Dynamics of the quantized radiation field in a cavity vibrating at the fundamental frequency

Ying Wu,[1,2,3] M.-C. Chu,[1] and P. T. Leung[1]
[1]Department of Physics, The Chinese University of Hong Kong, Hong Kong
[2]Physics Department, Huazhong University of Science and Technology, Wuhan 430074, People's Republic of China
[3]Laboratory of Magnetic Resonance and Atomic and Molecular Physics, Wuhan Institute of Physics and Mathematics, The Chinese Academy of Sciences, Wuhan 430071, People's Republic of China

(Received 22 May 1998; revised manuscript received 13 October 1998)

We present a method to solve exactly a one-dimensional model of quantized radiations in a cavity oscillating in the fundamental resonance, using the effective Hamiltonian derived by C. K. Law [Phys. Rev. A **49**, 433 (1994)]. With this method, we derived explicit analytical expressions for the diagonalized Hamiltonian, the time-varying annihilation, creation, and photon number operators for the radiation field, which completely specify the dynamics of the system. [S1050-2947(99)00604-6]

PACS number(s): 42.50.Dv, 42.50.Lc, 03.65.−w

I. INTRODUCTION

Since Moore's pioneering work in 1970 [1], there have been intensive studies focused on the quantum theory of the electromagnetic field in a cavity with moving boundaries [2–8]. The topic is of fundamental theoretical interest in that it reveals a number of delicate features of quantum physics such as the dynamical modification of the Casimir force [3] and the vacuum emission of photons with nonclassical photon statistics [4–7]. On the other hand, the subject is also of practical importance since it is closely related to sonoluminescence [8], high precision optical interferometry [9], the generation of squeezed light [10], and quantum nondemolition measurements [11], etc.

The dynamics of the electromagnetic field in a cavity with time-varying boundaries can be studied by constructing and solving an effective Hamiltonian for the system [6,7], which allows for a Schrödinger-picture description and provides a convenient basis for investigating the physics of the cavity field. Unfortunately, the derived Hamiltonians are usually too complicated to allow one to obtain an explicit analytical form of the state of the field. Although some progress has been made in applying perturbation theory techniques to study the small-oscillation-amplitude regime [5,12], to our knowledge, no one has succeeded in solving exactly any one of the models described by previously derived effective Hamiltonians in the resonance cases. This greatly hinders a general understanding of the system, especially since perturbation theory is expected to break down for a system on resonance at long time or for large oscillation amplitudes [5,12], when interesting and nontrivial physics show up. Exact solutions of the effective Hamiltonians will also greatly facilitate the investigation of both the field statistics and the resonant emission and absorption of photons by an atom placed in an oscillating cavity. It is therefore important and desirable to develop a method to solve the effective Hamiltonians systematically and nonperturbatively.

We make a first step towards this goal in this paper by presenting an exact solution to the effective Hamiltonian of the radiation fields in a cavity driven to oscillate in the fundamental resonance. The corresponding dynamics by explicitly obtaining analytical expressions of the diagonalized Hamiltonian, the time-varying annihilation, creation, and photon number operators for the radiation field. The exact analytical solution to this model provides a very convenient basis for studying the photon statistics as well as resonant photon emission and absorption properties of an atom placed in such an oscillating cavity. Besides, the method presented here will be helpful in solving other models for higher resonances.

This paper is organized as follows. In Sec. II, we briefly describe the effective Hamiltonian formalism by Law [6] for the quantized radiation modes in a a one-dimensional cavity oscillating in resonances with particular emphasis paid on the rotating wave approximation. In Sec. III, we develop a method to exactly diagonalize the effective Hamiltonian when the cavity boundary oscillates in the fundamental resonance. In Sec. IV, we investigate the corresponding dynamics by obtaining explicitly the exact analytical expressions of the time-varying annihilation, creation, and photon number operators, and Sec. V concludes the paper with some discussions.

