Ambiguities of arrival-time distributions in quantum theory

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We consider the definition that might be given, in quantum theory, to the time at which a particle arrives at a given place. We discuss an ambiguity that arises in three, but not in one, spatial dimensions. We first express this ambiguity within the ontology of Bohmian quantum theory, but we also show that it can be expressed independently of that ontology. [S1050-2947(99)02505-6]

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I. INTRODUCTION

In classical mechanics, a particle can be said to follow a definite trajectory, and so it is clear what is meant by the time at which a particle arrives in a given place. If one considers an ensemble of particles, it is then easy to say what is meant by a distribution of arrival times. In standard quantum theory, on the other hand, particles are not said to follow trajectories, and so the meaning of arrival time in quantum theory has been rather controversial. Some elements of this controversy include statements such as ‘‘the time of arrival cannot be precisely defined and measured in quantum mechanics’’ (quoted from [1]; see also [2–4]); statements that time of arrival *must* be definable, for example ‘‘Since the distribution of arrival times at a given spatial point is, in principle, a measurable quantity that can be determined via a time-of-flight experiment, it is reasonable to ask for an apparatus-independent theoretical prediction’’ [5]; specific proposals for defining a time-of-arrival distribution (for example, in [6–8]), and suggestions (for example, in [7,9]) that experiments could determine which if any of these proposals is correct. Time-of-arrival distributions in quantum theory are reviewed in [10,11].

In the causal theory of Bohm [12,13], particles *do* follow definite trajectories, and so the definition of arrival-time distributions is again unambiguous. Leavens, most recently in [14], has studied the arrival-time distribution of a free particle in Bohmian theory, and found results which differ from the proposal made in [7]. Deotto and Ghirardi [15], and also Holland [16] have proposed what I shall call Bohm-like theories: theories in which particles follow trajectories which differ from the trajectories of Bohmian theory, but which nevertheless reproduce all of the observational results of standard quantum theory, in the same way that Bohmian theory does. In this paper I will study a simple example of a Bohm-like theory, and demonstrate that in certain cases this theory will produce arrival-time distributions which are different than those produced by (standard) Bohmian theory. I will also use this same demonstration to discuss an ambiguity in the meaning of arrival-time distributions for three-dimensional problems that does not depend on the adoption of the ontology of Bohmian (or of Bohm-like) theories.

I begin by using terminology which is appropriate in classical mechanics. Consider a particle with position coordinates (x,y,z), which at the initial time $t=0$ has $x<0$. Let S_+ denote the region of space in which $x\geqslant 0$, S_- the region in which $x<0$, and S_0 the $x=0$ plane. Let T be the first time after $t=0$ at which the particle arrives at S_0 (equivalently, crosses from S_- to S_+); by definition, $T>0$. I will mostly be concerned with the integrated arrival-time distribution, which I denote as $P(t)$; that is, $P(t)$ is the probability that $T\leqslant t$.

The question to be discussed is whether this distribution P can be precisely and unambiguously defined in quantum theory. If it can be, it could be expected to satisfy, at least, the following properties (see also the discussion in [17]).

(i) $P(t)$ is monotonically increasing: $P(t)\geqslant P(t')$ for $t\geqslant t'\geqslant 0$.

(ii) Define $A(t)=dP(t)/dt$; from (i), we have $A(t)\geqslant 0$. Then $A(t)dt$ represents the probability that $T=t$.

(iii) Let $\overline{T}$ be the average value of T (averaged over those cases in which the particle does eventually arrive at S_0). From (ii), this is given by

$$\overline{T}=\frac{\int_0^\infty tA(t)dt}{\int_0^\infty A(t)dt}. \tag{1}$$

This can be rewritten, with $P_\infty := \lim_{t\to\infty} P(t)$, as

$$\overline{T}=\frac{\int_0^\infty dt[P_\infty - P(t)]}{P_\infty}. \tag{2}$$

(iv) Now let $Q(t)$ be the probability that the particle would be found, at time t, in S_+; that is,

$$Q(t)=\int\int\int |\Psi(x,y,z,t)|^2\theta(x)dx\ dy\ dz. \tag{3}$$

The initial condition we are assuming means that $Q(t=0)=0$. We expect that

$$P(t)\geqslant Q(t). \tag{4}$$

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If the particle could surely be found in S_+ at time t if it had arrived there at time t' (with $0 \leqslant t' \leqslant t$), then the relation (4) could be replaced by an equality.

