Photon wave functions in a localized coordinate space basis

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A first quantized theory of the photon is developed that incorporates all of the usual rules of quantum mechanics. The state vector is argued to be proportional to the four-vector potential. Scalar products and the probability amplitude are invariant under gauge transformations that satisfy the Lorentz gauge condition. The eigenvectors of a recently constructed Hermitian position operator with commuting components provide a basis for the position representation. [S1050-2947(99)02605-0]

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I. INTRODUCTION

In interactions with matter where photons are created and destroyed a second quantized formalism is required. However, where annihilation and creation are not involved, a first quantized description should exist. There have been several attempts to construct a first quantized description of the photon, but none of these allow full use of the formal structure of quantum mechanics. Considering the current interest in one-photon states and localizability [1–3], it is timely to make yet another attempt to solve this long standing problem.

If position is to be an observable, the formal structure of quantum mechanics requires a position operator. It has long been claimed that there is no suitable position operator for the photon [4]. Recently a photon position operator with commuting components has been constructed [5]. This removes a very significant obstacle to the full realization of photon quantum mechanics.

It is a fluke of nature that Planck's constant occurs to the same power in all the terms of Maxwell's equations and thus $\hbar$ does not appear explicitly. For nonrelativistic particles, energy occurs to the first power while momentum is squared and $\hbar$ cannot be divided out. In the case of massive relativistic particles, $E^2-p^2c^2=m^2c^4$ again results in a wave equation in which $\hbar$ cannot be eliminated. In spite of the absence of $\hbar$, by analogy with massive particles, the nineteenth century Maxwell equations should describe the wave mechanics of a photon.

Pryce constructed a six-component wave function from the electric and magnetic fields **E** and **B** [6], rejecting the vector potential **A** based on the need for gauge invariance. Using **E** and **B**, Maxwell's theory in vacuum can be stated in a form that closely parallels Dirac's theory of the electron [7]. Recently, there has been renewed interest in this approach [8–11]. The wave function was written by Bialynicki-Birula as the six-vector

$$\psi\equiv\sqrt{\epsilon}\begin{pmatrix}\mathbf{E}^{(+)}+ic\mathbf{B}^{(+)}\\ \mathbf{E}^{(+)}-ic\mathbf{B}^{(+)}\end{pmatrix},\tag{1}$$

where $+$ and $-$ describe photon helicity [8] and the superscript $(+)$ denotes the positive energy part. In this and a related three-vector formulation due to Sipe [9] where the ±1 helicity components are added, the concept of photon probability density is given up, and replaced with energy density.

State vectors can be defined in **p** space using the Landau-Peierls wave function [12] that differs from the fields by a factor $\sqrt{p}$. Expectation values then take the familiar quantum mechanical form [13]. However, since these functions are not fundamental to the electromagnetic field, the source terms become nonlocal functions of the charge and current densities [14] and the photon probability and current densities do not transform as Lorentz objects [8] and are not gauge invariant.

Photon quantum mechanics based on Eq. (1) has been explored in considerable detail [11]. If $\hat{H}$ is the Hamiltonian, a scalar product

$$\langle\psi|\phi\rangle_{\mathrm{BB}}=\int d^3r\,\psi^\dagger\hat{H}^{-1}\phi\tag{2}$$

can be defined and momentum and angular momentum eigenvectors found. (Here the subscript BB denotes the Bialynicki-Birula scalar product, and is used to distinguish it from the scalar product defined in the present paper.) The limitation is that a position operator was assumed not to exist and, as a consequence, no projection operator onto a region of **r** space was available. Probability density was defined as $\psi^\dagger\psi$ divided by the average energy, as in [9].

The wave function given by Eq. (1) is a natural choice if it is thought necessary to base probability amplitudes on energy density, which equals $\psi^\dagger\psi$. The existence of a position operator whose eigenvectors define a projection operator $|\mathbf{r}\rangle\langle\mathbf{r}|$ suggests that it may be possible to base photon quantum mechanics on number density as is usual. Since charges are coupled to the vector potential in the photon-matter Hamiltonian, **A** may provide a simpler basis for photon quantum mechanics than $\mathbf{E}\pm ic\mathbf{B}$, provided problems associated with gauge invariance can be overcome.

