Quantum-mechanical model for particles carrying electric charge and magnetic flux in two dimensions

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We propose a simple quantum-mechanical equation for n particles in two dimensions, each particle carrying electric charge and magnetic flux. Such particles appear in (2+1)-dimensional Chern-Simons field theories as charged vortex soliton solutions, where the ratio of charge to flux is a constant independent of the specific solution. As an approximation, the charge-flux interaction is described here by the Aharonov-Bohm potential, and the charge-charge interaction by the Coulomb one. The equation for two particles, one with charge and flux (q, Φ/Z) and the other with ($-Zq$, $-\Phi$) where Z is a pure number is studied in detail. The bound-state problem is solved exactly for arbitrary q and Φ when $Z>0$. The scattering problem is exactly solved in parabolic coordinates in special cases when $q\Phi/2\pi\hbar c$ takes integers or half integers. In both cases the cross sections obtained are rather different from that for pure Coulomb scattering. [S1050-2947(99)03705-1]

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I. INTRODUCTION

Field theories with Chern-Simons (CS) term in (2+1)-dimensional space time admit soliton solutions carrying both electric charge and magnetic flux [1–8]. These solutions are often called CS vortices or vortex solitons, as compared with Nielsen-Olesen vortices [9], which are electrically neutral. They appear in both relativistic and nonrelativistic field theories, and regardless of whether the gauge-field action involves both Maxwell and CS terms or only a pure CS term. The ratio of electric charge q to magnetic flux Φ depends only on the parameters in the field theoretical model, not on the specific solution. Such solutions are not only of interest in field theories, but also expected to be useful in condensed matter physics. However, the interaction of these vortex solitons is very complicated. A single soliton solution is available in analytic form only for nonrelativistic theory and when the Maxwell term is absent. It seems difficult to find multisoliton solutions in closed forms, especially when both Maxwell and CS terms are present. Therefore, a simple quantum-mechanical model for the interaction of such vortex solitons may be of interest. The purpose of the present paper is to study such a model.

The real CS vortices have finite sizes. The electric charge density and the magnetic flux density (the magnetic field) depend on the specific solution. As a simple approximation, we use pointlike particles to represent them in this paper. Both the magnetic flux and the electric charge are then confined to a region of infinitesimal area, in other words, to a point where the particle is located. The vector potential associated with the flux is the Aharonov-Bohm (AB) potential [10]. (see also Refs. [11,12] for some more works on the subject.) This is responsible for the charge-flux interaction. As for the charge-charge interaction, we make use of the Coulomb potential. Note that in two-dimensional space there are two kinds of Coulomb potentials. The first one satisfies the two-dimensional Poisson equation with point source and is proportional to $\ln r$, where r is the distance between the two point charges. The second simply imitates the form of the three-dimensional one and is proportional to $1/r$. It should be remarked that the real interaction between the CS vortices may be very complicated, and it depends on whether the field-theoretical model involves both Maxwell and CS terms or only a CS term. Neither of the above forms can be expected to be capable of well describing the real situation. Either one is in any case a rough approximation. We prefer the latter one since it is easier to obtain exact solutions in this case. This is the potential adopted in the study of the so-called two-dimensional hydrogen atom (2H) [13–18].

In this paper we confine ourselves to the framework of nonrelativistic quantum mechanics. Now that the forms of the interaction potentials are established, we can write down an n-body Schrödinger equation for these particles carrying magnetic flux as well as electric charges. This is done in Sec. II. The ath particle has charge and flux $(q_a,\ \Phi_a)$, where $a=1,2,\ldots,n$. It should be emphasized that the ratio q_a/Φ_a does not depend on a, as pointed out in the first paragraph. After the time variable is separated out to obtain a stationary Schrödinger equation, we concentrate our attention on the two-body problem. This is separable into two equations. One governs the center-of-mass motion, which is free, and the other governs the relative motion, which is of main interest to us and is the main subject of the remaining part of this paper. It is remarkable that the separability of the two-body equation crucially depends on the condition $q_1/\Phi_1=q_2/\Phi_2$. We then denote $(q_1,\Phi_1)=(q,\Phi/Z)$, $(q_2,\Phi_2)=(-Zq,-\Phi)$, where Z is a nonvanishing real number. The relative Hamiltonian has the same form as that for a particle of reduced mass moving in the composite field of a vector AB potential and a scalar Coulomb one. This may be called an Aharonov-Bohm-Coulomb system. Although the so-called ABC system has been dealt with by numerous works [19–24] in the literature, the Coulomb potential considered

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there is a three-dimensional one. Thus the situation is quite different from that studied here. In other words, the model studied in the above cited works is a three-dimensional ABC system, while that encountered here is a two-dimensional one.

In Sec. III we study the bound-state problem. Bound states are possible only when $Z>0$, i.e., when the Coulomb field represents attractive force, regardless of whether an AB potential is present. When $\Phi=0$, the spectrum is just those of the 2H. The level E_N has degeneracy $2N+1(N=0,1,2,\ldots)$. If $q\Phi/2\pi\hbar c$ takes nonvanishing integers, the spectrum is roughly the same except that the ground state has energy E_1 and the level E_N has degeneracy $2N(N=1,2,\ldots)$ since some solutions are not acceptable. In the general case each level E_N of the 2H splits into two, each with lower degeneracy. When $q\Phi/2\pi\hbar c$ takes half integers, however, some of the splitted levels coincide and we have again a high degeneracy. The degeneracy implies that the system should have SU(2) symmetry in this case, as the SO(3) symmetry of the ordinary 2H [13,16,17]. But this has not been explicitly proved.