(v) From (i) and (iv), we have

$$P(t) \geqslant \max_{t \geqslant t' \geqslant 0} Q(t'). \tag{5}$$

In the next section I will present an example in which a Bohm-like theory produces a different result for $P(t)$ than does the usual Bohmian theory. The implications of this example are discussed in the final section.

II. EXAMPLE

Consider a free ($V=0$) particle which, at the initial time $t=0$, is described by the following minimum-uncertainty wave packet centered at the point $x=-x_1$, $y=0$, $z=0$:

$$\Psi(x,y,z,t=0) = \left[\frac{1}{\pi^3 a^2 b^2 c^2}\right]^{1/4} \exp[ikx] \times \exp\left[-\frac{(x+x_1)^2}{2a^2}\right] \exp\left[-\frac{y^2}{2b^2}\right] \times \exp\left[-\frac{z^2}{2c^2}\right], \tag{6}$$

where a, b, c, and k are positive constants, and where I have set both the mass of the particle and the value of $\hbar$ to one. We want the particle to start out with $x<0$; this corresponds to $x_1>0$. Strictly speaking, this wave packet does not satisfy the condition $Q(t=0)=0$, because its tail extends to positive values of x. However, $Q(0)$ can be made arbitrarily small by taking (x_1/a) large [see Eq. (9) below].

Define $\alpha=(a^2+it)^{1/2}$; $\beta=(b^2+it)^{1/2}$; $\gamma=(c^2+it)^{1/2}$. Then

$$\Psi(x,y,z,t) = \left[\frac{a^2 b^2 c^2}{\pi^3}\right]^{1/4} \frac{\exp[i(kx-k^2 t/2)]}{\alpha\beta\gamma} \times \exp\left[-\frac{(x+x_1-kt)^2}{2\alpha^2}\right] \exp\left[-\frac{y^2}{2\beta^2}\right] \times \exp\left[-\frac{z^2}{2\gamma^2}\right]. \tag{7}$$

At time t, the center of the wave packet is at $x=-x_1+kt$ (with $x_1>0$ and $k>0$), $y=z=0$. It is straightforward to calculate

$$Q(t) = \tfrac{1}{2}\,\mathrm{erfc}(\eta), \tag{8}$$

where $\eta := a(x_1-kt)/|\alpha|^2$, and erfc is the complementary error function: $\mathrm{erfc}(\eta) = 2\pi^{-1/2}\int_\eta^\infty \exp(-\xi^2)\,d\xi$. It follows from Eq. (8) that

$$Q(0) = \tfrac{1}{2}\,\mathrm{erfc}(x_1/a), \tag{9}$$

and that

$$\lim_{t\to\infty} Q(t) = \tfrac{1}{2}\,\mathrm{erfc}(-ak). \tag{10}$$

We see from Eq. (10) that Q does not approach one even as $t\to\infty$; this is because the wave packet spreads out as its center moves toward large x, so that a finite fraction of the tail of the packet remains in S_-.

In Bohmian theory [12,13] the particle is considered to have a definite position, which will be denoted as $\mathbf{r}=(X,Y,Z)$. Let $\mathbf{v}_b$ denote the Bohmian velocity of the particle (that is, $\mathbf{v}_b = d\mathbf{r}/dt$); then if we write $\Psi = R\exp(iS)$, in Bohmian theory $\mathbf{v}_b$ is given by

$$\mathbf{v}_b = \nabla S. \tag{11}$$

It then follows that

$$\nabla\cdot(|\Psi|^2 \mathbf{v}_b) = -\frac{\partial|\Psi|^2}{\partial t}. \tag{12}$$

In fact, the product $(|\Psi|^2\mathbf{v}_b)$ is just the usual quantum probability current, and Eq. (12) is just the equation of conservation of probability in standard quantum theory. One associates with a given wave function Ψ an ensemble of particles, whose distribution agrees with the quantum probability density $|\Psi|^2$; Eq. (12) assures that this agreement, if it exists at the initial time, persists for all time, and this in turn means that Bohmian theory will reproduce all of the experimental predictions of standard quantum theory [18]. Thus Bohmian theory is not in conflict with, but rather is a completion of, standard quantum theory.

