Generalized coherent states

Ronald F. Fox
School of Physics, Georgia Institute of Technology, Atlanta, Georgia 30332-0430
(Received 3 November 1998)

Generalized coherent states are constructed for the Coulomb problem. Following a construction procedure proposed by Klauder [J. Phys. A **29**, L293 (1996)], Rydberg atom coherent states are defined and analyzed. The relationship between decorrelation in time and delocalization in space is elucidated. Keplerian orbits are discussed. The connection with sharp Gaussian wave packets used to explain pump-probe experiments is made. This is achieved by introducing genuine Gaussian Klauder coherent states that are overcomplete, and permit a resolution of the identity operator. They decorrelate comparatively slowly, and remain spatially localized for many Keplerian periods. [S1050-2947(99)05605-X]

PACS number(s): 03.65.−w

I. INTRODUCTION

Ever since Schrödinger [1] introduced coherent states for the harmonic oscillator, attempts to generalize this idea have been made. The su(2) generalized coherent states [2] are an especially nice example of a successful extension of the coherent state idea. More challenging has been the objective of obtaining generalized coherent states for the Coulomb potential problem, as was originally proposed by Schrödinger [1]. Recently, significant progress has been made in this direction [3,4]. Nevertheless, criticism of this approach has been raised [5,6]. It was motivated by comparison with experiment using a pump-probe technique to detect the periodic return of a wave packet to a nucleus along an elliptical orbit [7,8]. These experiments have been refined [9–11] and decay and revival have been observed as well as fractional revivals. Gaussian wave packets [12–14] have successfully accounted for these fascinating observations. Gaussian wave packets, *per se*, are not generalized coherent states and lack the property of resolution of the identity operator that is so useful for genuine coherent states [2]. The Majumdar-Sharatchandra [4] states for the hydrogen atom do have a Gaussian approximation (see Sec. IV D below) for a large principal quantum number, but its variance is predetermined by the structure of these states and is much larger than for the *ad hoc* Gaussian wave packets [12–14] that are consistent with experimental observations. The purpose of the present paper is to present genuine Gaussian generalized coherent states, and to critique the recent literature. These states allow a resolution of the identity operator and can have very small variances for selected operators. They should prove useful in contexts other than the present, such as for quantum-classical correspondence theory via Husimi-Wigner distributions [15–17] semiclassical theory [18], and wavelets for signal processing [19,20].

This paper is organized as follows. In Sec. II, a review of coherent states for the harmonic oscillator and of generalized coherent states for angular momentum is presented. In Sec. III, Klauder's construction of generalized coherent states for Hamiltonians with discrete spectra is given. Section IV, the longest section of the paper, is devoted to Rydberg atom coherent states, in accord with Klauder's construction [3] but in parallel with the particular rendering given by Majumdar and Sharatchandra [4]. Section IV is separated into seven subsections. Section IV D contains a quantum-mechanical derivation of Kepler's third law for circular Rydberg coherent states. Section V deals with temporal decorrelation and the criticisms of Bellomo and Stroud [5,6]. Finally, Sec. VI contains our construction of genuine Gaussian generalized coherent states and natural generalizations of them. Section VI could be read directly after Secs. I, II, and III, since the intervening sections essentially provide motivation and context only.

II. HARMONIC OSCILLATOR AND su(2) COHERENT STATES

The paradigms for generalized coherent states are the harmonic-oscillator coherent states $|\alpha\rangle$ and the su(2) coherent states $|\theta,\phi\rangle$ [2]. The harmonic-oscillator coherent state $|\alpha\rangle$ for complex parameter α is defined by

$$|\alpha\rangle=\exp\left[-\frac{|\alpha|^2}{2}\right]\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle, \tag{1}$$

where $|n\rangle$ denotes an eigenstate of the harmonic-oscillator Hamiltonian, and the sum is over integer n's. These states are normalized

$$\langle\alpha|\alpha\rangle=1, \tag{2}$$

because

$$\sum_{n=0}^{\infty}\frac{|\alpha|^{2n}}{n!}=\exp[|\alpha|^2], \tag{3}$$

and they provide a resolution of the identity operator:

$$\frac{1}{\pi}\int d^2\alpha|\alpha\rangle\langle\alpha|=\sum_{n=0}^{\infty}|n\rangle\langle n|=1 \tag{4}$$

because

$$2\int_0^{\infty}r\,dr\,\exp[-r^2]\frac{r^{2n}}{n!}=1 \tag{5}$$

for all n where $\alpha=r\exp[i\phi]$. If H is the harmonic-oscillator Hamiltonian, then

$$\exp\left[-\frac{i}{\hbar}Ht\right]|\alpha\rangle=\exp[-i\omega t/2]|\alpha e^{-i\omega t}\rangle$$
$$=\exp[-i\omega t/2]|re^{i(\phi-\omega t)}\rangle, \qquad (6)$$

which exhibits Klauder's definition of ''temporal stability'' [3]. The su(2) generalized coherent states $|\theta,\phi\rangle$ are defined by [2,17,21]

$$|j,\theta,\phi\rangle=\exp\left[i\theta\frac{1}{\hbar}(\sin(\phi)J_x-\cos(\phi)J_y)\right]|j,j\rangle$$
$$=\exp\left[-\frac{\theta}{2\hbar}(J_+e^{-i\phi}-J_-e^{i\phi})\right]|j,j\rangle$$
$$=\sum_{p=0}^{2j}\frac{e^{ip\phi}}{p!}\cos^{2j-p}(\theta)\sin^p(\theta)$$
$$\times\left(\frac{(2j)!p!}{(2j-p)!}\right)^{1/2}|j,j-p\rangle, \qquad (7)$$

where $|j,m\rangle$ denotes an eigenstate of J^2 and J_z for the su(2) algebra of angular momentum operators. These operators satisfy the commutation identities

$$[J_i,J_j]=i\hbar\varepsilon^{ijk}J_k, \qquad (8)$$

$$[J_z,J_\pm]=\pm\hbar J_\pm, \quad [J_+,J_-]=2\hbar J_z \qquad (9)$$

for $J_\pm=J_x\pm iJ_y$, and where ε^{ijk} is completely antisymmetric and repeated indices are summed. These states are normalized

$$\langle\theta,\phi|\theta,\phi\rangle=1, \qquad (10)$$

and provide a resolution of the identity operator:

$$\frac{2j+1}{4\pi}\int d\Omega|\theta,\phi\rangle\langle\theta,\phi|=1, \qquad (11)$$

where $d\Omega$ is differential solid angle. They are localized for large j in the sense that

$$\langle\theta,\phi|J_z|\theta,\phi\rangle=\hbar j\cos(\theta), \qquad (12)$$

$$\langle\theta,\phi|J_\pm|\theta,\phi\rangle=\hbar je^{\pm i\phi}\sin(\theta), \qquad (13)$$

$$\frac{1}{\hbar^2j^2}[\langle\theta,\phi|J^2|\theta,\phi\rangle-\langle\theta,\phi|\vec{J}|\theta,\phi\rangle^2]=\frac{1}{j}. \qquad (14)$$

Thus $|\theta,\phi\rangle$ points in the direction of $\hat{n}\hat{k}\cos(\theta)+(\hat{i}\cos(\phi)+\hat{j}\sin(\phi))\sin(\theta)$ with a ratio of its standard deviation to its average that vanishes with increasing j like $1/\sqrt{j}$.

III. KLAUDER COHERENT STATES

Klauder's construction of generalized coherent states [3] for Hamiltonians with discrete spectra may be represented as follows. Let the Hamiltonian H have eigenstates and eigenenergies satisfying

$$H|n\rangle=E_n|n\rangle=\hbar\omega e_n|n\rangle, \qquad (15)$$

so that the e_n's are dimensionless for some energy scale $\hbar\omega$, and wherein for definiteness $e_0<e_1<e_2<\cdots$. We define the generalized Klauder coherent state by

$$|n_0,\phi_0\rangle=(N(n_0))^{-1/2}\sum_{n=0}^{\infty}\frac{n_0^{n/2}}{\sqrt{\rho_n}}e^{ie_n\phi_0}|n\rangle, \qquad (16)$$

in which $-\infty<\phi_0<\infty$. The parameters ρ_n are moments of a positive weight function $K(n_0)$ such that

$$\rho_n=\int_0^\infty dn_0\frac{K(n_0)}{N(n_0)}n_0^n, \qquad (17)$$

$N(n_0)$ is the normalization factor satisfying

$$N(n_0)=\sum_{n=0}^{\infty}\frac{n_0^n}{\rho_n}. \qquad (18)$$

This guarantees that

$$\langle n_0,\phi_0|n_0,\phi_0\rangle=1. \qquad (19)$$

The resolution of the identity operator is given by

$$\int_0^\infty dn_0K(n_0)\lim_{\Phi\to\infty}\frac{1}{2\Phi}\int_{-\Phi}^{\Phi}d\phi_0|n_0,\phi_0\rangle\langle n_0,\phi_0|$$
$$=\int_0^\infty dn_0\frac{K(n_0)}{N(n_0)}\sum_{n=0}^{\infty}\frac{n_0^n}{\rho_n}|n\rangle\langle n|=\sum_{n=0}^{\infty}|n\rangle\langle n|=1 \qquad (20)$$

because

$$\lim_{\Phi\to\infty}\frac{1}{2\Phi}\int_{-\Phi}^{\Phi}d\phi_o e^{i(e_n-e_{n'})\phi_0}=\delta_{nn'}. \qquad (21)$$

One natural choice of weight function [3] $K(n_0)$ is $K(n_0)=1$ for which $\rho_n=n!$. In this case, $N(n_0)=e^{n_0}$, and we have precisely the Poisson coefficients used in the harmonic oscillator coherent states of Eq. (1).