In Sec. IV we study the scattering problem. In the general case partial-wave expansion in the polar coordinates should be employed. However, as the asymptotic form of the partial wave involves logarithmic distortion due to the long range nature of the Coulomb field, it is somewhat difficult to handle the partial-wave expansion. In this paper we restrict our discussion to special cases where $q\Phi/2\pi\hbar c$ takes integers or half integers. In these cases the scattering problem can be exactly solved in parabolic coordinates, as the ordinary Coulomb scattering in two dimensions [25]. Note that what we use here are parabolic coordinates on the plane, and thus they are quite different from the rotational parabolic coordinates used in the discussion of the ordinary three-dimensional Coulomb problem in the text books of quantum mechanics. The latter are also used in the study of the three-dimensional ABC system [19]. When $\Phi=0$ the cross section is just that for the Coulomb scattering in two dimensions. When $q\Phi/2\pi\hbar c$ takes nonzero integers, the cross section gains an additional term, which comes from the interference of the scattered wave with an additional stationary wave present in the scattering solution. Without the stationary wave term the solution would become meaningless at the origin. To the best of our knowledge, such circumstances are not encountered previously in the literature. When $q\Phi/2\pi\hbar c$ takes half integers, the result is simple but, of course, rather different from that for pure Coulomb scattering. Without the Coulomb field our results reduce to those for pure AB scattering [10,11]. The classical limit of the results is also discussed.

Section V is devoted to a brief summary and some more remarks.

II. THE MODEL

Consider n pointlike particles carrying magnetic flux as well as electric charges in two-dimensional space. The ath particle has mass μ_a, carries electric charge and magnetic flux (q_a, Φ_a), $a=1,2,\ldots,n$. The position of the ath particle is denoted by $\mathbf{r}_a=(x_a, y_a)$. As remarked in the introduction, the ratio q_a/Φ_a is independent of a. More precisely, we have

$$\frac{q_1}{\Phi_1}=\frac{q_2}{\Phi_2}=\ldots=\frac{q_n}{\Phi_n}. \tag{1}$$

We describe the charge-flux interactions among the particles by the vector AB potentials and the charge-charge interactions by the scalar Coulomb ones. The n-body wave function is denoted by $\Psi^{(n)}(t,\mathbf{r}_1,\ldots,\mathbf{r}_n)$. In this paper we work in the domain of nonrelativistic quantum mechanics. The Schrödinger equation for the wave function is then

$$i\hbar\frac{\partial\Psi^{(n)}}{\partial t}=H_{\mathrm{T}}\Psi^{(n)}, \tag{2a}$$

where H_{T} is the Hamiltonian of the system given by

$$H_{\mathrm{T}}=-\sum_{a=1}^{n}\frac{\hbar^2}{2\mu_a}\left[\boldsymbol{\nabla}_a-\frac{iq_a}{\hbar c}\mathbf{A}_a(\mathbf{r}_1,\ldots,\mathbf{r}_n)\right]^2+\sum_{a<b}\frac{q_aq_b}{|\mathbf{r}_a-\mathbf{r}_b|}, \tag{2b}$$

where the second term (the Coulomb interaction) involves a double summation subject to the condition $a<b$, and $\mathbf{A}_a(\mathbf{r}_1,\ldots,\mathbf{r}_n)$ is the AB vector potential at the position $\mathbf{r}_a$. Note that all particles, except the ath one, contribute to $\mathbf{A}_a$. Thus the components of $\mathbf{A}_a$ are given by

$$A_{ax}(\mathbf{r}_1,\ldots,\mathbf{r}_n)=-\sum_{b\neq a}\frac{\Phi_b}{2\pi}\frac{y_a-y_b}{|\mathbf{r}_a-\mathbf{r}_b|^2},$$

$$A_{ay}(\mathbf{r}_1,\ldots,\mathbf{r}_n)=\sum_{b\neq a}\frac{\Phi_b}{2\pi}\frac{x_a-x_b}{|\mathbf{r}_a-\mathbf{r}_b|^2}. \tag{2c}$$

Since the Hamiltonian H_{T} does not involve t, the time-dependent factor in $\Psi^{(n)}$ can be separated out. Let

$$\Psi^{(n)}(t,\mathbf{r}_1,\ldots,\mathbf{r}_n)=e^{-iE_{\mathrm{T}}t/\hbar}\psi^{(n)}(t,\mathbf{r}_1,\ldots,\mathbf{r}_n). \tag{3}$$