In this paper a case is first presented for use of the vector potential as the wave function of the photon. The scalar product of two state vectors and the expectation value and adjoint of an operator are defined in Sec. II. It is then demonstrated in Sec. III that the eigenvectors of the position operator form a basis for the position representation. In Sec.

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IV the relationship to second quantized photon operators and other photon wave functions is examined. Système International units are used throughout.

II. EXPECTATION VALUES

Expressions for momentum and angular momentum of the field based on symmetries of the standard Lagrangian are of the form

$$O=\sum_{j=1}^{3}\frac{i\epsilon_0}{\hbar}\int d^3r E_j\hat{O}A_j \tag{3}$$

[15]. Here A_j and E_j are the Cartesian components of the vector potential and electric field, respectively, and $-\epsilon_0 E_j$ arises as the variable conjugate to A_j. In the absence of matter, **E** is transverse and thus only real photons contribute. The density $i\epsilon_0\mathbf{E}^{(-)}\cdot A^{(+)}/\hbar$ was examined previously [16] and arguments presented for its use as a number operator also apply to a first quantized theory.

A covariant formalism will be used here, both for fundamental reasons and to allow flexibility in the calculation of potentials and fields in the presence of current sources. Einstein notation where repeated indices are summed over will be used for conciseness. The four-vector potential is $A^\mu=(A^0,\mathbf{A})$ where cA^0 is the scalar potential and **A** is the vector potential. A covariant four-vector is obtained from its contravariant counterpart as $A_\mu=g_{\mu\nu}A^\nu$. The metric tensor g is diagonal with zeroth entry $+1$ and first to third entry -1 and attaches a minus sign to the first to third terms of a covariant-contravariant product. The gradient four-vector is $\partial_\mu=((1/c)\partial/\partial t,\boldsymbol{\nabla})$ so that the derivative with respect to contravariant components gives a covariant vector.

The Lorentz gauge condition

$$\partial_\mu A^\mu=0 \tag{4}$$

will be imposed to ensure that A transforms as a four-vector. The Lagrangian density $\mathcal{L}=-\frac{1}{2}\varepsilon_0 c^2(\partial_\mu A^\nu)(\partial^\mu A_\nu)$ gives the equations of motion $\partial_\mu\partial^\mu A^\nu=0$ in a charge and current free region. Analogous arguments to those in [15] based on translational and rotational invariance of this covariant Lagrangian then result in conserved quantities of the form

$$O=\frac{i\epsilon_0}{\hbar}\int d^3r\frac{\partial A_\mu}{\partial t}\hat{O}A^\mu. \tag{5}$$

In the Coulomb gauge, $\boldsymbol{\nabla}\cdot\mathbf{A}=0$ requires **A** to be transverse and the zeroth equation of motion becomes $\nabla^2A^0=0$ which implies $A^0=0$ for a free photon. Then $(\partial A_\mu/\partial t)A^\mu$ reduces to $\mathbf{E}\cdot\mathbf{A}$, making Eqs. (3) and (5) the same. Free photon expressions in the Coulomb gauge can be obtained by setting the longitudinal and scalar components of the four-vector potential equal to zero.

As with the versions of photon quantum mechanics discussed in the introductory paragraphs, the positive energy components will be used here. The notation $A^{(+)\mu}$ will denote the positive frequency part of the four-vector potential. The Schrödinger equation

$$\hat{H}A^{(+)\mu}=i\hbar\partial A^{(+)\mu}/\partial t, \tag{6}$$

with Hamiltonian

$$\hat{H}=c\sqrt{\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}} \tag{7}$$

will be used to eliminate the time dependence.

A state vector, proportional to the four-vector potential but normalized to give unity, will be defined as

$$|\psi^\mu\rangle\equiv\frac{\sqrt{2\epsilon_0}}{\hbar}|A^{(+)\mu}\rangle, \tag{8}$$

where the factor $\sqrt{2}$ comes from the reduction to positive frequencies. It will be assumed that Maxwell's equations are satisfied and that $\mathbf{B}=\boldsymbol{\nabla}\times\mathbf{A}$ and $\mathbf{E}=-\partial\mathbf{A}/\partial t-c\boldsymbol{\nabla}A^0$. This together with Eq. (4) ensures that the field vectors have the usual orthogonality properties. Since they are based on A, the state vectors are transverse in the Coulomb gauge, but include the possibility of longitudinal and scalar components in the Lorentz gauge.