Notice that the extension of the ϕ_0 domain from $[-\pi,\pi]$ to $(-\infty,\infty)$ is essential for the resolution of the identity operator because it is required for the identity of Eq. (21). This is a key step in the Klauder construction. In order to obtain Gaussian generalized coherent states below (see Sec. VI), a similar extension will be required for the n_0 domain.

IV. RYDBERG ATOM COHERENT STATES

In this section, a detailed account of Pauli's su(2)×su(2) algebra [22] for the quantum Coulomb problem is given. It produces the Klauder states for Rydberg atoms in the form given by Mujumdar and Sharatchandra [4]. The special case of circular orbits is elucidated, and Kepler's third law is derived quantum mechanically. This is followed by a study of dephasing in the azimuthal angle. These results enable us to critique the recent criticisms of Bellomo and Stroud [5,6]. The critique is presented in Sec. V.

A. Pauli's algebra

A Rydberg atom is described by the Hamiltonian

$$H=\frac{p^2}{2m_0}-\frac{Ze^2}{r}, \tag{22}$$

angular momentum

$$\vec{L}=\vec{r}\times\vec{p}, \tag{23}$$

and eccentricity vector (also called the Runge-Lenz vector [23])

$$\vec{\varepsilon}=\hat{r}-\frac{1}{2Ze^2m_0}(\vec{p}\times\vec{L}-\vec{L}\times\vec{p})=\hat{r}-\frac{a_0}{Z}(\vec{\nabla}+r\partial_r\vec{\nabla}-\vec{r}\nabla^2), \tag{24}$$

which is rendered in spherical polar coordinates for later use. Instead of $\vec{\varepsilon}$, we will use a renormalized variant defined by

$$\vec{K}=\left(\frac{(Ze^2)^2m_0}{2|E|}\right)^{1/2}\vec{\varepsilon}=\hbar n_{\rm op}\vec{\varepsilon}, \tag{25}$$

where E is the Rydberg atom energy given by

$$E=-\frac{(Ze^2)^2m_0}{2\hbar^2(n_{\rm op})^2}, \tag{26}$$

in which the number operator appears and is defined by

$$n_{\rm op}=\frac{1}{\hbar}\sqrt{L^2+K^2+\hbar^2} \tag{27}$$

and has the property

$$n_{\rm op}|n,l,m\rangle=n|n,l,m\rangle, \tag{28}$$

in which $|n,l,m\rangle$ denotes a standard Rydberg atom state of the form

$$|n,l,m\rangle=R_{nl}(r)Y_l^m(\theta,\phi), \tag{29}$$

in which the spherical harmonics have the standard form [24] and the radial functions are the standard hydrogenlike functions [25] for $Z\neq 1$.

The operators $\vec{L}$ and $\vec{K}$ satisfy the commutation relations

$$[L_i,L_j]=i\hbar\varepsilon^{ijk}L_k, \tag{30}$$

$$[K_i,K_j]=i\hbar\varepsilon^{ijk}L_k, \tag{31}$$

$$[L_i,K_j]=i\hbar\varepsilon^{ijk}K_k, \tag{32}$$

$$\begin{gathered}[L^2,L_i]=[K^2,L_i]=0,\\ [L^2+K^2,K_i]=0\Rightarrow[n_{\rm op},L_i]=[n_{\rm op},K_i]=0.\end{gathered} \tag{33}$$

Using the well-known formulas

$$L_z=-i\hbar\partial_\phi \tag{34}$$

and

$$L_\pm=\pm\hbar\exp[\pm i\phi](\partial_\theta\pm i\cot an(\theta)\partial_\phi), \tag{35}$$

the well-known matrix element formulas follow:

$$\langle n',l',m'|L_z|n,l,m\rangle=\delta_{n'n}\delta_{l'l}\delta_{m'm}\hbar m, \tag{36}$$

$$\begin{aligned}&\langle n',l',m'|L_\pm|n,l,m\rangle\\ &\quad=\delta_{n'n}\delta_{l'l}\delta_{m'm\pm1}\hbar\sqrt{(l\mp m)(l\pm m+1)}.\end{aligned} \tag{37}$$

These are paralleled by the following formulas:

$$K_z=\hbar n_{\rm op}\left(\cos(\theta)+\frac{a_0}{Z}\left[(\cos(\theta)+\sin(\theta)\partial_\theta)\partial_r+\frac{\cos(\theta)}{r}\left(-\frac{L^2}{\hbar^2}\right)\right]\right), \tag{38}$$

$$K_\pm=\hbar n_{\rm op}\left(\sin(\theta)e^{\pm i\phi}+\frac{a_0}{Z}e^{\pm i\phi}\left[\left(\sin(\theta)-\cos(\theta)\partial_\theta\mp\frac{i}{\sin(\theta)}\partial_\phi\right)\partial_r+\frac{\sin(\theta)}{r}\left(-\frac{L^2}{\hbar^2}\right)\right]\right), \tag{39}$$

$$\langle n',l',m'|K_z|n,l,m\rangle=-\delta_{n'n}\delta_{m'm}\hbar\left[\delta_{l'l-1}\left(\frac{(n^2-l^2)(l-m)(l+m)}{(2l+1)(2l-1)}\right)^{1/2}+\delta_{l'l+1}\left(\frac{(n^2-(l+1)^2)(l-m+1)(l+m+1)}{(2l+1)(2l+3)}\right)^{1/2}\right], \tag{40}$$

$$\begin{aligned}\langle n',l',m'|K_\pm|n,l,m\rangle=\delta_{n'n}\delta_{m'm\pm1}\hbar\Bigg[&\pm\delta_{l'l+1}\left(\frac{(n^2-(l+1)^2)(l\pm m+2)(l\pm m+1)}{(2l+1)(2l+3)}\right)^{1/2}\\ &\mp\delta_{l'l-1}\left(\frac{(n^2-l^2)(l\mp m)(l\mp m-1)}{(2l+1)(2l-1)}\right)^{1/2}\Bigg].\end{aligned} \tag{41}$$

Equations (36) and (37) imply

$$\langle n',l',m'|L^2|n,l,m\rangle=\delta_{n'n}\delta_{l'l}\delta_{m'm}\hbar^2 l(l+1), \tag{42}$$

and Eqs. (40) and (41) imply

$$\langle n',l',m'|K^2|n,l,m\rangle=\delta_{n'n}\delta_{l'l}\delta_{m'm}\hbar^2(n^2-(l^2+l+1)). \tag{43}$$

Together, these identities imply

$$\langle n',l',m'|(L^2+K^2+\hbar^2)|n,l,m\rangle=\delta_{n'n}\delta_{l'l}\delta_{m'm}\hbar^2 n^2, \tag{44}$$

which justifies Eqs. (27) and (28).

Introduce operators $\vec{M}$ and $\vec{N}$ defined by [23]

$$\vec{M}=\tfrac{1}{2}(\vec{L}+\vec{K}) \quad \text{and} \quad \vec{N}=\tfrac{1}{2}(\vec{L}-\vec{K}). \tag{45}$$

These operators satisfy the commutation relations of the algebra su(2)×su(2):

$$[M_i,M_j]=i\hbar\varepsilon^{ijk}M_k, \tag{46}$$

$$[N_i,N_j]=i\hbar\varepsilon^{ijk}N_k, \tag{47}$$

$$[M_i,N_j]=0. \tag{48}$$

Since

$$M^2-N^2=(\vec{M}+\vec{N})\cdot(\vec{M}-\vec{N})=\vec{L}\cdot\vec{K}=0, \tag{49}$$

the eigenstates of $\vec{M}$ and $\vec{N}$ are labeled by $|j_M,m_M\rangle$ and $|j_N,m_N\rangle$, respectively, with $j_M=j_N=j$. The last equality in Eq. (49) follows directly from the differential operator representations of $\vec{L}$ and $\vec{K}$. While the eigenstates of L^2 and L_s depend only on the angles θ and ϕ, the eigenstates of K^2 and K_z depend on r as well. Thus we may express the states for the Rydberg atom as product states [4],

$$|j,m_M\rangle|j,m_N\rangle=\sum_{l=0}^{2j} C^{l\,m_M+m_N}_{j\,m_M\,j\,m_N}|2j+1,l,m_M+m_N\rangle, \tag{50}$$

in which the right-hand side gives the Clebsch-Gordon expansion in terms of the Rydberg states of Eq. (29). The fact that these Rydberg states all have principal quantum number $2j+1$ follows from the operator n_{op}. According to Eq. (28),

$$n_{\text{op}}^2|2j+1,l,m_M+m_N\rangle=(2j+1)^2|2j+1,l,m_M+m_N\rangle, \tag{51}$$

whereas, according to Eqs. (27), (45), and (48)

$$\begin{aligned} n_{\text{op}}^2|j,m_M\rangle|j,m_N\rangle &= \frac{1}{\hbar^2}(L^2+K^2+\hbar^2)|j,m_M\rangle|j,m_N\rangle=\frac{1}{\hbar^2}((\vec{M}+\vec{N})^2+(\vec{M}-\vec{N})^2+\hbar^2)|j,m_M\rangle|j,m_N\rangle \\ &= \frac{1}{\hbar^2}(2M^2+2N^2+\hbar^2)|j,m_M\rangle|j,m_N\rangle=(2j(j+1)+2j(j+1)+1)|j,m_M\rangle|j,m_N\rangle \\ &= (2j+1)^2|j,m_M\rangle|j,m_N\rangle. \end{aligned} \tag{52}$$