II. HAMILTONIAN FORMALISM FOR AN OSCILLATING CAVITY

In investigating the field quantization and the effective Hamiltonian formalism, Law [6] considered a one-dimensional cavity formed by two perfectly reflecting mirrors with one of the mirrors fixed at the position $x=0$ and the other moving in a prescribed trajectory $x=q(t)$. Expanding the vector potential and its conjugate momentum in a set of "instantaneous" mode functions $\{\phi_k(x;t)\}$,

$$\hat{A}(x,t)=\sum_k \hat{Q}_k(t)\phi_k(x;t),$$

$$\hat{\pi}(x,t)=\epsilon(x,t)\sum_k \hat{P}_k(t)\phi_k(x;t), \tag{1}$$

where $\epsilon(x,t)$ is the dielectric constant, and $\phi_k(x;t)$ are solutions of the wave equation subjected to the boundary conditions

$$\phi_k(0;t)=\phi_k(q(t);t)=0,$$

the system can be quantized by imposing the appropriate commutation relations on the generalized position and momentum operators $\hat{Q}_k$ and $\hat{P}_k$. The equations of motion for $\hat{Q}_k$ and $\hat{P}_k$ can be derived from the wave equation, and the effective Hamiltonian can then be constructed accordingly. The final form of the effective Hamiltonian is given in terms of the ''instantaneous'' creation and annihilation operators $a_k^\dagger$, a_k, constructed from the appropriate linear combinations of $\hat{Q}_k$ and $\hat{P}_k$. In the case of $\epsilon(x,t)=1$, the effective Hamiltonian reads [6]

$$H=\sum_k \omega_k(t)a_k^\dagger a_k+i\sum_k \frac{\dot{q}(t)}{4q(t)}(a_k^{\dagger 2}-a_k^2)+\frac{i}{2}\sum_{j,k} g_{jk}\frac{\dot{q}(t)}{q(t)}(a_k^\dagger a_j^\dagger+a_k^\dagger a_j-a_j a_k-a_j^\dagger a_k), \quad (2)$$

where $g_{jk}=(-1)^{j+k}k\sqrt{kj}/(j^2-k^2)$ as $k\neq j$, $g_{jk}=0$ for $j=k$, $\dot{q}(t)=dq(t)/dt$, and $\omega_k(t)=k\pi/q(t)$. Under the rotating-wave approximation (RWA), Law explicitly wrote down the ''interaction'' part of the resonant effective Hamiltonians [i.e., the part derived from Eq. (2) in the absence of its first summation term $\Sigma_k\omega_k(t)a_k^\dagger a_k$] for a particular choice of the prescribed trajectory,

$$q(t)=L\exp[q_0\cos(\Omega t)/L],$$

and for $\Omega=m\pi/L$, $m=1,2,3$ where q_0 and Ω characterize, respectively, the amplitude and frequency of the oscillation around a natural cavity length L. Here we supply the expression for the ''free'' part of the resonant effective Hamiltonians under the RWA. It reads $\omega I_0(q_0/L)\Sigma_k k a_k^\dagger a_k$, where $\omega=\pi/L$, and $I_0(q_0/L)$ is the modified Bessel function of order zero derived from the relation $I_0(q_0/L)=\int_0^{2L}\exp[-q_0\cos(\Omega t)/L]dt/(2L)$ with $\Omega=m\pi/L$. Let us prove this result. In order to get the resonant effective Hamiltonian by applying the RWA to the Hamiltonian in Eq. (2), one needs to expand formally all c-number functions in Eq. (2) into Fourier series. Note that $q(t)$ enters the last two summations of Eq. (2) only in the form $\dot{q}(t)/q(t)$, which is already a purely sinusoidal form and equals $\dot{q}(t)/q(t)=-q_0\Omega\sin(\Omega t)/L=i\Omega(q_0/2L)[\exp(i\Omega t)-\exp(-i\Omega t)]$. This is the reason why Law made the above particular choice of $q(t)$. The quantity $\omega_k(t)=k\pi/q(t)\equiv k\omega\exp[-q_0\cos(\Omega t)/L]$ appears only in the first summation in Eq. (2). It can be expressed as the Fourier series, as $\Omega=m\omega$, $\omega=\pi/L$,