The scalar product will be written as

$$\langle\psi|\phi\rangle\equiv-\langle\psi_\mu|\hat{H}|\phi^\mu\rangle, \tag{9}$$

which equals $-\langle\phi^\mu|\hat{H}|\psi_\mu\rangle^*$, as suggested by Eq. (5) and the complex conjugate of Eq. (6). The minus sign is required since the spacelike components of A_μ are negative. In momentum space $\hat{H}$ becomes just the measure pc and Eq. (9) reduces to

$$\langle\psi|\phi\rangle=-\int d^3p\,\psi_\mu^*(\mathbf{p},t)pc\,\phi^\mu(\mathbf{p},t). \tag{10}$$

Conservation of total probability is achieved if $\psi_\mu\hat{H}\psi^\mu$ is the first component of a four-vector satisfying a continuity equation. In **r** space, the four-vector

$$j^\nu\equiv-\frac{i\hbar c}{2}[\psi_\mu^*(\partial^\nu\psi^\mu)-(\partial^\nu\psi_\mu^*)\psi^\mu] \tag{11}$$

is a candidate. Evaluation of $\partial_\nu j^\nu$ using the equation of motion, $\partial_\mu\partial^\mu A^\nu=0$, gives $\partial_\nu j^\nu=0$. This shows that j^ν is a conserved four-current density, making the spatial integral of its 0th component time independent. While the first and second terms of Eq. (11) are not identical, their integrals over **r** space are equal, and the integral of j^0 reduces to Eq. (9). This can be verified by substitution of the general free photon wave function, $\psi^\mu(\mathbf{r},t)\propto\int d^3p\,\psi^\mu(\mathbf{p})e^{i(\mathbf{p}\cdot\mathbf{r}-pct)/\hbar}$. Thus an equivalent scalar product to Eq. (9) is the integral of j^0.

The longitudinal and scalar contributions to Eq. (10) cancel, leaving only the transverse components, and $\langle\psi|\phi\rangle$ can be seen to be invariant under gauge transformations that satisfy the Lorentz gauge condition. For a free particle whose wave function goes as $e^{i(\mathbf{p}\cdot\mathbf{r}-pct)/\hbar}$, a general gauge transformation is necessarily of the Lorentz form.

Following Eq. (5) and using Eq. (9) with $\phi=\hat{O}\chi$, the expectation value of an operator, $\hat{O}$, becomes

$$\langle\psi|\hat{O}|\chi\rangle\equiv-\langle\psi_\mu|\hat{H}\hat{O}_\nu^\mu|\chi^\nu\rangle. \tag{12}$$

The operator $\hat{O}$ can be scalar or vector, and in general is a matrix. The 4×4 unit matrix is implied if none is explicitly written. Adjoints will be required to satisfy the usual relationship,

$$\langle\chi|\hat{O}^{\dagger}|\psi\rangle=\langle\psi|\hat{O}|\chi\rangle^{*}. \tag{13}$$

We will show that these definitions result in a first quantized description of the photon consistent with all the rules of quantum mechanics.

III. EIGENVECTORS

The momentum observable will be considered first. In the position representation, $\hat{\mathbf{p}}=-i\hbar\boldsymbol{\nabla}$ and the μth component of its eigenfunction with eigenvalue $\mathbf{p}'$ and polarization λ' is

$$\phi^{\mu}_{\mathbf{p}'\lambda'}(\mathbf{r})=e^{\mu}_{\mathbf{p}'\lambda'}e^{i\mathbf{p}'\cdot\mathbf{r}/\hbar}/\sqrt{p'c(2\pi\hbar)^{3}}. \tag{14}$$