B. Highest weight and Helgason's identity

In order to construct coherent states, we follow the procedure used to generate generalized, su(2) coherent states [17]. This requires obtaining the ''highest weight'' state, which we now prove is given by

$$|j,j\rangle|j,j\rangle=|2j+1,2j,2j\rangle. \tag{53}$$

In su(2)×su(2), the highest weight state satisfies

$$M_+|j,j\rangle|j,j\rangle=0 \quad \text{and} \quad N_+|j,j\rangle|j,j\rangle=0 \tag{54}$$

and

$$M_z|j,j\rangle|j,j\rangle=\hbar j|j,j\rangle|j,j\rangle$$

and (55)

$$N_z|j,j\rangle|j,j\rangle=\hbar j|j,j\rangle|j,j\rangle.$$

From Eq. (45), it follows that

$$L_+|j,j\rangle|j,j\rangle=0 \quad \text{and} \quad K_+|j,j\rangle|j,j\rangle=0 \tag{56}$$

and

$$L_z|j,j\rangle|j,j\rangle=2\hbar j|j,j\rangle|j,j\rangle$$

and (57)

$$K_z|j,j\rangle|j,j\rangle=0.$$

The four conditions of Eqs. (56) and (57) imply that

$$|j,j\rangle|j,j\rangle=|2j+1,2j,2j\rangle. \tag{58}$$

The proof of this assertion involves explicit calculation using the differential forms in Eqs. (34), (35), (38), and (39) and the functional form [23–25]

$$|2j+1,2j,2j\rangle=\frac{1}{\sqrt{\pi}}\left(\frac{Z}{a_0}\right)^{2j+3/2}\frac{1}{(2j+1)^{2j+2}(2j)!}r^{2j}\exp\left[-\frac{Zr}{(2j+1)a_0}\right]\sin^{2j}(\theta)e^{i2j\phi}. \tag{59}$$

For fixed j, we may construct a coherent state factor using Helgason's identity [17,26] to expand the generator:

$$\begin{aligned}|j,\theta_M,\phi_M,\theta_N,\phi_N\rangle&=\exp\left[i\theta_M\frac{1}{\hbar}(\sin\phi_M M_x-\cos\phi_M M_y)\right]\exp\left[i\theta_N\frac{1}{\hbar}(\sin\phi_N N_x-\cos\phi_N N_y)\right]|j,j\rangle|j,j\rangle\\&=\exp\left[-\frac{\theta_M}{2\hbar}(M_+e^{-i\phi_M}-M_-e^{i\phi_M})\right]\exp\left[-\frac{\theta_N}{2\hbar}(N_+e^{-i\phi_N}-N_-e^{i\phi_N})\right]|j,j\rangle|j,j\rangle\\&=\sum_{p=0}^{2j}\sum_{q=0}^{2j}\frac{e^{ip\phi_M+iq\phi_N}}{p!q!}\cos^{2j-p}(\theta_M)\cos^{2j-q}(\theta_N)\sin^p(\theta_M)\sin^q(\theta_N)\\&\quad\times\left(\frac{(2j)!p!(2j)!q!}{(2j-p)!(2j-q)!}\right)^{1/2}|j,j-p\rangle|j,j-q\rangle.\end{aligned} \tag{60}$$

Equation (50) can be used to convert the ket outer product in the last line of Eq. (60), but this requires application of the Racah formula [27] for the construction of the Clebsch-Gordon coefficients which are not otherwise given in closed form. An alternative construction utilizes the properties of the operators, M_-, N_-, L_-, and K_-. From

$$M_-|j,m_M\rangle=\hbar\sqrt{(j+m_M)(j-m_M+1)}|j,m_M-1\rangle, \tag{61}$$

it follows that

$$M_-^k|j,j\rangle=\hbar^k\left(\frac{(2j)!k!}{(2j-k)!}\right)^{1/2}|j,j-k\rangle, \tag{62}$$

and similarly for N_-. Therefore [recall Eq. (48)],

$$|j,j-p\rangle|j,j-q\rangle=\frac{1}{\hbar^{p+q}}\left(\frac{(2j-p)!(2j-q)!}{(2j)!p!(2j)!q!}\right)^{1/2}M_-^pN_-^q|j,j\rangle|j,j\rangle. \tag{63}$$

Equations (30), (31), and (32) imply

$$[L_-,K_-]=0. \tag{64}$$

Using Eq. (45), we may convert the right-hand side of Eq. (63) into

$$|j,j-p\rangle|j,j-q\rangle=\frac{1}{(2\hbar)^{p+q}}\left(\frac{(2j-p)!(2j-q)!}{(2j)!p!(2j)!q!}\right)^{1/2}\sum_{a=0}^{p}\frac{p!}{a!(p-a)!}\sum_{b=0}^{q}\frac{q!}{b!(q-b)!}(-1)^qL_-^{p+q-a-b}K_-^{a+b}|2j+1,2j,2j\rangle. \tag{65}$$

The actions of L_- and K_- are given by Eqs. (37) and (41), respectively. By inspection of these formulas, it is clear that the equality of the m components, i.e., $j-p+j-q=2j-p-q$, is guaranteed, and is consistent with Eq. (50).

Following Klauder's lead [3,4] for the j sum, we obtain the Rydberg coherent state (we have scaled the phase ϕ_0 slightly differently than in Sec. III in anticipation of Kepler's third law below)

$$\begin{aligned}|\text{Ryd},n_0,\phi_0,t\rangle=&\sum_{j=0}^{\infty}\exp\left[-\frac{n_0}{2}\right]\frac{n_0^j}{\sqrt{(2j)!}}\exp\left[i\frac{n_0^3\phi_0}{2(2j+1)^2}\right]\exp\left[i\frac{Z^2R_y}{\hbar(2j+1)^2}t\right]\sum_{p=0}^{2j}\sum_{q=0}^{2j}\frac{\exp[ip\phi_M+iq\phi_N]}{p!q!}\\&\times\cos^{2j-p}\left(\frac{\theta_M}{2}\right)\cos^{2j-q}\left(\frac{\theta_N}{2}\right)\sin^p\left(\frac{\theta_M}{2}\right)\sin^q\left(\frac{\theta_N}{2}\right)\left(\frac{(2j)!p!(2j)!q!}{(2j-p)!(2j-q)!}\right)^{1/2}|j,j-p\rangle|j,j-q\rangle,\end{aligned} \tag{66}$$

in which the j sum is over *half-integer* values of j, and the terminal ket product may be replaced by standard Rydberg atom $|n,l,m\rangle$ states in accord with the Clebsch-Gordon expansion in Eq. (50), or by the method of Eq. (65). The j coefficients must satisfy two roles simultaneously [3]. They must guarantee normalization of $|\mathrm{Ryd},n_0,\phi_0,t\rangle$, and provide a resolution of the identity operator for the Hilbert space of bound states [3,4]. The time evolution operator for the Hamiltonian of Eq. (22) evolves the state $|\mathrm{Ryd},n_0,\phi_0\rangle = |\mathrm{Ryd},n_0,\phi_0,0\rangle$ into $|\mathrm{Ryd},n_0,\phi_0+2\Omega t/n_0^3\rangle = |\mathrm{Ryd},n_0,\phi_0,t\rangle$, where $\Omega = Z^2R_y/\hbar$, and Ryd is the Rydberg constant.

C. Properties of Rydberg atom coherent states

The following expectation values are exact consequences of Eq. (66), albeit after considerable computation:

$$\langle \mathrm{Ryd},n_0,\phi_0,t|L^2|\mathrm{Ryd},n_0,\phi_0,t\rangle = \hbar^2(\tfrac{1}{2}(n_0+n_0^2)+n_0+\tfrac{1}{2}(n_0+n_0^2)\hat{n}_M\cdot\hat{n}_N), \tag{67}$$

$$\langle \mathrm{Ryd},n_0,\phi_0,t|K^2|\mathrm{Ryd},n_0,\phi_0,t\rangle = \hbar^2(\tfrac{1}{2}(n_0+n_0^2)+n_0-\tfrac{1}{2}(n_0+n_0^2)\hat{n}_M\cdot\hat{n}_N), \tag{68}$$

$$\langle \mathrm{Ryd},n_0,\phi_0,t|\vec{L}|\mathrm{Ryd},n_0,\phi_0,t\rangle = \hbar(\tfrac{1}{2}n_0(\hat{n}_M+\hat{n}_N)), \tag{69}$$

$$\langle \mathrm{Ryd},n_0,\phi_0,t|\vec{K}|\mathrm{Ryd},n_0,\phi_0,t\rangle = \hbar(\tfrac{1}{2}n_0(\hat{n}_M-\hat{n}_N)), \tag{70}$$

$$\begin{aligned}\langle \mathrm{Ryd},n_0,\phi_0,t|\varepsilon^2|\mathrm{Ryd},n_0,\phi_0,t\rangle &= \frac{1}{n_0}(1+(\gamma-1)e^{-n_0}-\mathrm{Ei}(n_0)e^{-n_0}+\ln(n_0)e^{-n_0}\\ &\quad +(1-\vec{n}_M\cdot\vec{n}_N)[n_0-2+(2-\gamma)e^{-n_0}\\ &\quad +\mathrm{Ei}(n_0)e^{-n_0}-\ln(n_0)e^{-n_0}]),\end{aligned} \tag{71}$$

$$\langle \mathrm{Ryd},n_0,\phi_0,t|\vec{\varepsilon}|\mathrm{Ryd},n_0,\phi_0,t\rangle = \left(\left(\frac{1}{2}-\frac{1}{2n_0}(1-\exp[-n_0])\right)(\hat{n}_M-\hat{n}_N)\right), \tag{72}$$

in which $\hat{n}_M$ and $\hat{n}_N$ are radial unit vectors given in terms of θ_M and ϕ_M and θ_N and ϕ_N, respectively. Equations (70) and (72) differ by more than a factor of $\hbar n_0$, because Eqs. (25), (26), and (28) imply that the j sum in Eq. (66) is affected. In Eq. (71), γ is the Euler constant, and Ei is the exponential integral function given by

$$\mathrm{Ei}(z) = \gamma + \ln(z) + \sum_{n=1}^{\infty}\frac{z^n}{n!n} \tag{73}$$

for positive z.