We have for $\psi^{(n)}$ the stationary Schrödinger equation,

$$H_{\mathrm{T}}\psi^{(n)}=E_{\mathrm{T}}\psi^{(n)}. \tag{4}$$

In the following we concentrate our attention on the two-body problem, since this is the only case where exact analysis is possible. In this case the first summation in Eq. (2b) contains two terms and the second contains only one. We introduce the relative position $\mathbf{r}$ and the center-of-mass position $\mathbf{R}$ defined by

$$\mathbf{r}=\mathbf{r}_1-\mathbf{r}_2, \quad \mathbf{R}=\frac{\mu_1\mathbf{r}_1+\mu_2\mathbf{r}_2}{M}, \tag{5}$$

where $M=\mu_1+\mu_2$ is the total mass of the system. Note that both $\mathbf{A}_1$ and $\mathbf{A}_2$ depend only on $\mathbf{r}$; it is not difficult to recast H_{T} in the form

$$H_{\rm T} = -\frac{\hbar^2}{2\mu_1}\left(\nabla_r - \frac{iq_1}{\hbar c}\mathbf{A}_1\right)^2 - \frac{\hbar^2}{2\mu_2}\left(\nabla_r + \frac{iq_2}{\hbar c}\mathbf{A}_2\right)^2 + \frac{q_1q_2}{r} - \frac{\hbar^2}{2M}\nabla_R^2 + \frac{i\hbar}{Mc}(q_1\mathbf{A}_1 + q_2\mathbf{A}_2)\cdot\nabla_R, \tag{6}$$

where $r=|\mathbf{r}|$. Using Eqs. (1) and (2c), it can be shown that

$$q_1\mathbf{A}_1 = -q_2\mathbf{A}_2. \tag{7}$$

Thus Eq. (6) reduces to

$$H_{\rm T} = -\frac{\hbar^2}{2\mu}\left(\nabla_r - \frac{iq_1}{\hbar c}\mathbf{A}_1\right)^2 + \frac{q_1q_2}{r} - \frac{\hbar^2}{2M}\nabla_R^2, \tag{8}$$

where $\mu=\mu_1\mu_2/(\mu_1+\mu_2)$ is the reduced mass of the system. Now Eq. (4) can be separated into two equations. Let

$$\psi^{(2)}(\mathbf{r}_1,\mathbf{r}_2) = \psi_{\rm cm}(\mathbf{R})\psi(\mathbf{r}). \tag{9}$$

We have

$$-\frac{\hbar^2}{2M}\nabla_R^2\psi_{\rm cm} = E_{\rm cm}\psi_{\rm cm}, \tag{10}$$

$$H\psi = E\psi, \tag{11a}$$

where

$$H = -\frac{\hbar^2}{2\mu}\left(\nabla_r - \frac{iq_1}{\hbar c}\mathbf{A}_1\right)^2 + \frac{q_1q_2}{r}, \tag{11b}$$

$$A_{1x} = -\frac{\Phi_2}{2\pi}\frac{y}{r^2}, \quad A_{1y} = \frac{\Phi_2}{2\pi}\frac{x}{r^2}, \tag{11c}$$

and $E_{\rm cm}+E=E_{\rm T}$. Equation (10) governs the center-of-mass motion of the system, which is obviously free and will not be discussed any further. Equation (11) governs the relative motion of the two particles, which is of essential interest to us and is the main subject of the remaining part of this paper. In the following we omit the subscript r of ∇_r. We also denote $(q_1,\Phi_1)=(q,\Phi/Z)$ and $(q_2,\Phi_2)=(-Zq,-\Phi)$, where Z is a nonvanishing real number. The Hamiltonian (11b) can be written as

$$H = -\frac{\hbar^2}{2\mu}\left(\nabla + i\frac{q\Phi}{2\pi\hbar c}\nabla\theta\right)^2 - \frac{Zq^2}{r}, \tag{12}$$

where (r,θ) are polar coordinates on the xy plane and r has been used above.

As pointed out in the Introduction, the Hamiltonian (12) is the same, as that governs the motion of a charged particle in the combined field of a vector AB potential and a scalar Coulomb one. However, it is quite different from that for the so-called ABC system studied in the literature, since that is a three-dimensional model while ours is a two-dimensional one. More precisely, in their Coulomb potential, $r=(x^2+y^2+z^2)^{1/2}$, whereas in ours $r=(x^2+y^2)^{1/2}$. In fact, everything is independent of z here, or, if one prefers, there is no z component here.

To conclude this section we emphasize that the separability of Eq. (4) (for $n=2$) crucially depends on the relation (7) and thus on the condition (1).

III. BOUND STATES

In this section we study bound states of the two-body system. These are solutions vanishing at infinity of Eq. (11). It is convenient to solve Eq. (11a), with the Hamiltonian written in the form of Eq. (12), in polar coordinates. We denote

$$\frac{q\Phi}{2\pi\hbar c} = m_0 + \nu, \tag{13}$$

where m_0 is an integer and $0\leqslant\nu<1$. Equation (11) can be written in polar coordinates as

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\left(\frac{\partial}{\partial\theta} + im_0 + i\nu\right)^2\psi + \left(\frac{2\mu E}{\hbar^2} + \frac{2\mu Zq^2}{\hbar^2 r}\right)\psi = 0. \tag{14}$$

We write ψ as

$$\psi(r,\theta) = R(r)e^{i(m-m_0)\theta}, \quad m=0,\pm1,\pm2,\ldots. \tag{15}$$

Then $R(r)$ satisfies the equation

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \left[\frac{2\mu E}{\hbar^2} + \frac{2\mu Zq^2}{\hbar^2 r} - \frac{(m+\nu)^2}{r^2}\right]R = 0. \tag{16}$$