These functions have been normalized according to Eq. (9). The two transverse unit vectors will be called $\mathbf{e}_{\mathbf{p}1}$ and $\mathbf{e}_{\mathbf{p}2}$, while $\mathbf{e}_{\mathbf{p}3}\equiv\mathbf{p}/p$ and e_0 denotes the scalar component. In momentum space, $\hat{\mathbf{p}}=\mathbf{p}$ and the probability amplitude for finding the photon with momentum $\mathbf{p}$ and polarization λ is, by Eq. (9),

$$\langle\mathbf{p}\lambda|\mathbf{p}'\lambda'\rangle=\delta(\mathbf{p}-\mathbf{p}')\delta_{\lambda,\lambda'} \tag{15}$$

as expected. The δ-function normalized state vector is

$$\phi^{\mu}_{\mathbf{p}'\lambda'}(\mathbf{p})=e^{\mu}_{\mathbf{p}'\lambda'}\delta(\mathbf{p}-\mathbf{p}')/\sqrt{p'c}. \tag{16}$$

There is clearly a problem with the zeroth basis vector, since its norm is negative and it represents a negative probability density. This is a well-known difficulty associated with the nonphysical scalar photons [15]. Here the problem will be dealt with in the following way: In the reference frame where $\phi^3=\phi^0=0$ the Coulomb gauge condition is satisfied and there are only two photon polarizations or, after taking linear combinations, two photon helicities. Viewed from an arbitrary inertial frame, the states have equal longitudinal and scalar components but they still describe one-photon states. Eigenvectors with a nonzero $\lambda=1$ or $\lambda=2$ component and with equal $\lambda=0$ and $\lambda=3$ components describe the above two transverse states in a general inertial reference frame and have positive norms. This basis is complete for the description of real photon states in the Coulomb or the Lorentz gauge. The gauge condition reduces the number of independent vectors to three, while the remaining orthogonal vector with zero $\lambda=1$ and 2 components has zero norm and is equivalent to the vacuum.

In the case of the position operator, the usual momentum space choice $\hat{\mathbf{r}}=i\hbar\boldsymbol{\nabla}_{\mathbf{p}}$, where $\boldsymbol{\nabla}_{\mathbf{p}}$ is the gradient operator in $\mathbf{p}$ space, is not Hermitian and does not have transverse eigenvectors. Since normalization according to Eq. (9) results in a factor $p^{-1/2}$ and the states are vectors, $\hat{\mathbf{r}}$ must differentiate $\mathbf{e}_{\mathbf{p}\lambda}$ and $p^{-1/2}$, and this produces unwanted terms that must be subtracted. The position operator

$$\hat{\mathbf{r}}=i\hbar\nabla_{\mathbf{p}}+i\hbar\frac{\mathbf{p}}{2p^{2}}+\frac{\hbar}{p^{2}}\mathbf{p}\times\mathbf{S}-\frac{\hbar}{p}\cot\theta_{\mathbf{p}}\mathbf{e}_{\mathbf{p}2}S_{\mathbf{p}3} \tag{17}$$

was obtained in [5] for the case $\alpha=-\frac{1}{2}$ that applies here. This operator is Hermitian and its eigenvectors are transverse, longitudinal or scalar.

In momentum space the normalized eigenvectors of $\hat{\mathbf{r}}$ given by Eq. (17), $|\mathbf{r}'\lambda'\rangle$, with eigenvalues $\mathbf{r}'$ are

$$\psi^{\mu}_{\mathbf{r}'\lambda'}(\mathbf{p})=e^{\mu}_{\mathbf{p}\lambda'}e^{-i\mathbf{p}\cdot\mathbf{r}'/\hbar}/\sqrt{pc(2\pi\hbar)^{3}}. \tag{18}$$

These functions describe physical photons for $\lambda=1$ and 2. They satisfy

$$\langle\mathbf{r}'\lambda'|\mathbf{r}''\lambda''\rangle=\delta(\mathbf{r}'-\mathbf{r}'')\delta_{\lambda'\lambda''}, \tag{19}$$

that is, they are orthogonal and δ-function normalized.

State vectors can be expanded in a position or momentum basis using the unit closure operator in the usual way. Since the photon's energy is known in the momentum basis,

$$|\psi(t)\rangle=\sum_{\lambda=\pm1}\int d^{3}p|\mathbf{p}\lambda\rangle\langle\mathbf{p}\lambda|\psi(0)\rangle e^{-ipct/\hbar}. \tag{20}$$