To obtain these results, we have repeatedly used the fundamental identity

$$\begin{aligned}&\langle j',j'-q'|\langle j',j'-p'|(\vec{M}\pm\vec{N})|j,j-p\rangle|j,j-q\rangle\\ &= \delta_{j'j}\langle j,j-q'|\langle j,j-p'|(\vec{M}\pm\vec{N})|j,j-p\rangle|j,j-q\rangle\\ &= \delta_{j'j}\Bigg(\hat{k}\delta_{p'p}\delta_{q'q}\hbar(j-p\pm(j-q))+\hat{i}\Bigg[\delta_{p'p-1}\delta_{q'q}\frac{\hbar}{2}\sqrt{p(2j-p+1)}\pm\delta_{p'p}\delta_{q'q-1}\frac{\hbar}{2}\sqrt{q(2j-q+1)}\\ &\quad +\delta_{p'p+1}\delta_{q'q}\frac{\hbar}{2}\sqrt{(2j-p)(p+1)}\pm\delta_{p'p}\delta_{q'q+1}\frac{\hbar}{2}\sqrt{(2j-q)(q+1)}\Bigg]\\ &\quad +\hat{j}\Bigg[\delta_{p'p-1}\delta_{q'q}\frac{\hbar}{2i}\sqrt{p(2j-p+1)}\pm\delta_{p'p}\delta_{q'q-1}\frac{\hbar}{2i}\sqrt{q(2j-q+1)}\\ &\quad -\delta_{p'p+1}\delta_{q'q}\frac{\hbar}{2i}\sqrt{(2j-p)(p+1)}\mp\delta_{p'p}\delta_{q'q+1}\frac{\hbar}{2i}\sqrt{(2j-q)(q+1)}\Bigg]\Bigg).\end{aligned} \tag{74}$$

In performing the p and q sums, care must be taken with the limits of the summations since, for example, $\delta_{p'p-1}$ requires that $p\geqslant 1$, so that p' is not less than zero. After carefully adjusting the limits and shifting the indices appropriately, we then use two identities [26] to finish the computations:

$$\sum_{p=0}^{2j-1}(2j-p)\frac{(2j)!}{(2j-p)!p!}x^{2p} = 2j(1+x^2)^{2j-1}, \tag{75}$$

$$\sum_{p=0}^{2j}(j-p)\frac{(2j)!}{(2j-p)!p!}x^{2p} = j(1-x^2)(1+x^2)^{2j-1}. \tag{76}$$

We may choose to have the conserved angular momentum along the z axis and the conserved eccentricity vector along the x axis. It is straightforward to show that this can be achieved by setting

$$\theta_M=\theta_N=\overline{\theta} \quad \text{and} \quad \phi_M=0 \quad \text{and} \quad \phi_N=\pi, \tag{77}$$

using $\langle\cdots\rangle$ to denote the Rydberg coherent state expectation value, from Eqs. (67)–(72) we obtain

$$\langle\vec{L}\rangle=\hbar n_0\cos(\overline{\theta})\hat{k}, \tag{78}$$

$$\langle\vec{K}\rangle=\hbar n_0\sin(\overline{\theta})\hat{i}, \tag{79}$$

$$\langle\vec{\varepsilon}\rangle=\left(1-\frac{1}{n_0}(1-e^{-n_0})\right)\sin(\overline{\theta})\hat{i}, \tag{80}$$

$$\langle L^2\rangle=\hbar^2 n_0(1+(1+n_0)\cos^2(\overline{\theta})), \tag{81}$$

$$\langle K^2\rangle=\hbar^2 n_0(1+(1+n_0)\sin^2(\overline{\theta})), \tag{82}$$

$$\langle\varepsilon^2\rangle=\frac{1}{n_0}(1-e^{-n_0})-\cos^2(\overline{\theta})\times\frac{1}{n_0}(\mathrm{Ei}(n_0)-\gamma-\ln(n_0))e^{-n_0}+\sin^2(\overline{\theta})\left(1-\frac{2}{n_0}(1-e^{-n_0})\right), \tag{83}$$

$$\langle L^2\rangle-\langle\vec{L}\rangle\cdot\langle\vec{L}\rangle=\hbar^2 n_0(1+\cos^2(\overline{\theta})), \tag{84}$$

$$\langle K^2\rangle-\langle\vec{K}\rangle\cdot\langle\vec{K}\rangle=\hbar^2 n_0(1+\sin^2(\overline{\theta})), \tag{85}$$

$$\langle\varepsilon^2\rangle-\langle\vec{\varepsilon}\rangle\cdot\langle\vec{\varepsilon}\rangle=\frac{1}{n_0}(1-e^{-n_0})-\cos^2(\overline{\theta})\frac{1}{n_0}(\mathrm{Ei}(n_0)-\gamma-\ln(n_0))\times e^{-n_0}-\sin^2(\overline{\theta})\frac{1}{(n_0)^2}(1-e^{-n_0})^2. \tag{86}$$

D. Circular Rydberg atom coherent states

A circle is produced when $\overline{\theta}=0$ is chosen. The general Rydberg coherent state in Eq. (66) simplifies considerably (only the $p=0$ and $q=0$ terms need to be kept), becoming

$$|\mathrm{circ},n_0,\phi_0,t\rangle=\sum_{j=0}^{\infty}\exp\left[-\frac{n_0}{2}\right]\frac{n_0^j}{\sqrt{(2j)!}}\exp\left[i\frac{n_0^3\phi_0}{2(2j+1)^2}\right]\exp\left[i\frac{Z^2R_y}{\hbar(2j+1)^2}t\right]|2j+1,2j,2j\rangle, \tag{87}$$

in which the j sum is again over *half-integers*. The position vector expectation value is now

$$\langle\mathrm{circ},n_0,\phi_0,t|r\hat{n}|\mathrm{circ},n_0,\phi_0,t\rangle=\sum_{j'=0}^{\infty}\sum_{j=0}^{\infty}e^{-n_0}\frac{(n_0)^{j+j'}}{\sqrt{(2j)!(2j')!}}\exp\left[i\left(\Omega t+\frac{n_0^3\phi_0}{2}\right)\left(\frac{1}{(2j+1)^2}-\frac{1}{(2j'+1)^2}\right)\right]\times\langle 2j'+1,2j',2j'|r\hat{n}|2j+1,2j,2j\rangle, \tag{88}$$

in which $\Omega=Z^2R_y/\hbar$. The matrix elements, by lengthy but straightforward computation, yield

$$\langle 2j'+1,2j',2j'|r\hat{n}|2j+1,2j,2j\rangle=\frac{a_0}{Z}\left[\left(\frac{\hat{i}}{2}+\frac{\hat{j}}{2i}\right)\delta_{2j'2j+1}\frac{(2j+1)^{2j+4}(2j+2)^{2j+3}}{(2j+3/2)^{4j+5}}+\left(\frac{\hat{i}}{2}-\frac{\hat{j}}{2i}\right)\delta_{2j'2j-1}\frac{(2j+1)^{2j+2}(2j)^{2j+3}}{(2j+1/2)^{4j+3}}\right]. \tag{89}$$

Thus Eq. (88) becomes

$$\langle\mathrm{circ},n_0,\phi_0,t|r\hat{n}|\mathrm{circ},n_0,\phi_0,t\rangle=\frac{a_0}{Z}e^{-n_0}\sum_{j=0}^{\infty}P(j)\left[\hat{i}\cos\left(\left(\Omega t+\frac{n_0^3\phi_0}{2}\right)\left[\frac{1}{(2j+1)^2}-\frac{1}{(2j+2)^2}\right]\right)+\hat{j}\sin\left(\left(\Omega t+\frac{n_0^3\phi_0}{2}\right)\left[\frac{1}{(2j+1)^2}-\frac{1}{(2j+2)^2}\right]\right)\right], \tag{90}$$

in which

$$P(j)=\frac{(n_0)^{2j+1/2}}{(2j)!\sqrt{2j+1}}\frac{(2j+1)^{2j+4}(2j+2)^{2j+3}}{(2j+3/2)^{4j+5}}$$

$$\approx P(n_0)\left(\frac{\pi n_0}{2}\right)^{1/2}\frac{\exp\left[-\frac{1}{2}\frac{\left(j-\frac{n_0}{2}\right)^2}{(n_0/4)}\right]}{\left(\frac{\pi n_0}{2}\right)^{1/2}},\qquad (91)$$

wherein

$$P(n_0)\xrightarrow[n_0\gg 1]{}\frac{e^{n_0}}{\sqrt{2\pi n_0}}.\qquad (92)$$

These limiting approximations permit us to replace the sum in Eq. (90) by an integral, provided that we observe that the half-integer values for j in the sum imply a ''density-of-states'' factor of 2, i.e.,

$$\sum_{j=0}^{\infty}f(j)\rightarrow 2\int_0^{\infty}dy\,f(y).\qquad (93)$$

For the circle case, we obtain (for $n_0\gg 1$)

$$\langle \mathrm{circ},n_0,\phi_0,t|r\hat{n}|\mathrm{circ},n_0,\phi_0,t\rangle =\frac{a_0}{Z}n_0^2\left[\hat{i}\cos\left(\frac{2Z^2R_y}{\hbar n_0^3}t+\phi_0\right)+\hat{j}\sin\left(\frac{2Z^2R_y}{\hbar n_0^3}t+\phi_0\right)\right].\qquad (94)$$

Kepler's third law relates the period τ to the radius r:

$$\tau=2\pi\left(\frac{m_0}{k}\right)^{1/2}r^{3/2}\qquad (95)$$

where k is the strength of the $1/r$ potential. In the present case,

$$\tau=2\pi\frac{\hbar n_0^3}{2Z^2R_y}=2\pi\frac{\hbar^3 n_0^3}{Z^2 m_0 e^4},\qquad (96)$$

$$r=\frac{a_0}{Z}n_0^2=\frac{\hbar^2 n_0^2}{Zm_0e^2},\qquad (97)$$

$$\left(\frac{m_0}{k}\right)^{1/2}=\left(\frac{m_0}{Ze^2}\right)^{1/2}.\qquad (98)$$

Even the coefficient agrees exactly.