Now it can be shown that $E>0$ gives scattering solutions, which will not be discussed in this section. Thus bound states have $E<0$. It will also become clear in the following that bound states are possible only when $Z>0$, i.e., when the Coulomb potential represents attraction. These are all familiar conclusions in the pure Coulomb problem in three or two dimensions. Note that the factorized form of the solution (15) itself requires

$$R(0) = 0, \tag{17}$$

except for $m=m_0$. This is because θ is not well-defined at the origin. It will exclude some well-behaved solutions of Eq. (16). As $E<0$, we introduce a dimensionless variable ρ defined as

$$\rho = \alpha r, \quad \alpha = \frac{\sqrt{-8\mu E}}{\hbar}, \tag{18}$$

and a new parameter

$$\lambda = \frac{Zq^2}{\hbar}\sqrt{-\frac{\mu}{2E}}; \tag{19}$$

then Eq. (16) can be written as

$$\frac{d^2R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} + \left[-\frac{1}{4} + \frac{\lambda}{\rho} - \frac{(m+\nu)^2}{\rho^2}\right]R = 0. \tag{20}$$

Now we define a new function $u(\rho)$ through the relation

$$R(\rho) = e^{-\rho/2}\rho^{|m+\nu|}u(\rho); \tag{21}$$

then we have for $u(\rho)$ the equation

$$\rho\frac{d^2u}{d\rho^2}+(2|m+\nu|+1-\rho)\frac{du}{d\rho}-\left(|m+\nu|+\frac{1}{2}-\lambda\right)u=0. \tag{22}$$

This is the confluent hypergeometric equation. It is solved by the confluent hypergeometric function

$$u(\rho)=CF(|m+\nu|+\tfrac{1}{2}-\lambda,2|m+\nu|+1,\rho), \tag{23}$$

where C is a normalization constant to be determined below. The other solution to Eq. (22) makes $R(r)$ infinite at $r=0$ and is thus dropped. The above solution, though well behaved at $r=0$, diverges when $r\to\infty$: $u(\rho)$ behaves like e^{ρ} and $R(\rho)$ behaves like $e^{\rho/2}$. Therefore, it is still not acceptable in general. Physically acceptable solutions appear when E or λ takes special values so that the confluent hypergeometric series terminates. This happens when

$$|m+\nu|+\tfrac{1}{2}-\lambda=-n_r,\quad n_r=0,1,2,\ldots \tag{24}$$

and $u(\rho)$ becomes a polynomial of order n_r. From Eq. (19) we see that this can be satisfied only when $Z>0$, and the energy levels are given by

$$E=-\frac{\mu Z^2q^4}{2\hbar^2(n_r+|m+\nu|+1/2)^2}. \tag{25}$$

The corresponding wave function is

$$\psi_{n_rm}(r,\theta)=C_{n_rm}e^{-\rho/2}\rho^{|m+\nu|}\times F(-n_r,2|m+\nu|+1,\rho)e^{i(m-m_0)\theta}. \tag{26}$$

There are degeneracies in the energy levels. This is why we have not attached any subscript to E. The degeneracy depends on the values of ν and m_0. The various cases are discussed as follows.

(1) $\nu=m_0=0$. This is the case of a pure Coulomb problem, or the 2H. We introduce the principal quantum number

$$N=n_r+|m|. \tag{27}$$

Then the energy levels are written as

$$E_N=-\frac{\mu Z^2q^4}{2\hbar^2(N+1/2)^2},\quad N=0,1,2,\ldots. \tag{28}$$

With a given N, the possible values for (n_r,m) are $(N,0)$, $(N-1,\pm1)$, ..., $(0,\pm N)$, and the degeneracy is $d_N=2N+1$. These results are well known [13–18].

(2) $\nu=0$, $m_0\neq0$. In other words, $q\Phi/2\pi\hbar c$ takes nonzero integers. In this case the energy levels are roughly the same. However, from Eq. (26) we see that the solution with $m=0$ is not acceptable, regardless of the value of n_r, because the radial part of the wave function does not satisfy Eq. (17). Therefore, the ground state has energy E_1 and the level E_N has degeneracy $d_N=2N$ $(N=1,2,\ldots)$.

(3) $0<\nu<\frac{1}{2}$ or $\frac{1}{2}<\nu<1$. In this case each level of the 2H splits into two. When $m\geqslant0$, we have

$$E_N^+=-\frac{\mu Z^2q^4}{2\hbar^2(N+\nu+1/2)^2},\quad N=0,1,2,\ldots \tag{29a}$$

while when $m<0$, we have

$$E_N^-=-\frac{\mu Z^2q^4}{2\hbar^2(N-\nu+1/2)^2},\quad N=1,2,\ldots. \tag{29b}$$