Since each particle in the Bohmian ensemble follows a definite trajectory, the interpretation of arrival-time distributions is unambiguous. The quantity $P(t)$ defined above is simply the fraction of particles in the ensemble which have $X \geqslant 0$ for any time t' with $0 \leqslant t' \leqslant t$, and of course $Q(t)$ is the fraction which have $X \geqslant 0$ at time t. For the wave function in Eq. (7), the components of the Bohmian velocity turn out to be

$$v_{bx}(X,Y,Z,t) = [k+(X+x_1)t]/|\alpha|^4, \tag{13}$$

$$v_{by}(X,Y,Z,t) = Yt/|\beta|^4, \tag{14}$$

$$v_{bz}(X,Y,Z,t) = Zt/|\gamma|^4. \tag{15}$$

Since $x_1>0$, we see from Eq. (13) that, for all $t \geqslant 0$,

$$v_{bx}(X=0,Y,Z,t) > 0. \tag{16}$$

This means that if the particle does enter S_+, it can never leave. Since that was the condition which gives equality in the relation (4), we see that in this example, Bohmian theory gives

$$P(t) = Q(t), \tag{17}$$

where $Q(t)$ is given by Eq. (8).

Because of the factorized form of Ψ in this example, the x component of the Bohmian motion is the same as in the one-dimensional example of a minimum-uncertainty packet studied in [14]. In fact we can, without having to solve for the Bohmian trajectories in detail, recover one of the main results of [14], namely, that a finite fraction of the Bohmian ensemble never makes it to the region S_+. That fraction is just $(1-P_\infty)$; by Eq. (17) this equals $[1-\lim_{t\to\infty}Q(t)]$, which we saw in Eq. (10) is not zero.

It is Eq. (12) which ensures, for Bohmian theory, that an ensemble of particles with initial distribution given by $|\Psi|^2$ reproduces the experimental predictions of standard quantum theory. One can formulate an alternative theory, which I will refer to as a Bohm-like theory, in which a particle again has a definite position, but in which the velocity (call it $\mathbf{v}_{bl}$) may differ from the Bohmian velocity $\mathbf{v}_b$ [given in Eq. (11)]. Let $\delta\mathbf{v}$ denote the difference between $\mathbf{v}_{bl}$ and $\mathbf{v}_b$:

$$\mathbf{v}_{bl}=\mathbf{v}_b+\delta\mathbf{v}. \tag{18}$$

Then in order for this Bohm-like theory to agree with standard quantum theory in the same sense that Bohmian theory does, one must require that $\mathbf{v}_{bl}$ also satisfy Eq. (12); that is, one must require

$$\boldsymbol{\nabla}\cdot(|\Psi|^2\delta\mathbf{v})=0. \tag{19}$$

Deotto and Ghirardi [15] have shown that it is possible to choose $\mathbf{v}_{bl}$ in such a way as to satisfy several requirements that one may reasonably expect, in particular what they call ''genuine'' Galilean covariance. I will consider a simplified form of the theory suggested in [15]; I will take

$$\delta\mathbf{v}=\lambda(\boldsymbol{\nabla}|\Psi|^2)\times\mathbf{v}_b, \tag{20}$$

where λ is a constant and $\mathbf{v}_b$ is still given by Eq. (11); $\mathbf{v}_{bl}$ is then given by Eq. (18). This $\mathbf{v}_{bl}$ will certainly not satisfy all of the requirements imposed in [15]; I will argue in the next section that this makes this simple example of a Bohm-like theory implausible, but not demonstrably incorrect. For now, I will proceed to discuss the consequences of the choice (20). This choice does at least satisfy Eq. (19); to see that, note that $\mathbf{v}_b=\boldsymbol{\nabla}S$ and that $\boldsymbol{\nabla}\cdot(|\Psi|^2\boldsymbol{\nabla}|\Psi|^2\times\boldsymbol{\nabla}S)$ vanishes identically for any $|\Psi|^2$ and any S.