The scalar product $\langle\mathbf{p}\lambda|\psi\rangle$ is the probability amplitude for finding it with momentum $\mathbf{p}$. Similarly, the scalar product $\langle\mathbf{r}\lambda|\psi\rangle$ is the probability amplitude for finding a photon at position $\mathbf{r}$ with polarization λ, and $|\psi(t)\rangle$ can be expanded in a position basis as

$$|\psi(t)\rangle=\sum_{\lambda=\pm1}\int d^{3}r|\mathbf{r}\lambda\rangle\langle\mathbf{r}\lambda|\psi(t)\rangle. \tag{21}$$

The probability amplitude $\langle\mathbf{r}\lambda|\psi\rangle$ is consistent with any photon wave function whose transverse components give the above scalar product and whose longitudinal and scalar components are equal. Since the longitudinal and scalar components must be equal to satisfy the gauge condition, Eq. (4), $e^{\mu}_{\mathbf{p}g}=e^{\mu}_{\mathbf{p}3}+\delta^{\mu}_{0}$ will be defined. If a gauge term proportional to this four-vector is added to any state vector, scalar products are unaffected, implying that the new vector is equivalent to the original one. Here equivalence refers to the equality of scalar products in quotient space [17].

In a position basis, using Eqs. (20) and (14) and

$$\langle\mathbf{p}\lambda|\mathbf{r}'\lambda'\rangle=\delta_{\lambda\lambda'}e^{-i\mathbf{p}\cdot\mathbf{r}'/\hbar}/\sqrt{(2\pi\hbar)^{3}}, \tag{22}$$

the localized states become

$$\psi^{\mu}_{\mathbf{r}'\lambda'}(\mathbf{r})=\int d^{3}p\,e^{\mu}_{\mathbf{p}\lambda'}e^{i\mathbf{p}\cdot(\mathbf{r}-\mathbf{r}')/\hbar}/[\sqrt{pc}(2\pi\hbar)^{3}]. \tag{23}$$

If the spherical polar unit vectors $\hat{\theta}_{\mathbf{p}}$ and $\hat{\phi}_{\mathbf{p}}$ are defined as the two transverse directions 1 and 2, respectively, the normalized state vector

$$\psi^{\mu}_{\mathbf{r}'z}(\mathbf{p})=\psi^{\mu}_{\mathbf{r}'1}(\mathbf{p})-\cot\theta_{\mathbf{p}}e^{\mu}_{\mathbf{p}g}e^{-i\mathbf{p}\cdot\mathbf{r}'/\hbar}/\sqrt{pc(2\pi\hbar)^{3}} \tag{24}$$

has a fixed direction $\hat{\mathbf{z}}$ in space such as would result from a current source of known direction. The state described by Eq. (24), while not an eigenfunction of $\hat{\mathbf{r}}$, is equivalent to one. In the position representation the space part of $\psi_{\mathbf{r}'z}$ goes as $\hat{\mathbf{z}}|\mathbf{r}-\mathbf{r}'|^{-5/2}$, implying an extended wave function [18,19].

IV. DISCUSSION

In this section, the photon wave function will be discussed in relation to the one- to zero-photon transition amplitude of the annihilation operator and the use of positive frequencies. The present vector potential based wave function will be compared with photon wave functions based on $\mathbf{E}\pm ic\mathbf{B}$ and the Landau-Peierls wave function. Gauge invariance and the photon probability density will be discussed and the results summarized.

Following second quantized theory, a photon's wave function in the momentum representation can be obtained from the positive frequency part of the field operator as $\psi_{\mathbf{r}'\lambda'}=\langle 0|\hat{\psi}^{(+)}_{\mathbf{r}'}|\mathbf{p}\lambda'\rangle$ [20]. For photons the annihilation operator is the usual vector potential operator rescaled using Eq. (8) to give

$$\hat{\psi}^{(+)}_{\mathbf{r}'}=\sum_{\lambda=1,2}\int d^3p\,\mathbf{e}_{\mathbf{p}\lambda}e^{i\mathbf{p}\cdot\mathbf{r}'/\hbar}a_{\mathbf{p}\lambda}/\sqrt{pc(2\pi\hbar)^3}. \qquad (25)$$

Then $\psi_{\mathbf{r}'\lambda'}(\mathbf{p})$ agrees with Eq. (18). The operator given by Eq. (25) destroys a photon at $\mathbf{r}'$ while its adjoint creates one in the usual position space photon-matter Hamiltonian. In field theory an interaction Hamiltonian creates particles at $\mathbf{r}'$ which propagate to $\mathbf{r}''$ where they are destroyed. This is consistent with the physical picture obtained here, that Eq. (18) describes a photon localized at $\mathbf{r}'$.