The transition from reciprocal squares of the principal quantum number in the exponentials of Eq. (90) to reciprocal cubes in Eq. (94) results from the interference of adjacent energy levels in the expansion of Eq. (87) caused by the couplings of $2j$ to $2j\pm 1$ created by the matrix elements on the right-hand side of Eq. (88). This is a manifestation of the traditional Bohr correspondence principle [28].

E. Slightly eccentric Rydberg atom coherent states

To obtain a slightly eccentric elliptical orbit, we choose $\overline{\theta}$ slightly larger than 0, and keep the $p=1$ and $q=0$ and $q=1$ and $p=0$ terms in Eq. (66) as well as the $p=0$ and $q=0$ term used for the circle case. The equivalent of the Clebsch-Gordon coefficients can be obtained by using properties of the $\vec{L}$, $\vec{K}$, $\vec{M}$, and $\vec{N}$ operators expressed in Eq. (65). In particular,

$$|j,j-1\rangle|j,j\rangle=\frac{1}{\sqrt{2}}(|2j+1,2j,2j-1\rangle+|2j+1,2j-1,2j-1\rangle),\qquad (99)$$

$$|j,j\rangle|j,j-1\rangle=\frac{1}{\sqrt{2}}(|2j+1,2j,2j-1\rangle-|2j+1,2j-1,2j-1\rangle).\qquad (100)$$

Therefore, a slightly eccentric coherent state is given by

$$|\mathrm{ellip},n_0,\phi_0,t\rangle=\sum_{j=0}^{\infty}e^{-n_0/2}\frac{n_0^j}{\sqrt{(2j)!}}\exp\left[i\frac{\Omega t+\frac{n_0^3\phi_0}{2}}{(2j+1)^2}\right][|2j+1,2j,2j\rangle+\overline{\theta}\sqrt{j}|2j+1,2j-1,2j-1\rangle].\qquad (101)$$

We now need variations of the matrix element given in Eq. (89):

$$\langle 2j'+1,2j'-1,2j'-1|r\hat{n}|2j+1,2j-1,2j-1\rangle=\frac{a_0}{Z}\left[\left(\frac{\hat{i}}{2}+\frac{\hat{j}}{2i}\right)\delta_{2j'2j+1}\frac{(2j+1)^{2j+4}(2j+2)^{2j+2}}{(2j+3/2)^{4j+4}}\left(\frac{j}{j+1/2}\right)^{1/2}+\left(\frac{\hat{i}}{2}-\frac{\hat{j}}{2i}\right)\delta_{2j'2j-1}\frac{(2j+1)^{2j+1}(2j)^{2j+3}}{(2j+1/2)^{4j+2}}\left(\frac{j-1/2}{j}\right)^{1/2}\right],\qquad (102)$$

$$\langle 2j'+1,2j',2j'|r\hat{n}|2j+1,2j-1,2j-1\rangle=\frac{a_0}{Z}\left[\left(\frac{\hat{i}}{2}+\frac{\hat{j}}{2i}\right)\delta_{2j'2j}(-3(2j+1)\sqrt{j})+\left(\frac{\hat{i}}{2}-\frac{\hat{j}}{2i}\right)\delta_{2j'2j-2}\frac{(2j-1)^{2j+2}(2j+1)^{2j+1}}{(2j)^{4j+2}}\sqrt{j}\right],\qquad (103)$$

$$\langle 2j+1,2j,2j|r\hat{n}|2j'+1,2j'-1,2j'-1\rangle=\frac{a_0}{Z}\left[\left(\frac{\hat{i}}{2}+\frac{\hat{j}}{2i}\right)\delta_{2j'2j}(-3(2j+1)\sqrt{j})+\left(\frac{\hat{i}}{2}-\frac{\hat{j}}{2i}\right)\delta_{2j'2j+2}\frac{(2j+1)^{2j+4}(2j+3)^{2j+3}}{(2j+2)^{4j+6}}\sqrt{j+1}\right]. \tag{104}$$

In the sums for $\delta_{2j'2j-1}$ and $\delta_{2j'2j-2}$, lower limit restrictions on j are required so that $j'\geqslant 0$. When these are imposed, j can be shifted so that the new j runs from 0 to ∞ as before. After lengthy computation, the result is

$$\langle \text{ellip},n_0,\phi_0,t|r\hat{n}|\text{ellip},n_0,\phi_0,t\rangle=\frac{a_0}{Z}n_0^2\left[\hat{i}\left(\cos\left(\frac{2Z^2R_y}{\hbar n_0^3}t+\phi_0\right)+\frac{\varepsilon}{2}\cos\left(\frac{4Z^2R_y}{\hbar n_0^3}t+2\phi_0\right)-\frac{3\varepsilon}{2}\right)+\hat{j}\left(\sin\left(\frac{2Z^2R_y}{\hbar n_0^3}t+\phi_0\right)+\frac{\varepsilon}{2}\sin\left(\frac{4Z^2R_y}{\hbar n_0^3}t+2\phi_0\right)\right)\right]. \tag{105}$$

Because eccentricity only introduces simple harmonics of the fundamental frequency, $2\Omega/n_0^3$, Kepler's third law remains exact.

We can show that Eq. (105) represents an ε perturbation of the circular orbit described by Eq. (94). By changing variables from r to $u=1/r$, one may show that the classical equation of motion is ($k=Ze^2$, and L is the angular momentum) [30]

$$\frac{d^2}{dt^2}u=4k^2\frac{\varepsilon^2-1}{2L^2}u^3+5\frac{k}{m_0}u^4-3\frac{L^2}{m_0^2}u^5. \tag{106}$$

Writing $u=u_0+\varepsilon u_1+\varepsilon^2 u_2$, we find

$$u_0=\frac{m_0 k}{L^2}=\frac{1}{r_c},\quad u_1=\frac{1}{r_c}\cos(\omega_c t),$$

$$u_2=\frac{1}{r_c}(\cos(2\omega_c t)-1), \tag{107}$$

in which r_c is the classical radius and ω_c is the classical frequency. The boundary conditions used for the solution just given are that this solution agrees with the orbital equation at $t=0$, i.e., with

$$r=\frac{a(1-\varepsilon^2)}{1+\varepsilon\cos(\theta)}, \tag{108}$$

where a is the semimajor axis and the numerator is equal to the classical radius [29]. Using Eq. (97) for the classical radius and $2\Omega/n_0^3$ for the classical frequency, Eq. (105) may be used to show that the Rydberg atom electron radius magnitude is

$$r=r_c\sqrt{1+\tfrac{5}{2}\varepsilon^2-2\varepsilon\cos(\omega_c t)-\tfrac{3}{2}\varepsilon^2\cos(2\omega_c t)}$$

$$\cong r_c(1-\varepsilon\cos(\omega_c t)-\varepsilon^2\cos(2\omega_c t)+\varepsilon^2), \tag{109}$$

wherein we have used $\sqrt{1+x}\cong 1+\frac{1}{2}x-\frac{1}{8}x^2$, in which x stands for all of the ε terms. This is precisely the first-order inversion of the results in Eq. (107), i.e., $1/(1+x)\cong 1-x$. So far, we have been unable to obtain comparable closed-form results for arbitrary eccentricity. However, the results here strongly suggest that higher powers of the eccentricity and higher harmonics of the fundamental frequency will make up such general results.

F. Dephasing of the azimuthal angle

While the results above show that the expected value of the position executes circular or slightly eccentric orbital motion, it is also important to determine the rate at which uncertainty in the coordinates grows. In this section, we investigate this issue for the circular Rydberg coherent states. We show that these states remain tightly compact in both r and θ, but exhibit dephasing in ϕ. To do this, we need the explicit coordinate dependence given by Eqs. (59) and (87).

Define $\psi_{\text{circ}}(r,\theta,\phi,t)$ by

$$\psi_{\text{circ}}(r,\theta,\phi,t)=\langle r,\theta,\phi|\text{circ},n_0,\phi_0,t\rangle. \tag{110}$$

The probability density associated with $\psi_{\text{circ}}(r,\theta,\phi,t)$ is given by

$$P(r,\theta,\phi,t)=\sum_{j'=0}^{\infty}\sum_{j=0}^{\infty} e^{-n_0}\frac{n_0^{j+j'}}{\sqrt{(2j)!(2j')!}}\exp\left[i\left(\Omega t+\frac{n_0^3\phi_0}{2}\right)\left(\frac{1}{(2j+1)^2}-\frac{1}{(2j'+1)^2}\right)\right]$$
$$\times\frac{1}{\pi}\left(\frac{Z}{a_0}\right)^{2j+2j'+3}\frac{1}{(2j+1)^{2j+2}(2j)!}\,\frac{1}{(2j'+1)^{2j'+2}(2j')!}\,e^{i(2j-2j')\phi}\sin^{2j+2j'}(\theta)$$
$$\times r^{2j+2j'}\exp\left[-\frac{Zr}{a_0}\left(\frac{1}{2j+1}+\frac{1}{2j'+1}\right)\right]. \tag{111}$$

We can reduce this to distributions in one coordinate at a time by integrating the other two coordinates. The required integrals are

$$\int_0^{2\pi}d\phi\, e^{i(2j-2j')\phi}=2\pi\delta_{jj'}, \tag{112}$$

$$\int_0^{\pi}d\theta\sin^{2j+2j'+1}(\theta)=2\,\frac{(2j+2j')!!}{(2j+2j'+1)!!}, \tag{113}$$

$$\int_0^{\infty}dr\, r^{2j+2j'+2}\exp\left[-\frac{Zr}{a_0}\left(\frac{1}{2j+1}+\frac{1}{2j'+1}\right)\right]$$
$$=(2j+2j'+2)!\left(\frac{a_0}{Z}\right)^{2j+2j'+3}$$
$$\times\left(\frac{(2j+1)(2j'+1)}{2j+2j'+2}\right)^{2j+2j'+3}. \tag{114}$$

In Eq. (113) $j+j'$ is even; for $j+j'$ odd multiply by $\pi/2$.