The possible values of (n_r,m) that correspond to E_N^+ are $(N,0),(N-1,1),\ldots,(0,N)$; thus the degeneracy is $d_N^+=N+1$. Those that correspond to E_N^- are $(N-1,-1)$, $(N-2,-2)$, ..., $(0,-N)$; thus the degeneracy is $d_N^-=N$. The difference between the case $0<\nu<\frac{1}{2}$ and the case $\frac{1}{2}<\nu<1$ lies in the order of the energy levels. In the first case the order of the levels is

$$E_0^+<E_1^-<E_1^+<\cdots<E_N^-<E_N^+<E_{N+1}^-<\cdots. \tag{30a}$$

In the second case it is

$$E_1^-<E_0^+<E_2^-<\cdots<E_N^-<E_{N-1}^+<E_{N+1}^-<\cdots. \tag{30b}$$

(4) $\nu=\frac{1}{2}$. In other words, $q\Phi/2\pi\hbar c$ takes half integers. In this case we have

$$E_N^+=E_{N+1}^-=-\frac{\mu Z^2q^4}{2\hbar^2(N+1)^2},\quad N=0,1,2,\ldots. \tag{31}$$

The degeneracy of the level is $d_N^++d_{N+1}^-=2N+2$. This implies that the system has higher dynamical symmetry than the geometrical SO(2). It is well known that the 2H possesses SO(3) symmetry, just like the ordinary three-dimensional hydrogen atom possesses SO(4) symmetry. It seems that the symmetry for the present case is SU(2), and the above energy level corresponds to the value $(N+\frac{1}{2})(N+\frac{3}{2})$ for the Casimir operator of the SU(2) algebra. But this has not been explicitly proved. One can construct the Runge-Lenz vector in a way similar to that in the case of 2H [16]. However, the conservation of it and the closure of the algebra involve some difficulty due to the singularity of the AB potential at the origin. Perhaps some other method should be employed to deal with the problem.

Both bound state and scattering problems of the two-dimensional Coulomb field can be solved in parabolic coordinates [17,18,25]. Here we point out that the case (2) and (4) discussed above can also be solved in parabolic coordinates. As no new result can be obtained, we will not discuss the solutions in detail. In the next section we will deal with the scattering problem. It is in these two cases that exact solutions are available.

Finally, we give the value of the normalization constant C_{n_rm} in the wave function (26):

$$C_{n_rm}=\frac{4\mu Zq^2}{\hbar^2(2n_r+2|m+\nu|+1)\Gamma(2|m+\nu|+1)}\times\left[\frac{\Gamma(n_r+2|m+\nu|+1)}{2\pi n_r!(2n_r+2|m+\nu|+1)}\right]^{1/2}. \tag{32}$$

IV. SCATTERING PROBLEM

In this section we study the scattering problem of the two-body system. Here the Coulomb field may be either attractive or repulsive. We denote $\kappa = Zq^2$, which may be positive or negative. For general value of m_0 and ν, one may employ the method of partial-wave expansion in polar coordinates. Then the starting point may be Eqs. (15) and (16). However, the asymptotic form of $R(r)$ when $r\to\infty$ involves the $\ln r$ distortion, due to the long-range nature of the Coulomb field. This may be more clearly seen in the following. Thus it is not easy to treat the partial-wave expansion and to obtain the scattering cross section in a closed form. For this reason we confine ourselves in this paper to two special cases where exact analysis can be carried out in parabolic coordinates, and defer the general discussion to subsequent study.

Consider Eqs. (11a) and (12). Let us make a transformation,

$$\psi(r,\theta) = e^{-i(m_0+\nu)\theta}\psi_0(r,\theta). \tag{33}$$

The new wave function $\psi_0(r,\theta)$ satisfies the Schrödinger equation with a pure Coulomb field:

$$-\frac{\hbar^2}{2\mu}\nabla^2\psi_0 - \frac{\kappa}{r}\psi_0 = E\psi_0. \tag{34}$$

In parabolic coordinates this equation can be separated into two ordinary differential equations while Eq. (11) cannot be separated. The probability current density,

$$\mathbf{j} = \frac{\hbar}{2i\mu}(\psi^*\nabla\psi - \psi\nabla\psi^*) + \frac{(m_0+\nu)\hbar}{\mu}\psi^*\psi\nabla\theta, \tag{35}$$

can be written in terms of ψ_0 as

$$\mathbf{j} = \frac{\hbar}{2i\mu}(\psi_0^*\nabla\psi_0 - \psi_0\nabla\psi_0^*). \tag{36}$$

Although ψ_0 satisfies a simpler equation, the problem does not become easier since ψ_0 must satisfy a nontrivial boundary condition,

$$\psi_0(r,\theta+2\pi) = e^{i2\pi\nu}\psi_0(r,\theta), \tag{37}$$

such that $\psi(r,\theta)$ is single valued. Moreover, $\psi_0(r,\theta)$ should have proper behavior at the origin, so that ψ is well defined there. The latter condition also imposes a constraint on the solution.

It is, in general, difficult to deal with Eq. (37). In the following we only consider two special cases. The first is $\nu = 0$, or $q\Phi/2\pi\hbar c$ takes integers. In this case Eq. (37) becomes

$$\psi_0(r,\theta+2\pi) = \psi_0(r,\theta) \quad (\nu=0), \tag{38}$$

which means ψ_0 is single valued. This is because the first factor in Eq. (33) is also single valued in the present case. The second case we are to consider is $\nu=\frac{1}{2}$, or $q\Phi/2\pi\hbar c$ takes half integers. In this case Eq. (37) becomes

$$\psi_0(r,\theta+2\pi) = -\psi_0(r,\theta) \quad (\nu=\tfrac{1}{2}). \tag{39}$$

Though this is not convenient in polar coordinates, it may be easily treated in parabolic coordinates.