It is possible to discuss the distribution $P(t)$ that this Bohm-like theory will imply for the example given by Eq. (7) without having to find the trajectories explicitly. If it were the case that the x component of $\mathbf{v}_{bl}$ were positive everywhere on the plane S_0 for all times $t\geq 0$, we could conclude that $P(t)=Q(t)$, just as we did in Eq. (17) for the standard Bohmian theory. As we shall see below, if this condition on the x component of $\mathbf{v}_{bl}$ is *not* satisfied, then this Bohm-like theory will necessarily imply a *different* distribution $P(t)$ than does standard Bohmian theory. From Eqs. (7), (14), (15), and (20), the x component of $\delta\mathbf{v}$ at $X=0$ is

$$\delta v_x(0,Y,Z,t)=2\lambda|\Psi(0,Y,Z,t)|^2(c^2-b^2)YZt/(|\beta|^4|\gamma|^4), \tag{21}$$

while from Eq. (13),

$$v_{bx}(0,Y,Z,t)=(k+x_1t)/|\alpha|^4. \tag{22}$$

Let me now take $\lambda>0$ and $(c^2-b^2)<0$. Then in the two quadrants of the plane S_0 with the product YZ negative, δv_x will be positive, and since v_{bx} is positive everywhere on S_0, we will have $v_{blx}(=v_{bx}+\delta v_x)>0$. On the other hand, in the quadrants with YZ positive, δv_x is negative, and so v_{blx} will be positive if and only if $|\delta v_x|\leq v_{bx}$. For a fixed value of t, the maximum value of $|\delta v_x(0,Y,Z,t)|$ occurs at the points $Y=\pm|\beta|^2/(\sqrt{2}b)$, $Z=\pm|\gamma|^2/(\sqrt{2}c)$. This maximum value is

$$|\delta v_x|_{\max}=\lambda\,\frac{a(b^2-c^2)t}{\pi^{3/2}e|\alpha|^2|\beta|^4|\gamma|^4}\exp\left[\frac{-a^2(x_1-kt)^2}{|\alpha|^4}\right]. \tag{23}$$

From Eq. (23), $|\delta v_x|_{\max}$ is zero at $t=0$ and is proportional to t^{-4} as $t\to\infty$, while from Eq. (22), v_{bx} is nonzero at $t=0$ and is proportional to t^{-1} as $t\to\infty$. Therefore it is possible to have a value of λ sufficiently small so that $|\delta v_x|_{\max}<v_{bx}$ for all times $t\geq 0$. In that case, v_{blx} would be positive everywhere on S_0 for all $t\geq 0$, and so the values of $P(t)$ in this Bohm-like theory and in the standard Bohmian theory would agree.

Now let me take λ to be sufficiently large so that $|\delta v_x|_{\max}>v_{bx}$ for some time $t>0$. This means that, for some values of Y, Z, and t, $v_{blx}(0,Y,Z,t)<0$, which implies that some members of the Bohmian ensemble are returning from S_+ to S_-. Let t_r be within an interval of time in which this return is occurring. At any time t, the fraction of the ensemble in S_+ equals $Q(t)$, but at t_r there is an additional fraction of ''returned'' members, which are in S_- at t_r but were in S_+ at some time prior to t_r. This means that $P(t_r)$ (which is the total fraction of ensemble members that were in S_+ at *any* time $t'\leq t$) must be greater than $Q(t_r)$.

To be certain of this conclusion, we must show that, of the ensemble members which returned from S_+ prior to t_r, at least a finite fraction still are in S_- at t_r. Let D be an open, bounded region of the plane S_0, such that at every point of D and for an interval of time around t_r, v_{blx} is negative; such a region must exist, if λ is sufficiently large. For sufficiently small ϵ, it must be possible to find a subset $D_\epsilon\subset D$ such that the distance between any point in D_ϵ and any point on S_0 not in D is at least ϵ. Now it can be shown that, for Y and Z bounded, the magnitude of the component of $\mathbf{v}_{bl}$ parallel to S_0 is bounded, independently of X and t; call such a bound $|v_\parallel|_{\max}$. Thus any member of the ensemble which returns to S_- through D_ϵ must spend at least an amount of time $\tau=\epsilon/|v_\parallel|_{\max}$ in S_- (because it takes at least time τ for it to clear the region D). Thus all members of the ensemble which return to S_- through D_ϵ in the time interval $[t_r-\tau,t_r]$ will still be in S_- at time t_r.