It is now clear that the ϕ integration produces reduced distributions that are independent of time. The reduced distribution for r and θ is given by

$$Q(r,\theta)=\int_0^{2\pi}d\phi\, P(r,\theta,\phi,t)=2\sum_{j=0}^{\infty} e^{-n_0}\frac{n_0^{2j}}{(2j)!}\left(\frac{Z}{a_0}\right)^{4j+3}$$
$$\times\frac{1}{(2j+1)^{4j+4}[(2j)!]^2}$$
$$\times r^{4j}\exp\left[-\frac{Zr}{a_0}\left(\frac{2}{2j+1}\right)\right]\sin^{4j}(\theta). \tag{115}$$

In parallel with Eqs. (91) and (92), we find

$$e^{-n_0}\frac{n_0^{2j}}{(2j)!}\approx\frac{1}{2}\,\frac{\exp\left[-\frac{1}{2}\,\frac{(j-n_0/2)^2}{(n_0/4)}\right]}{\sqrt{2\pi(n_0/4)}}. \tag{116}$$

This implies that

$$\sin^{4j}(\theta)\approx\sin^{2n_0}(\theta)=\exp[2n_0\ln\sin(\theta)]$$
$$\approx\exp\left[-\frac{1}{2}\,\frac{(\theta-\pi/2)^2}{(1/2n_0)}\right]. \tag{117}$$

This means that the root-mean-square deviation compared to the mean is

$$\frac{\sqrt{\langle(\Delta\theta)^2\rangle}}{\pi/2}=\frac{\sqrt{2}}{\pi\sqrt{n_0}}. \tag{118}$$

Thus, for sufficiently large n_0, θ is confined to be very close to $\pi/2$, i.e., in the azimuthal plane. Equation (116) also implies that

$$\left(\frac{Zr}{a_0}\right)^{4j}\exp\left[-\frac{Zr}{a_0}\left(\frac{2}{2j+1}\right)\right]\approx\left(\frac{Zr}{z_0}\right)^{2n_0}\exp\left[-\frac{Zr}{a_0}\left(\frac{2}{n_0}\right)\right]=\exp\left[-\frac{2Zr}{n_0a_0}+2n_0\ln(Zr/a_0)\right]$$
$$\approx\exp[-2n_0+2n_0\ln n_0^2]\exp\left[-\frac{1}{2}\,\frac{(r-r_0)^2}{(n_0^3a_0^2/2Z^2)}\right], \tag{119}$$

wherein

$$r_0=\frac{a_0}{z}n_0^2, \tag{120}$$

and the variance is clearly $n_0^3a_0^2/2Z^2$. This means that the root-mean-square deviation compared to the mean is

$$\frac{\sqrt{\langle(\Delta r)^2\rangle}}{r_0}=\frac{1}{\sqrt{2n_0}}. \tag{121}$$

Thus, for sufficiently large n_0, r is confined to relatively very close to the circle radius of Eq. (97).

In contrast to these time-independent results for r and θ, the reduced distribution for the angle ϕ is time dependent. Using Eqs. (111), (113), and (114), we obtain

$$\Phi(\phi,t)=\int_0^\infty dr\, r^2\int_0^\pi d\theta \sin(\theta)P(r,\theta,\phi,t)=\frac{2}{\pi}e^{-n_0}\sum_{j'=0}^{\infty}\sum_{j=0}^{\infty}\frac{n_0^{j+j'}}{\sqrt{(2j)!(2j')!}}\exp\left[i\left(\Omega t+\frac{n_0^3\phi_0}{2}\right)\left(\frac{1}{(2j+1)^2}-\frac{1}{(2j'+1)^2}\right)\right]$$
$$\times(2j+2j'+2)[(2j+2j')!!]^2\frac{(2j+1)^{2j'+1}(2j'+1)^{2j+1}}{(2j+2j'+2)^{2j+2j'+3}}\frac{1}{(2j)!(2j')!}e^{i(2j-2j')\phi}, \tag{122}$$

wherein we have used the identity

$$(2j+2j'+2)!\frac{(2j+2j')!!}{(2j+2j'+1)!!}=(2j+2j'+2)[(2j+2j')!!]^2. \tag{123}$$

We now use the following approximations:

$$e^{-n_0}\frac{n_0^{j+j'}}{\sqrt{(2j)!(2j')!}}\approx\frac{\pi n_0}{\sqrt{2\pi n_0}}\frac{\exp\left[-\frac{1}{2}\frac{(j-n_0/2)^2}{(n_0/2)}\right]}{\sqrt{\pi n_0}}\frac{\exp\left[-\frac{1}{2}\frac{(j'-n_0/2)^2}{(n_0/2)}\right]}{\sqrt{\pi n_0}}, \tag{124}$$

$$(2j+2j'+2)[(2j+2j')!!]^2\frac{(2j+1)^{2j'+1}(2j'+1)^{2j+1}}{(2j+2j'+2)^{2j+2j'+3}(2j)!(2j')!}\approx\frac{[(2n_0)!!]^2}{(n_0!)^2}\frac{(n_0+1)^{n_0+1}(n_0+1)^{n_0+1}}{(2n_0+2)^{2n_0+2}}$$
$$\approx 2^{2n_0}2^{-(2n_0+2)}=2^{-2}, \tag{125}$$

$$\exp\left[i\left(\Omega t+\frac{n_0^3\phi_0}{2}\right)\left(\frac{1}{(2j+1)^2}-\frac{1}{(2j'+1)^2}\right)\right]\approx\exp\left[i\left(\Omega t+\frac{n_0^3\phi_0}{2}\right)\left(\frac{1}{n_0^2(1+(2j-n_0+1)/n_0)^2}-\frac{1}{n_0^2(1+(2j'-n_0+1)/n_0)^2}\right)\right]$$
$$\approx\exp\left[-i\left(\Omega t+\frac{n_0^3\phi_0}{2}\right)\frac{8}{n_0^3}(2j-2j')+i\left(\Omega t+\frac{n_0^3\phi_0}{2}\right)\frac{3}{n_0^4}((2j)^2-(2j')^2)\right], \tag{126}$$

wherein we have used $1/(1+x)^2\sim 1-2x+3x^2+\cdots$. Replacing the two sums by integrals in accord with Eq. (93), we find

$$\Phi(\phi,t)\approx\left(\frac{2n_0}{\pi}\right)^{1/2}\int_0^\infty dx'\frac{\exp\left[-\frac{1}{2}\frac{(x'-n_0/2)^2}{(n_0/2)}-i\left(\Omega t+\frac{n_0^3\phi_0}{2}\right)\frac{3}{n_0^4}(2x')^2\right]}{\sqrt{\pi n_0}}$$
$$\times\exp\left[-i2x'\left(\phi-\left(\Omega t+\frac{n_0^3\phi_0}{2}\right)\frac{8}{n_0^3}\right)\right]\int_0^\infty dx\frac{\exp\left[-\frac{1}{2}\frac{(x-n_0/2)^2}{(n_0/2)}+i\left(\Omega t+\frac{n_0^3\phi_0}{2}\right)\frac{3}{n_0^4}(2x)^2\right]}{\sqrt{\pi n_0}}$$
$$\times\exp\left[i2x\left(\phi-\left(\Omega t+\frac{n_0^3\phi_0}{2}\right)\frac{8}{n_0^3}\right)\right]. \tag{127}$$

Now shift the integration variables to $y=x-n_0/2$ and $y'=x'-n_0/2$, and obtain

$$\Phi(\phi,t)\approx\left(\frac{2n_0}{\pi}\right)^{1/2}\int_{-\infty}^{\infty}dy'\int_{-\infty}^{\infty}dy\frac{\exp\left[-\frac{1}{2}\frac{(y')^2}{(n_0/2)}-i\left(\Omega t+\frac{n_0^3\phi_0}{2}\right)\frac{3}{n_0^4}(2y')^2\right]}{\sqrt{\pi n_0}}\frac{\exp\left[-\frac{1}{2}\frac{(y)^2}{(n_0/2)}+i\left(\Omega t+\frac{n_0^3\phi_0}{2}\right)\frac{3}{n_0^4}(2y)^2\right]}{\sqrt{\pi n_0}}$$
$$\times\exp\left[i(2y-2y')\left(\phi-\Omega t\frac{2}{n_0^3}-\phi_0\right)\right]. \tag{128}$$

By performing the Gaussian integrals and rearranging the results, a normalized Gaussian reduced distribution is produced:

$$\Phi(\phi,t)\approx\frac{\exp\left[-\frac{1}{2}(\phi-\Omega t2/n_0^3-\phi_0)^2\frac{1}{\left(\frac{1}{4n_0}+\frac{9}{n_0}(\Omega t2/n_0^3+\phi_0)^2\right)}\right]}{\left[2\pi\left(\frac{1}{4n_0}+\frac{9}{n_0}(\Omega t2/n_0^3+\phi_0)^2\right)\right]^{1/2}}. \tag{129}$$

This distribution clearly shows that the averaged value of ϕ changes linearly with time in accord with the result in Eq. (94). At first glance, it would appear that the variance grows quadratically in time with a $1/n_0^7$ dependence. This would seem to be negligible for sufficiently large n_0. However, we have expressed this growing term in a form that shows that after exactly one period of the orbital revolution, the variance increases from $1/4n_0$ to $(1/4n_0)+(9/n_0)(2\pi)^2$, or by a factor of 1421.5. Thus, for typically obtained experimental Rydberg atom states with $50\leqslant n_0\leqslant 200$, say, there will be complete dephasing in the angle ϕ after less than one orbital period. If n_0 could be made as large as 10^6, say, then about 2^5 orbital periods would be required before the variance grew to order unity. While this may be impossible to achieve for Rydberg atoms, in Sec. IV G we show that it is trivial to achieve for celestial bodies.

