$$\begin{aligned} A_{\mu}(\theta) &= \cosh(\omega\mu)\cos\theta + \sinh(\omega\mu)\sin(\theta+\sigma_0), \\ B_{\mu}(\theta) &= \cosh(\omega\mu)\sin\theta - \sinh(\omega\mu)\cos(\theta+\sigma_0), \\ C_{\mu}(\theta) &= -\cosh(\omega\mu)\sin\theta - \sinh(\omega\mu)\cos(\theta+\sigma_0), \\ D_{\mu}(\theta) &= \cosh(\omega\mu)\cos\theta - \sinh(\omega\mu)\sin(\theta+\sigma_0). \end{aligned} \tag{A7}$$

Finally, $H_{\rm rev}^{(0)}(s)$ is obtained by squaring the operators of Eq. (A6). The result is just Eq. (6), with $a=0$, which refers to the evolution of the state in the subintervals $I_0(\mu)$, of duration σ_0, where the Schrödinger Hamiltonian has the unperturbed form [first Eq. (2)]. We now wish to sketch the evolution in the complementary intervals $I_1(\mu)$. For $s\in I_1(\mu)$, one can define the Heisenberg Hamiltonian $H_{\rm rev}^{(1)}(s) = \mathcal{U}_1(s)H_0\mathcal{U}_1^{\dagger}(s)$, and the state $|n,s\rangle_1 = \mathcal{U}_1(s)|n\rangle$. Starting from Eq. (5), a structure similar to Eq. (A2) does emerge, the only difference being a ''residual'' evolution operator $U_1(\theta)$ instead of $U_0(\theta)$. The corresponding transfer matrix is given by the first Eq. (A3), with θ replacing σ_1. The resulting operators $P_{\rm tr}(s), Q_{\rm tr}(s)$ are still linear combinations of P and Q. Their squares yield the result reported in Eq. (6) for $a=1$.

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[15] The $aO(\xi^2)$ terms that appear in Eqs. (8) (and related following equations) are linear combinations of $\cosh(2\omega\mu)$ and $\sinh(2\omega\mu)$, just like the zeroth- and first-order terms, with coefficients of order ξ^2. The square roots $\sqrt{1-\xi}$ in the trigonometric functions have not been expanded, since when $\theta=\sigma_1$, one has a direct connection with the quasiresonance condition (9).
[16] It can be easily seen (see also Ref. [10]) that the complete set of quasiresonant values is $\sigma'=\pi j$ $(j\in\mathbb{Z})$. We have selected the even ones just for the sake of simplicity, since the odd ones simply change the sign of ω.
[17] See, for example, A. Messiah, *Quantum Mechanics* (North-Holland, Amsterdam, 1970), Vol. 2, Chap. XVII.
[18] The reason why the integral of $y_a(\theta,\beta)$ in θ is exponentially decreasing in β is that the peak in Fig. 2 [whose height is proportional to $\exp(\beta|\omega|)$] has a width vanishing like $\exp(-2\beta|\omega|)$, so that the integral contribution of each peak is proportional to $\exp(-\beta|\omega|)$. This can be seen from Eq. (12), on studying $d^2y_a/d\theta^2|_{\theta_0^*}$.
[19] See, for example, J. J. Sakurai, *Modern Quantum Mechanics*, edited by San Fu Tuan, revised ed. (Addison Wesley, Reading, MA, 1994).
[20] As shown in Ref. [4] and stressed in Sec. I, there is a ''band'' of values of σ', around each quasiresonant value, that yields an exponential increase of the mean energy, that is, a hyperbolic form of the functions depending on μ. The corresponding rates decrease from $|\omega|$ [Eq. (8d)] to zero at the band edges. Outside the bands, the functions depending on μ become *trigonometric*, and the effect of the frequency fluctuation is simply a periodic modulation of the mean energy.