We therefore see that, with the wave function as given in Eq. (7), the Bohm-like theory defined by Eq. (20) with a sufficiently large value of λ will imply that $P(t)>Q(t)$, for some values of t. Since with this wave function the standard Bohmian theory gives $P(t)=Q(t)$ for all t, we conclude that these two theories can give different arrival-time distributions $P(t)$.

III. DISCUSSION

The choice for $\delta\mathbf{v}$ made in Eq. (20) does not respect many of the conditions set out by Deotto and Ghirardi [15]. For example, the cross product of two vectors is a pseudovector, although a velocity must of course be a true vector. To take this choice seriously, one would have to say that Eq. (20) is only valid in a particular coordinate system; if you want to know $\delta\mathbf{v}$ in some other coordinate system, use Eq. (20) to calculate it in the particular system, and then transform. Deotto and Ghirardi require that there not be any preferred coordinate system; while this requirement is certainly quite reasonable, it is not, strictly speaking, necessary. As long as the Bohm-like theory reproduces the observational consequences of standard quantum theory, the preferred coordinate system remains hidden; its existence can be neither confirmed nor refuted by any experimental result.

Still, Deotto and Ghirardi, and in a different way Holland [16], have shown that it is possible to formulate a Bohm-like theory which is considerably more plausible than the one defined by Eq. (20). It is certainly an important question for the program of studying Bohmian theories, to judge which of the possible alternatives for the particle velocity is the most plausible [and the alternative given by Eq. (20) is surely *not* the most plausible]. One could certainly criticize the calculations presented here, because of the implausibility of the choice (20) or for that matter because the condition $Q(0)=0$ is not strictly satisfied. The calculation presented here does have the virtue of simplicity, and it is hard to believe that the result obtained [that $P(t)$ differs from that implied by the standard Bohmian theory] is an artifact either of the transformation properties of Eq. (20) or of the (arbitrarily small) tail of the initial wave function. Rather, this result gives one confidence to conjecture that for *any* Bohm-like theory (with nontrivial $\delta\mathbf{v}$) there exists an example of a wave function with $Q(0)=0$ exactly, for which that theory and standard Bohmian theory yield different $P(t)$.

The example presented here does not imply any additional ambiguity within the Bohmian program, beyond that already recognized in [15,16]. It is obvious that, when theories make different choices for $\delta\mathbf{v}$, there will be some quantities for which those theories will imply different results. What this example does show is that such theories will differ on a quantity, namely, the distribution of arrival times, that one might have hoped would be definable strictly in terms of the wave function, and so would be independent of any particular completion of standard quantum theory.

Of course, if one asks about the results of a particular experiment designed to measure times of arrival, quantum theory should be able to give an unambiguous answer, and Bohmian theory as well as any Bohm-like theory should agree with that answer. The issue we are considering is whether that answer can be stated, within standard quantum theory, in a way which is independent of the particular way in which the arrival times are to be measured. In standard quantum theory, no result is meaningful unless it is measured; the quantity $Q(t)$ defined in Eq. (3) must be interpreted as the probability that the particle be found in S_+ at time t, rather than the probability that it *is* there. Nevertheless, we do not have to consider the particular way in which the particle's position is measured; in terms of Bohm-like theories, we can say that they all must agree on the quantity $Q(t)$. One might have thought that $P(t)$ would enjoy the same status; after all, $P(t)$ is, roughly speaking, like the conjunction of $Q(t')$ for $0\leq t'\leq t$. Unfortunately, a determination of position at one time will disturb the determination at any other time, and different Bohm-like theories, while constrained to have identical ensembles of positions at any one time, differ precisely because they have different trajectories.