The validity of the Hamiltonian, Eq. (7), requires that $|\psi(t)\rangle$ include strictly positive energies. For a free photon, Eq. (25) generates only positive energies. However, a real current source drives both positive and negative frequency oscillations of $a_{\mathbf{p}\lambda}$. When interactions with charged matter are included, the correct positive frequency part discussed in [21,22] must be used.

The expectation value of a field operator such as is defined by Eq. (25) is zero for a state with a definite number of photons, in particular for one photon. Thus while the wave function of a localized photon is nonlocal in $\mathbf{r}$ space, the average field is zero and a state that is known to contain exactly one photon does not interact with charged matter. Fields are nonzero during emission and absorption because the number of photons is uncertain.

If a photon with momentum $\mathbf{p}$ is emitted at $\mathbf{r}'$, time t', its electric field must go as $\sqrt{p}$ to give the correct energy. At the instant of emission the fields for each $\mathbf{p}$ are in phase and their sum is nonlocal. The meaning of the nonlocal fields and wave functions associated with newly created and localized photons is not yet clear. However, it has been concluded that the field generated by a spontaneously emitting atom is causal [21,9].

There are alternative wave functions and operators that result in the same expectation value, Eq. (12), and hence the same predictions as the present theory. For example, $\langle\psi_\mu|\hat{H}\hat{O}^\mu_\nu|\phi^\nu\rangle$ can be written as

$$\langle\tilde{\psi}|\hat{\tilde{O}}|\tilde{\phi}\rangle\equiv\langle\hat{H}^{1/2}\psi_\mu|\hat{H}^{1/2}\hat{O}^\mu_\nu\hat{H}^{-1/2}|\hat{H}^{1/2}\phi^\nu\rangle. \qquad (26)$$

In p space and the Coulomb gauge, this gives the Landau-Peierls wave function, $f_{\mathbf{p}\alpha}\sim(pc)^{1/2}\phi^\alpha(\mathbf{p})$, considered in [13]. The advantage of the approach in [13] and Eq. (26) is that $\hat{H}=pc$ is eliminated to give the familiar form, $\int d^3p f^*_{\mathbf{p}\alpha}\hat{\tilde{O}}f_{\mathbf{p}\alpha}$ where $f_{\mathbf{p}\alpha}$ is the probability amplitude. However, $\mathbf{f}_{\mathbf{p}}$ does not bear any direct relationship to Maxwell's equations, and its relationship to the matter current density is complicated and nonlocal. It has no simple gauge and Lorentz transformation properties [8]. A second vector can be defined to give the analog of Maxwell's equations and probability four-current density [14], but its components are not gauge invariant and it is not a four-vector [19].

The scalar product,

$$\langle\tilde{\psi}|\hat{\tilde{O}}|\tilde{\phi}\rangle\equiv\langle\hat{H}\psi_\mu|\hat{H}^{-1}(\hat{H}\hat{O}^\mu_\nu\hat{H}^{-1})|\hat{H}\phi^\nu\rangle, \qquad (27)$$

is of the same form as Eq. (2) and results in the wave function $\tilde{\psi}=\hat{H}\psi$, and new operators $\hat{\tilde{O}}$. The Hamiltonian and momentum operators are unchanged, and the new position operator acquires a term $-i\hbar\mathbf{p}/p^2$ so that $\hat{\mathbf{r}}$ has eigenvectors proportional to $\sqrt{p}$ as in Eq. (1). This corresponds to $\alpha=\frac{1}{2}$ in [5].

If ψ and Eq. (9) are replaced by $\tilde{\psi}\sim\hat{H}\psi=i\hbar\partial\psi/\partial t$ and Eq. (27), an alternative version of photon quantum mechanics is obtained. Calculation of $\hat{H}^{-1}\tilde{\psi}$ is problematic in the Lorentz gauge, and the resulting formalism is probably best restricted to the Coulomb gauge. For a free photon, $\tilde{\psi}$ is then proportional to the electric field vector, and is equivalent to the wave function defined by Scully and Zubairy [23]. It is invariantly normalized, and some may prefer to treat $\tilde{\psi}$ as the primary photon wave function.