G. Celestial bodies as Rydberg coherent states

The issue of the correspondence principle can be approached by treating celestial dynamics by the Schrödinger equation, and comparing the resulting description with that of Newtonian classical mechanics. In this section, we do this for the Earth, Mars, and Saturn. The strength of attraction, Ze^2, for Rydberg atoms need only be replaced by GMm for celestial bodies where $G=6.67\times10^{-8}\,\mathrm{dyn\,cm^2/gm^2}$, Newton's gravitational constant; $M=1.89\times10^{33}\,\mathrm{gm}$, the mass of the Sun; and $m=m_e=5.98\times10^{27}\,\mathrm{gm}$, the mass of the Earth. The masses of Mars and Saturn are $0.108m_e$ and $95.2m_e$, respectively. This change in attractive strength is enormous: $Ze^2\sim Z\times23.04\times10^{-20}\,\mathrm{erg\,cm}$ and $GMm_e\sim7.538\times10^{53}\,\mathrm{erg\,cm}$, about 72 orders of magnitude larger. The Bohr radius $\hbar^2/m_0e^2$ is $5.29\times10^{-9}\,\mathrm{cm}$, whereas the celestial analog $\hbar^2/GMmm$ is 2.44×10^{-136} cm for $m=m_e$, about 127 orders of magnitude smaller. Similarly, the Bohr orbital period $2\pi\hbar^3/e^4m_0$ is 1.5×10^{-16} s, whereas the celestial analog $2\pi\hbar^3/G^2M^2m^2m$, is 2.14×10^{-216} s for $m=m_e$, about 200 orders of magnitude smaller. Since we know the orbital radius and period for the Earth (for the present purpose, we can ignore the eccentricity of the Earth's orbit), it is a simple matter to determine the principal quantum number in accord with the celestial analogs of Eqs. (96) and (97). For the Sun-Earth system we know that $\tau=3.16\times10^7$ s, and that $r=1.50\times10^{13}$ cm. Equation (96) implies that $n_{\mathrm{SE}}=2.53\times10^{74}$, and Eq. (97) implies that $n_{\mathrm{SE}}=2.53\times10^{74}$. This is an enormous principal quantum number. Corresponding results for Mars and Saturn yield $n_{\mathrm{SM}}=3.37\times10^{73}$ and $n_{\mathrm{SS}}=7.43\times10^{76}$, respectively.

Looking back at Eq. (129), we see that for the Earth the variance grows by a factor of about $1422\times(\text{square of the number of periods})$. Since each period is a year, the variance will not reach order unity for $n_{\mathrm{SE}}=2.53\times10^{74}$, until about 10^{36} years have elapsed. This is so much longer than the age of the universe that we can conclude that a Rydberg coherent state treatment of the Sun-Earth system yields a compact, localized state in all three spherical polar coordinates for the entire lifetime of the system. In this limit of extremely large principal quantum numbers, the quantum-mechanical treatment of celestial dynamics reproduces the classical mechanical description with very great precision.

V. TEMPORAL DECORRELATION

Bellomo and Stroud [5,6] used the time autocorrelation function proposed by Nauenberg [13,14],

$$C(t)=\left|\langle\psi|\exp\left[-\frac{i}{\hbar}Ht\right]|\psi\rangle\right|^2, \tag{130}$$

where $|\psi\rangle$ denotes either a generalized coherent state or a wave packet. For the circular Rydberg coherent states, this yields

$$\begin{aligned}C(t)&=\left|\langle\mathrm{circ},n_0,\phi_0|\exp\left[-\frac{i}{\hbar}Ht\right]|\mathrm{circ},n_0,\phi_0\rangle\right|^2\\&=\left|\sum_{j=0}^{\infty}e^{-n_0}\frac{n_0^{2j}}{(2j)!}\exp\left[i\frac{Z^2R_y}{\hbar(2j+1)^2}t\right]\right|^2\\&\cong\frac{1}{\left[1+\left(\frac{6\Omega t}{n_0^3}\right)^2\right]^{1/2}}\exp\left[-2n_0\left(\frac{\Omega t}{n_0^3}\right)^2\frac{2}{1+\left(\frac{6\Omega t}{n_0^3}\right)^2}\right],\end{aligned} \tag{131}$$

where again $\Omega = Z^2 R_y/\hbar$. In a Kepler period [see Eq. (94)], i.e., $t = T = n_0^3 \pi/\Omega$, the decorrelation is considerable for even modestly small n_0. Nevertheless, as already noted by Bellomo and Stroud [6], decorrelation does not necessarily imply a spreading of the wave packet. However, they went on to observe that the mean-square deviation in r for circular Rydberg states is proportional to n_0^3 (their R), which is very large as n_0 increases. In Sec. IV F it was shown that the relevant quantity is the ratio of the root-mean-square deviation and the mean radius which is given by Eq. (121). This quantity becomes very small with increasing n_0. Thus, as was shown above, the Majumdar-Sharatchandra Rydberg states are very well localized in both θ and r but, nevertheless, delocalize rapidly in ϕ unless n_0 is extremely large, as in the case of celestial mechanics (see Sec. IV G).

The Gaussian wave packets used earlier by Nauenberg [13,14] and many others, and by Mallalieu and Stroud [12], have the advantage that their variances are very small compared with the variances of order n_0 imposed by the Gaussian limit of the Majumdar-Sharatchandra Rydberg coherent states. Observed decay and revival, and even fractional revivials [12] can be explained using sharp Gaussian wave packets. This is achieved by expanding the energy denominators around the principal quantum number that is at the center of the sharp Gaussian. The incommensurate frequencies become almost perfectly uniformily distributed in this approximation. They are virtually in resonance with each other [12]. However, no resolution of the identity operator exists for these Gaussian wave packets. This deficiency is remedied in Sec. VI.

VI. GAUSSIAN GENERALIZED COHERENT STATES

A Gaussian generalized coherent state is constructed in parallel with the method used for Klauder states [3]. Given Eq. (15), we replace Eq. (16) with

$$|G,n_0,\phi_0\rangle = \sum_{n=0}^{\infty} \frac{\exp\left[-\frac{1}{4}\frac{(n-n_0)^2}{\sigma^2}\right]}{(N(n_0))^{1/2}} e^{ie_n\phi_0}|n\rangle \tag{132}$$

where

$$N(n_0) = \sum_{n=0}^{\infty} \exp\left[-\frac{(n-n_0)^2}{2\sigma^2}\right], \tag{133}$$

and this guarantees normalization

$$\langle G,n_0,\phi_0|G,n_0,\phi_0\rangle = 1. \tag{134}$$

Clearly, as $n_0 \to \infty$, $N(n_0) \to \sqrt{2\pi\sigma^2}$, but for finite n_0 and because the summation is discrete, $N(n_0)$ is generally not determined in closed form. The resolution of the identity operator is achieved by giving n_0 a domain of $-\infty$ to ∞ rather than just the positive values.

$$\int_{-\infty}^{\infty} dn_0 \lim_{\Phi\to\infty}\frac{1}{2\Phi}\int_{-\Phi}^{\Phi} d\phi_0 K(n_0)|G,n_0,\phi_0\rangle\langle G,n_0,\phi_0| = \int_{-\infty}^{\infty} dn_0 K(n_0)\frac{1}{N(n_0)}\sum_{n=0}^{\infty} \exp\left[-\frac{(n-n_0)^2}{2\sigma^2}\right]|n\rangle\langle n| = \sum_{n=0}^{\infty}|n\rangle\langle n| = 1, \tag{135}$$

provided $K(n_0)$ is given by

$$K(n_0) = \frac{N(n_0)}{\sqrt{2\pi\sigma^2}}. \tag{136}$$

The interesting and useful Gaussian coherent states are those with n_0 positive and reasonably large, but the states with negative n_0's are required for resolution of the identity operator. For highly negative n_0, $N(n_0)$ becomes very small, and the states contain all $|n\rangle$'s with slowly decreasing amplitudes. However, this permits sharpness in the variable conjugate to n_0. For large positive n_0's, the states contain almost exclusively those $|n\rangle$'s within three σ's of n_0.

It is easy to generalize these Gaussian states [31] to the form

$$|M,n_0,\phi_0\rangle = \sum_{n=0}^{\infty} n^M \frac{\exp\left[-\frac{1}{4}\frac{(n-n_0)^2 n^{4M}}{\sigma^2}\right]}{(N(n_0))^{1/2}} e^{ie_n\phi_0}|n\rangle, \tag{137}$$

where

$$N(n_0) = \sum_{n=0}^{\infty} n^{2M} \exp\left[-\frac{(n-n_0)^2 n^{4M}}{2\sigma^2}\right], \tag{138}$$

and the resolution of the identity operator takes the form

$$\int_{-\infty}^{\infty} dn_0 \lim_{\Phi\to\infty} \frac{1}{2\Phi} \int_{-\Phi}^{\Phi} d\phi_0 K(n_0) |M,n_0,\phi_0\rangle\langle M,n_0,\phi_0|$$
$$= \int_{-\infty}^{\infty} dn_0 K(n_0) \frac{1}{N(n_0)} \sum_{n=0}^{\infty} n^{2M} \times \exp\left[-\frac{(n-n_0)^2 n^{4M}}{2\sigma^2}\right] |n\rangle\langle n| = \sum_{n=0}^{\infty} |n\rangle\langle n| = 1, \tag{139}$$

provided $K(n_0)$ is given by

$$K(n_0) = \frac{N(n_0)}{\sqrt{2\pi\sigma^2}}, \tag{140}$$

since

$$\int_{-\infty}^{\infty} dn_0 \exp\left[-\frac{(n-n_0)^2 n^{4M}}{2\sigma^2}\right] = \left(\frac{2\pi\sigma^2}{n^{4M}}\right)^{1/2}. \tag{141}$$

The inclusion of n^M in Eq. (137) tends to suppress coefficients near $n=0$, which may be desirable for states with n_0 positive but small.