The ambiguity in the time-of-arrival distribution revealed by the example discussed here would not be present in a one-dimensional example. The analog of Eq. (19) for one dimension, namely, $\partial(|\Psi|^2\delta v)/\partial x=0$, together with suitable boundary conditions, would require $\delta v=0$. One would expect that general arguments for or against the definability of the arrival-time distribution, such as those quoted at the beginning of this paper, would be equally cogent in one and in three dimensions. It does seem, however, that there is an ambiguity in three dimensions which is not apparent if one only considers one-dimensional examples. Perhaps one dimension is misleadingly simple; one may be tempted by the isomorphism between configuration space and temporal space to ignore the special role played by time in nonrelativistic quantum mechanics.

The discussion above has been within the context of Bohmian and Bohm-like theories, but many of the same points can be discussed independently of the Bohmian ontology, by considering the quantum probability current (to be denoted $\mathbf{J}$). As Squires has pointed out [19], the freedom to choose alternative expressions for the velocity in Bohm-like theories is a direct reflection of the underdetermination of the quantum probability current in more than one spatial dimension. Mielnik [3] suggested that a reasonable first guess for a time-of-arrival density would be the component of $\mathbf{J}$ normal to the arrival surface; in our case this would mean identifying $J_x(x=0,y,z,t)$ as the probability for arriving at the point $(0,y,z)$ at time t. Mielnik then went on to show that this could not be correct in general, since there must exist examples in which this component becomes negative. It is sometimes suggested (for example, in [20]) that J_x does indeed give the correct arrival-time density, in those cases in which it is always positive.

Let $\mathbf{J}_c$ denote the customary form for the quantum probability current [which is just the product of $|\Psi|^2$ with $\mathbf{v}_b$ which is given in Eq. (11)]; then without now identifying $\mathbf{v}_b$ as the velocity of anything, we can see from Eq. (16) that for the wave function given in Eq. (7), J_{cx} is indeed positive everywhere on the plane S_0. So, if we follow the above suggestion, we would say that $J_{cx}(x=0,y,z,t)$ does indeed give the arrival-time density for this wave function.

Now define $\mathbf{J}_l$ to be $(|\Psi|^2\mathbf{v}_{bl})$, where $\mathbf{v}_{bl}$ is given by Eqs. (18) and (20) (and also need not be identified as the velocity of anything). Then from Eqs. (12) and (19) it follows that

$$\nabla\cdot\mathbf{J}_l=-\frac{\partial|\Psi|^2}{\partial t}, \tag{24}$$

which means that we can, if we wish, violate custom and call $\mathbf{J}_l$ (instead of $\mathbf{J}_c$) the quantum probability current. So we might as well say that J_{lx} gives the arrival-time density, in those cases in which J_{lx} is always positive.

The calculations of the preceding section show that, for the wave function given in Eq. (7), if λ happens to be small enough, then J_{lx} is positive everywhere on S_0. So, for small enough λ, the two possibilities for the probability current ($\mathbf{J}_c$ and $\mathbf{J}_l$) give us two possibilities for the arrival-time density (J_{cx} and J_{lx}) which disagree with each other [21]. One certainly can make an arbitrary choice between $\mathbf{J}_c$ and $\mathbf{J}_l$, that is, one can pick either one of them and choose to define that one to be the probability current. That choice, however, does not have any experimental implications—no experimental result can depend upon which definition one happens to make—so it would not make sense to expect the choice to be either confirmed or refuted by any experiment. For larger values of λ, we have seen that the Bohm-like theory defined by Eq. (20) gives a different distribution $P(t)$ than does the standard Bohm theory; still, we would not expect that any experiment could discriminate between these two theories, since they are constructed (more precisely, their multiparticle generalizations are constructed) to agree on all observational results. Similarly, in discussing time of arrival independently of the Bohmian ontology, one certainly can (and sometimes one does [7]) identify some quantity which can be calculated purely in terms of the wave function, and choose to call that quantity a time-of-arrival distribution. One can then discuss the question of whether this quantity does have [7] or does not have [22] properties that one might intuitively expect such a distribution to have. However, it might be that someone else would prefer a different choice, which would nevertheless lead to identical predictions for actually measured distributions, so that experiment could not discriminate between these different choices.

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