Now we introduce the parabolic coordinates (ξ,η) whose relation with (x,y) and (r,θ) are given by

$$x = \tfrac{1}{2}(\xi^2-\eta^2), \quad y = \xi\eta, \tag{40}$$

$$\xi = \sqrt{2r}\cos\frac{\theta}{2}, \quad \eta = \sqrt{2r}\sin\frac{\theta}{2}. \tag{41}$$

In these coordinates, Eqs. (38) and (39) become

$$\psi_0(-\xi,-\eta) = \psi_0(\xi,\eta) \quad (\nu=0) \tag{42}$$

and

$$\psi_0(-\xi,-\eta) = -\psi_0(\xi,\eta) \quad (\nu=\tfrac{1}{2}), \tag{43}$$

respectively, where for convenience we have used the same notation ψ_0 to denote the wave function in parabolic coordinates. It is easy to see that other values of ν in Eq. (37) renders $\psi_0(\xi,\eta)$ multivalued and thus are difficult to deal with. Though ψ_0 is double valued in polar coordinates in the case $\nu=\frac{1}{2}$, it becomes single valued in the parabolic coordinates. This is essentially because a $\xi\eta$ plane covers the xy plane twice, which is obvious from the relation $x+iy=(\xi+i\eta)^2/2$.

In the parabolic coordinates Eq. (34) becomes

$$(\partial_\xi^2+\partial_\eta^2)\psi_0 + k^2(\xi^2+\eta^2)\psi_0 + 4\beta k\psi_0 = 0, \tag{44}$$

where

$$k = \frac{\sqrt{2\mu E}}{\hbar}, \quad \beta = \frac{\mu\kappa}{\hbar^2 k}. \tag{45}$$

Note that $E>0$ since we are considering scattering states, and β is dimensionless. Equation (44) can be solved by separation of variables. Let

$$\psi_0(\xi,\eta) = v(\xi)w(\eta). \tag{46}$$

We have for v and w the following equations:

$$v'' + k^2\xi^2 v + \beta_1 k v = 0, \tag{47}$$

$$w'' + k^2\eta^2 w + \beta_2 k w = 0, \tag{48}$$

where $\beta_1+\beta_2 = 4\beta$, and primes denote differentiation with respect to argument. The general solution of Eq. (44) can be obtained by superposition of solutions of the form (46) over the parameter β_1. For the scattering problem at hand we will see, however, that a single β_1 is sufficient. No superposition is necessary. Specifically, we are looking for solutions that have the asymptotic property

$$\psi_0 \sim e^{ikx} \quad \text{for} \quad x\to-\infty. \tag{49}$$

This represents particles incident in the $+x$ direction, as is easily verified by using Eq. (36). In the parabolic coordinates it becomes

$$\psi_0 \sim e^{ik(\xi^2-\eta^2)/2} \quad \text{for} \quad \eta\to\infty \quad \text{and all} \quad \xi. \tag{50}$$

This can be satisfied only if

$$v(\xi)=e^{ik\xi^2/2} \tag{51}$$

and $w(\eta)$ has the asymptotic form

$$w(\eta)\sim e^{-ik\eta^2/2} \quad \text{for} \quad \eta\to\infty. \tag{52}$$

It is easy to verify that $v(\xi)$ given by Eq. (51) does satisfy Eq. (47) with $\beta_1=-i$. Then the constant β_2 in Eq. (48) is given by $\beta_2=4\beta+i$. The subsequent discussions depend on the value of ν, and we should distinguish between the two cases $\nu=0$ and $\nu=\frac{1}{2}$.

For $\nu=0$ we define a new function $u(\eta)$ by

$$w(\eta)=e^{-ik\eta^2/2}u(\eta); \tag{53}$$

then Eq. (48) becomes

$$u''-2ik\eta u'+4\beta ku=0. \tag{54}$$

On account of Eqs. (51) and (53), the condition (42) now simply means that $u(\eta)$ is an even function of η:

$$u(-\eta)=u(\eta). \tag{55}$$

It is easy to find the solution of Eq. (54) that satisfies this condition:

$$u(\eta)=c_1F(i\beta,\tfrac{1}{2},ik\eta^2), \tag{56}$$

where c_1 is a normalization constant. Collecting Eqs. (46), (51), (53), and (56) we obtain the solution

$$\psi_0=c_1e^{ik(\xi^2-\eta^2)/2}F(i\beta,\tfrac{1}{2},ik\eta^2)=c_1e^{ikx}F(i\beta,\tfrac{1}{2},ik\eta^2). \tag{57}$$

If in addition to $\nu=0$ we have $m_0=0$, i.e., for a pure Coulomb potential, this is the required solution. Taking the limit $r\to\infty$, and choosing the constant $c_1=e^{\beta\pi/2}\Gamma(1/2-i\beta)/\sqrt{\pi}$, we have for ψ_0 the asymptotic form

$$\psi_0\to\exp[ikx-i\beta\ln k(r-x)] + f_C(\theta)\frac{\exp(ikr+i\beta\ln 2kr)}{\sqrt{r}} \quad (r\to\infty) \tag{58}$$

up to the order $r^{-1/2}$, where

$$f_C(\theta)=\frac{\Gamma(1/2-i\beta)}{\Gamma(i\beta)}\frac{\exp(i\beta\ln\sin^2\theta/2-i\pi/4)}{\sqrt{2k\sin^2\theta/2}}. \tag{59}$$