Recent advocates of the photon wave function concept [8,9] have based it on $\sqrt{\varepsilon}(\mathbf{E}^{(+)}\pm ic\mathbf{B}^{(+)})$. The relationship of wave functions of this form to the present theory will now be considered. The Hamiltonian used in [8,10,11], $c\mathbf{S}\cdot\hat{\mathbf{p}}$, takes the curl of the vector on which it operates. Bialynicki-Birula's wave function, Eq. (1), describes photons with helicity $+1$ and -1. In $\mathbf{p}$ space, this implies that the electric field vector $\mathbf{E}^{(+)}_\pm$ is proportional to $\mathbf{e}_{\mathbf{p}1}\pm i\mathbf{e}_{\mathbf{p}2}$. Once $\mathbf{e}_{\mathbf{p}1}$ is selected, $\mathbf{e}_{\mathbf{p}2}=\mathbf{e}_{\mathbf{p}3}\times\mathbf{e}_{\mathbf{p}1}$ and it follows that $\mathbf{B}$, given by $i\omega_p\mathbf{B}=\boldsymbol{\nabla}\times\mathbf{E}$, must go as $\pm(\mathbf{e}_{\mathbf{p}1}\pm i\mathbf{e}_{\mathbf{p}2})$. This gives $\mathbf{E}^{(+)}_\pm\pm ic\mathbf{B}^{(+)}_\pm=2\mathbf{E}^{(+)}_\pm$ and $\mathbf{E}^{(+)}_\pm\mp ic\mathbf{B}^{(+)}_\pm=0$, that is, the ±1 helicity states can be equivalently described by $\mathbf{E}^{(+)}_\pm\pm ic\mathbf{B}^{(+)}_\pm$, $\mathbf{E}^{(+)}_\pm$, or $\mathbf{B}^{(+)}_\pm$ (but not by $\mathbf{E}^{(+)}_\pm\mp ic\mathbf{B}^{(+)}_\pm$). The Bialynicki-Birula scalar product, Eq. (2), requires $H^{-1}\psi=\pm\mathbf{E}^{(+)}_\pm/ipc$ which equals $\pm\mathbf{A}^{(+)}_\pm$. Thus Eqs. (2) and (27) are equivalent to Eq. (9) in the Coulomb gauge.

The four-vector j^ν in Eq. (11) is not gauge invariant, and this might be perceived as a fatal flaw in the present vector potential based formalism. However, expectation values calculated using Eq. (12) are independent of gauge. The result of a measurement of position is predicted by the expectation value of the projection operator $|\mathbf{r}\lambda\rangle\langle\mathbf{r}\lambda|$. Thus the probability density for finding a photon in state $|\psi\rangle$ at $\mathbf{r}$ with helicity λ is $\langle\psi|\mathbf{r}\lambda\rangle\,\langle\mathbf{r}\lambda|\psi\rangle$ and the density j^0 has no direct physical significance.

The vector potential plays a unique role in quantum mechanics. Using the minimal coupling Hamiltonian, interaction with a charged particle is described by $\mathbf{A}$, and an arduous transformation is required to express the interaction in terms of $\mathbf{E}$ and $\mathbf{B}$. The Aharanov-Bohm effect demonstrates that an electron can be affected by a magnetic field with which it has no local contact through a nonzero vector potential. While gauge invariance must be respected, **A** appears to have some physical significance. In the present formalism the state of the photon is also described by the vector potential.

In summary, using states proportional to the four-vector potential, a first quantized theory of the photon was developed. Expectation values and adjoints can be defined in the usual way. The eigenvectors of Hermitian momentum and position operators were shown to form a basis in momentum and position space, respectively. Scalar products and probability amplitudes are invariant under gauge changes that satisfy the Lorentz condition. State vectors consistent with this probability amplitude can be written in the Coulomb or the Lorentz gauge. The former choice results in strictly transverse components, while the latter form is covariant and allows a vector potential with a fixed direction in space. For eigenvectors with a definite helicity described in the Coulomb gauge, the Bialynicki-Birula-Sipe wave function and Bialynicki-Birula scalar product are regained and extended to include a configuration space basis. This new formalism provides a description of a single photon that is consistent with all of the usual formal rules of quantum mechanics.

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