Application of this construction to the Rydberg coherent states requires slight modifications to accommodate the summations over half-integer indices. For the Gaussian Rydberg coherent states, we obtain

$$|\mathrm{GR},n_0,\phi_0\rangle = \sum_{j=0}^{\infty} \frac{\exp\left[-\frac{(j-n_0/2)^2}{4\sigma^2}\right]}{(N(n_0))^{1/2}} \times \exp\left[i\frac{n_0^3\phi_0}{2(2j+1)^2}\right] |j,\theta_M,\phi_M,\theta_N,\phi_N\rangle, \tag{142}$$

where

$$N(n_0) = \sum_{j=0}^{\infty} \exp\left[-\frac{(j-n_0/2)^2}{2\sigma^2}\right]. \tag{143}$$

In both Eqs. (142) and (143), the summation is over half-integer j's. When approximating this sum by an integral, the density of states factor of 2 [see Eq. (93)] must be included. Thus, for large n_0, $N(n_0) \to 2\sqrt{2\pi\sigma^2}$ approximately. The resolution of the identity operator is given by

$$\int_{-\infty}^{\infty} dn_0 \lim_{\Phi\to\infty} \frac{1}{2\Phi} \int_{-\Phi}^{\Phi} d\phi_0 K(n_0) \int d\Omega_M \int d\Omega_N |\mathrm{GR},n_0,\phi_0\rangle\langle \mathrm{GR},n_0,\phi_0|$$
$$= \int_{-\infty}^{\infty} dn_0 K(n_0) \frac{1}{N(n_0)} \sum_{j=0}^{\infty} \sum_{m_M=-j}^{j} \sum_{m_N=-j}^{j} \exp\left[-\frac{(j-n_0/2)^2}{2\sigma^2}\right] |j,m_M\rangle|j,m_N\rangle\langle j,m_N|\langle j,m_M|$$
$$= \sum_{j=0}^{\infty} \sum_{m_M=-j}^{j} \sum_{m_N=-j}^{j} |j,m_M\rangle|j,m_N\rangle\langle j,m_N|\langle j,m_M| = 1, \tag{144}$$

provided $K(n_0)$ is given by

$$K(n_0) = \frac{N(n_0)}{\sqrt{2\pi 4\sigma^2}}. \tag{145}$$

For sufficiently large n_0, this weight approaches unity.

The correlation function defined in Eq. (130) is easily computed because of the temporal stability property of the Gaussian Rydberg coherent states, i.e.,

$$\exp\left[-\frac{i}{\hbar}Ht\right]|\mathrm{GR},n_0,\phi_0\rangle = \sum_{j=0}^{\infty} \frac{\exp\left[-\frac{(j-n_0/2)^2}{4\sigma^2}\right]}{(N(n_0))^{1/2}} \exp\left[i\frac{n_0^3\phi_0}{2(2j+1)^2}\right] \exp\left[i\frac{Z^2R_y}{\hbar(2j+1)^2}t\right] |j,\theta_M,\phi_M,\theta_N,\phi_N\rangle. \tag{146}$$

Therefore

$$C(t) = \left|\frac{1}{N(n_0)} \sum_{j=0}^{\infty} \exp\left[-\frac{(j-n_0/2)^2}{2\sigma^2}\right] \exp\left[i\frac{Z^2R_y}{\hbar(2j+1)^2}t\right]\right|^2$$
$$\cong \frac{1}{\left(1+\frac{576\sigma^4}{n_0^2}(\pi t/T)^2\right)^{1/2}} \exp\left[-16\sigma^2(\pi t/T)^2 \frac{1}{1+576\sigma^4(\pi t/T)^2/n_0^2}\right], \tag{147}$$

where T is again the Kepler period. For small σ and large n_0, this expression decays slowly compared with Eq. (131). By a similar analysis, the calculations leading to Eq. (129) for the azimuthal angle dephasing may be reevaluated for these Gaussian Rydberg coherent states. The result is

$$\Phi(\phi,t)\approx\frac{\exp\left[-\frac{1}{2}(\phi-\Omega t2/n_0^3-\phi_0)^2\left(\frac{1}{16\sigma^2}+\frac{36\sigma^2}{n_0^2}(\Omega t2/n_0^3+\phi_0)^2\right)^{-1}\right]}{\sqrt{2\pi\left(\frac{1}{16\sigma^2}+36\sigma^2(\Omega t2/n_0^3+\phi_0)^2/n_0^2\right)}}. \tag{148}$$

This result shows that many orbital periods may elapse before significant delocalization in the azimuthal angle occurs if σ is sufficiently small and n_0 is sufficiently large. For example, assume that n_0 is 320 and σ is 2.5 (these are the values used by Mallalieu and Stroud [12] in their Gaussian wave packets). With these values, the standard deviation, which is initially $\frac{1}{10}$ rad, doubles only after 72 orbital periods. When these same values for n_0 and σ are placed in Eq. (147), however, the decay is considerable even after only one period. This underscores the unreliability of decorrelation as a measure of delocalization. Moreover, if the infinite sum in Eq. (146) is approximated by a finite sum over just those j values within two or three σ's of $n_0/2$, then the result is almost periodic, and the correlation function computed therefrom will show revivals [12]. The use of the integral approximation in Eq. (147) smooths out and eliminates the revivals, much like in the case of the Jaynes-Cummings model [32], where revivals are a result of the discreteness of the fully quantum description.

ACKNOWLEDGMENTS

This work was supported by NSF Grant No. PHY-9514853. The author thanks Ray Flannery for bringing Ref. [4] to his attention, and for remarks regarding Bohr's correspondence principle, and Brian Kennedy and Li You for helpful discussions.

[1] E. Schrödinger, Naturwissenschaften **14**, 644 (1926).
[2] W.-M. Zhang, D. H. Feng, and R. Gilmore, Rev. Mod. Phys. **62**, 867 (1990).
[3] J. R. Klauder, J. Phys. A **29**, L293 (1996).
[4] P. Majumdar and H. S. Sharatchandra, Phys. Rev. A **56**, R3322 (1997).
[5] P. Bellomo and C. R. Stroud, J. Phys. A (to be published).
[6] P. Bellomo and C. R. Stroud, Phys. Rev. A **59**, 900 (1998).
[7] A. ten Wolde, L. D. Noordam, A. Lagendijk, and H. B. van Linden van den Heuvell, Phys. Rev. Lett. **61**, 2099 (1988).
[8] J. A. Yeazell, M. Mallalieu, J. Parker, and C. R. Stroud, Phys. Rev. A **40**, 5040 (1989).
[9] J. A. Yeazell, M. Mallalieu, and C. R. Stroud, Phys. Rev. Lett. **64**, 2007 (1990).
[10] D. R. Meacher, P. E. Meyler, I. G. Hughes, and P. Ewart, J. Phys. B **24**, L63 (1991).
[11] J. A. Yeazell and C. R. Stroud, Phys. Rev. A **43**, 5153 (1991).
[12] M. Mallalieu and C. R. Stroud, in *Coherent States: Past, Present and Future*, edited by D. H. Feng, J. R. Klauder, and M. R. Strayer (World Scientific, Singapore, 1994), pp. 301–314.
[13] M. Nauenberg, Phys. Rev. A **40**, 1133 (1989).
[14] M. Nauenberg, in *Coherent States: Past, Present and Future* (Ref. [12]), pp. 345–360.
[15] S.-J. Chang and K.-J. Shi, Phys. Rev. A **34**, 7 (1986).
[16] K. Nakamura, A. R. Bishop, and A. Shudo, Phys. Rev. B **39**, 12 422 (1989).
[17] R. F. Fox and T. C. Elston, Phys. Rev. E **50**, 2553 (1994).
[18] S. Tomsovic and E. J. Heller, Phys. Rev. E **47**, 282 (1993).
[19] D. Gabor, J. Inst. Electr. Eng. **93**, 429 (1946).
[20] I. Daubechies, Commun. Pure Appl. Math. **41**, 909 (1988).
[21] F. Haake, *Quantum Signatures of Chaos* (Springer-Verlag, Berlin, 1991), Chap. 7.
[22] W. Pauli, Z. Phys. **36**, 336 (1926).
[23] R. L. Liboff, *Introductory Quantum Mechanics*, 2nd ed. (Addison-Wesley, Reading, MA, 1992), p. 458.
[24] R. L. Liboff, *Introductory Quantum Mechanics* (Ref. [23]), pp. 366–372.
[25] R. L. Liboff, *Introductory Quantum Mechanics* (Ref. [23]), pp. 435–440.
[26] R. F. Fox, Phys. Rev. A **44**, 6193 (1991).
[27] A. Messiah, *Quantum Mechanics* (Wiley, New York, 1962), Vol. II, p. 1058.
[28] S. Gasiorowicz, *Quantum Physics* (Wiley, New York, 1996), pp. 20 and 21.
[29] H. Goldstein, *Classical Mechanics*, 2nd ed. (Addison-Wesley, Reading, MA, 1981), pp. 97 and 98.
[30] R. Adler, M. Bazin, and M. Schiffer, *Introduction to General Relativity* (McGraw-Hill, New York, 1975), pp. 201–209.
[31] J. Klauder (personal communication).
[32] D. F. Walls and G. J. Milburn, *Quantum Optics* (Springer-Verlag, Berlin, 1994), p. 205.