The first term in the above equation represents the incident wave while the second represents the scattered one. Both of them are distorted by a logarithmic term in the phase due to the long-range nature of the Coulomb field. Despite these distortions, it can be shown that the scattering cross section is given by

$$\sigma(\theta)=|f_C(\theta)|^2, \tag{60}$$

where the subscript C indicates pure Coulomb scattering. Using the mathematical formulas

$$|\Gamma(\pm i\beta)|^2=\frac{\pi}{\beta\sinh\beta\pi}, \quad |\Gamma(\tfrac{1}{2}\pm i\beta)|^2=\frac{\pi}{\cosh\beta\pi}, \tag{61}$$

we arrive at

$$\sigma_C(\theta)=\frac{\beta\tanh\beta\pi}{2k\sin^2\theta/2}. \tag{62}$$

This is the result obtained in Ref. [25]. If $m_0\neq0$, i.e., if $q\Phi/2\pi\hbar c$ takes nonzero integers, the solution (57) has a problem, however. This is because $\psi_0(\mathbf{r}=\mathbf{0})=c_1\neq0$, and according to Eq. (33), $\psi(\mathbf{r}=\mathbf{0})=c_1e^{-im_0\theta}$, which is not well defined since θ is not well defined at the origin. The correct solution for $m_0\neq0$ should be

$$\psi_0=c_1[e^{ikx}F(i\beta,\tfrac{1}{2},ik\eta^2)-e^{ikr}F(\tfrac{1}{2}-i\beta,1,-2ikr)], \tag{63}$$

where the second term in the square bracket also solves Eq. (34) with the condition (38), and does not affect the boundary condition (49). We have now $\psi_0(\mathbf{r}=\mathbf{0})=0$ and no problem arises. Due to this additional term, the solution now behaves at infinity like

$$\psi_0\to\psi_{\rm in}+\psi_{\rm sc}+\psi_{\rm st} \quad (r\to\infty), \tag{64}$$

where $\psi_{\rm in}$ and $\psi_{\rm sc}$ represent the incident and scattered waves that are given by the first and second terms in Eq. (58), respectively, and $\psi_{\rm st}$ represents a stationary wave that comes from the second term in Eq. (63) and is given by

$$\psi_{\rm st}=-e^{i\delta_0}\sqrt{\frac{2}{\pi k}}\frac{\cos(kr+\beta\ln 2kr+\delta_0-\pi/4)}{\sqrt{r}}, \tag{65}$$

where

$$\delta_0=\arg\Gamma(\tfrac{1}{2}-i\beta). \tag{66}$$

Since the second term in Eq. (63) is in fact the s-wave term in the partial-wave expansion for a pure Coulomb field, the logarithmic distortion in its asymptotic form mentioned before becomes clear here. Similar distortions appear in all partial waves regardless of whether the AB potential is present. The first term in Eq. (64) gives an incident current in the $+x$ direction (when $x\to-\infty$). The second gives a scattered one in the radial direction (the component in the θ direction can be ignored when $r\to\infty$) and leads to the cross section $\sigma_C(\theta)$ obtained above. The third term, as a stationary wave, contributes nothing to the cross section. There are, however, interference terms. The interference of the first term with the subsequent ones does not lead to physically significant results. However, the interference of the second and the third terms actually gives rise to an additional term in the cross section, which will be denoted by $\sigma_\times(\theta)$. The differential cross section in the present case is thus given by

$$\sigma_1(\theta)=\sigma_C(\theta)+\sigma_\times(\theta), \tag{67}$$

where

$$\sigma_\times(\theta) = -\frac{\sqrt{\beta \tanh \beta\pi}}{\sqrt{\pi}k} \frac{\cos(\delta_0 + \delta_1 - \beta \ln \sin^2 \theta/2)}{|\sin \theta/2|}, \tag{68}$$

and

$$\delta_1 = \arg\Gamma(i\beta). \tag{69}$$

In the neighborhood of $\theta=0$, $\sigma_\times(\theta)$ oscillates rapidly and thus the total contribution in a finite (but small) interval of θ may be neglected. For large θ, especially near $\theta=\pi$, however, $\sigma_\times(\theta)$ gives a considerable contribution. It is remarkable that $\sigma_\times(\theta)$ is not positive definite and thus $\sigma_1(\theta)$ may become negative somewhere. This means that the particles move toward the origin at some directions. To the best of our knowledge, similar results were not encountered previously in the literature. Though the differential cross section $\sigma_1(\theta)$ may become negative at some direction, it does not cause any trouble physically because the total cross section is positive (actually positively infinite due to the long-range nature of the potentials). Indeed, $\sigma_\times(\theta)$ gives a finite contribution (positive or negative) to the total cross section, while $\sigma_C(\theta)$ gives a positively infinite one.

Now we turn to the case $\nu=\frac{1}{2}$. In this case we make the transformation

$$w(\eta) = e^{-ik\eta^2/2}\eta u(\eta); \tag{70}$$

then the equation for u reads

$$\eta u'' + 2(1 - ik\eta^2)u' + 2k(2\beta - i)\eta u = 0. \tag{71}$$

The condition (43) means that $u(\eta)$ is an even function of η. The required solution can be found to be

$$u(\eta) = c_2 F(i\beta + \tfrac{1}{2}, \tfrac{3}{2}, ik\eta^2), \tag{72}$$

where c_2 is a normalization constant. Collecting Eqs. (46), (51), (70), and (72) we obtain the solution

$$\psi_0 = c_2 e^{ikx}\eta F(i\beta + \tfrac{1}{2}, \tfrac{3}{2}, ik\eta^2). \tag{73}$$

Here two remarks should be made. First, as a function of r and θ, ψ_0 is double valued, so that ψ is single valued [cf. Eq. (33) where now $\nu=\frac{1}{2}$] . Second, as a consequence of Eq. (43) and obvious from the above result, we have $\psi_0(\mathbf{r}=\mathbf{0}) = 0$ here, so that ψ is well defined at the origin. We choose

$$c_2 = 2\sqrt{\frac{k}{\pi}} \exp\left(\frac{\beta\pi}{2} - i\frac{\pi}{4}\right)\Gamma(1 - i\beta);$$

then the asymptotic form of ψ_0 is given by

$$\psi_0 \to \exp[ikx - i\beta \ln k(r-x)] \frac{\sin \theta/2}{|\sin \theta/2|} + f(\theta)\frac{\exp(ikr + i\beta \ln 2kr)}{\sqrt{r}} \quad (r\to\infty), \tag{74}$$

where

$$f(\theta) = \frac{\beta\Gamma(-i\beta)}{\Gamma(1/2 + i\beta)} \frac{\exp(i\beta \ln \sin^2\theta/2 + i3\pi/4)}{\sqrt{2k}\sin^2\theta/2}. \tag{75}$$

Again note that both terms are double valued. The double valueness does not cause much trouble in the calculation. Using the formulas (61) the cross section can be shown to be

$$\sigma_2(\theta) = |f(\theta)|^2 = \frac{\beta \coth \beta\pi}{2k\sin^2 \theta/2}. \tag{76}$$

This has the same angular distribution as $\sigma_C(\theta)$, but the dependence on other parameters is quite different.

If we ignore the relation $\kappa = Zq^2$ and treat κ as an independent parameter, we may set $\kappa=0$ in the above results (note that $Z=0$ is not allowed in our formalism). Then we have

$$\sigma_1(\theta) = 0, \quad \sigma_2(\theta) = \frac{1}{2\pi k \sin^2 \theta/2}. \tag{77}$$

These are the AB scattering cross sections for the corresponding values of ν.

Finally, we point out that the cross sections (62), (67), and (76), when expressed in terms of the classical velocity $v_c = \hbar k/\mu$ instead of k, involve $\hbar$ explicitly. In the classical limit, $\hbar\to 0$, $\beta = \kappa/\hbar v_c \to \infty$ (this is actually realized in the low-energy limit), we see that $\sigma_\times(\theta)$ is negligible when compared with $\sigma_C(\theta)$, and both tanh $\beta\pi$ and coth $\beta\pi$ tend to ± 1. So we have in this limit,

$$\sigma_C(\theta) = \sigma_1(\theta) = \sigma_2(\theta) = \frac{|\kappa|}{2\mu v_c^2 \sin^2 \theta/2}, \tag{78}$$

which is the classical scattering cross section for a pure Coulomb field in two dimensions. This result implies that the AB potential has no significant effect in the classical limit as expected.

V. SUMMARY AND DISCUSSIONS

In this paper we propose an n-body Schrödinger equation for particles carrying magnetic flux as well as electric charges. The ratio of electric charge to magnetic flux is the same for all particles. The two-body problem is studied in detail. The bound-state problem is exactly solved in the general case, while the scattering problem is exactly solved in two special cases.

The original intention of this paper is to describe the CS vortex solitons by a simple quantum-mechanical model. If the sizes of the solitons are small, the AB potential may be a good approximation in describing the charge-flux interaction. On the other hand, the real charge-charge interaction may be quite complicated; thus the Coulomb potential used here may be questionable. If a better form $V(q_a, q_b, |\mathbf{r}_a - \mathbf{r}_b|)$ can be found for the interaction potential of charge q_a at $\mathbf{r}_a$ and charge q_b at $\mathbf{r}_b$, then the n-body equation may be improved by substituting this potential for $q_a q_b/|\mathbf{r}_a - \mathbf{r}_b|$ in Eq. (2b). In this case the last term $q_1 q_2/r$ in the two-body relative Hamiltonian (11b) should be replaced by $V(q_1, q_2, r)$. With an improved potential, the Schrödinger equation might become more difficult to solve, however. Therefore, the model studied in this paper, even though it cannot well describe the interaction of the vortex solitons, may have some interest in itself since it allows exact analysis to some extent.

Several aspects of this model that need further studies may be: the dynamical symmetry of the two-body system, the scattering problem for general value of ν, and finally, the relativistic generalization of the model.

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