$$\omega_k(t)=k\omega\Sigma_{n=-\infty}^{n=\infty}B_n\exp(-inm\omega t),$$

where $B_0=\int_0^{2L}\exp[-q_0\cos(\Omega t)/L]dt/(2L)\equiv I_0(q_0/L)$ with $\Omega=m\pi/L$, and the other expansion coefficients can also be explicitly expressed. The RWA is to keep only those terms in Eq. (2) that are on resonance. This can be done as follows: substituting the above Fourier series of $\omega_k(t)$ and $\dot{q}(t)/q(t)$, as well as the transformation $a_k\to a_k\exp(-ik\omega t)$; $a_k^\dagger\to a_k^\dagger\exp(ik\omega t)$, $k=1,2,3,\ldots$ into Eq. (2), one then obtains the resonant effective Hamiltonians under the RWA by neglecting all the terms which still have the factor belonging to the fast time-varying set $\exp(\pm in\omega t)$, $n=1,2,3,\ldots$, and keeping only all terms that are time independent and/or slowly varying compared with the fast-varying terms $\exp(\pm i\omega t)$. In this way, it is easy to show that the resulted Hamiltonians (as $\Omega=m\omega$, $m=1,2,3,\ldots$) are $H_{RWA}=\omega I_0(q_0/L)\Sigma_k k a_k^\dagger a_k+H_{eff}^{(m)}$, $m=1,2,3,\ldots$ with $H_{int}^{(m)}$ $(m=1,2,3)$ identical to Law's Eqs. (3.4)–(3.6), respectively [6]. Law has claimed that the complicated form of the scattering terms in all three cases $(m=1,2,3)$ forbade one from finding the analytic solutions [6] while we shall present a method to solve exactly the effective model for the fundamental resonance case $(m=1)$ which, as discussed above, reads

$$H=\omega f\sum_{k=1}^{\infty} k a_k^\dagger a_k+\frac{q_0\omega}{4L}\sum_{k=1}^{\infty}\sqrt{k(k+1)}[a_k^\dagger a_{k+1}+a_{k+1}^\dagger a_k], \quad (3)$$

where $f\equiv I_0(q_0/L)=\int_0^{2L}\exp[-q_0\cos(\omega t)/L]dt/(2L)$, and I_0 is the modified Bessel function of order zero. In writing this equation, we have ultilized the simplified expression $f_\alpha(k)=\sqrt{k(k+\alpha)}$ for the function $f_\alpha(k)=k(k+\alpha)(2k+\alpha)^{-1}[\sqrt{(k+\alpha)/k}+\sqrt{k/(k+\alpha)}]$ in Eq. (3.7) of Ref. [6]. Let $x=\sqrt{(k+\alpha)/k}$, one sees that $x+x^{-1}=k(x^2+1)/(kx)=(2k+\alpha)/\sqrt{(k+\alpha)k}$ which immediately leads to the simplified relation.

III. METHOD TO DIAGONALIZE THE HAMILTONIAN

In order to diagonalize the effective Hamiltonian in the fundamental resonance case, we first introduce a fictitious harmonic oscillator described by the annihilation and creation operators A and $A^\dagger$ as well as the corresponding number operator $N=A^\dagger A$. The operators A and $A^\dagger$ satisfy the usual commutation relation $[A,A^\dagger]=1$, and they commute with all the operators a_k and $a_k^\dagger$ of the radiation field. Let $|n\rangle$, $n=0,1,2,\ldots$ denote the eigenkets of the number operator $N=A^\dagger A$ (not to be confused with the eigenkets of the photon number operators $a_k^\dagger a_k$), and use the relations $A|n\rangle=\sqrt{n}|n-1\rangle$ and $A^\dagger|n\rangle=\sqrt{n+1}|n+1\rangle$, we can rewrite the effective Hamiltonian in Eq. (3) as follows:

$$H/f=\omega\sum_{n,m=1}^{\infty} a_n^\dagger\langle n|\left[N+\frac{\overline{q_0}}{4L}(\sqrt{N}A+A^\dagger\sqrt{N})\right]|m\rangle a_m, \quad (4)$$

where $\overline{q_0}\equiv q_0/f$. This is the key step towards the diagonalization of the effective Hamiltonian, because it is then straightforward to see from this equation that we have turned this problem into the diagonalization of the operator $N+\overline{q_0}(\sqrt{N}A+A^\dagger\sqrt{N})/4L$, which can be put into another form,

$$N+\frac{\overline{q_0}}{4L}(\sqrt{N}A+A^\dagger\sqrt{N})=\sqrt{1-\left(\frac{\overline{q_0}}{2L}\right)^2}e^{\theta(\sqrt{N}A-A^\dagger\sqrt{N})}Ne^{-\theta(\sqrt{N}A-A^\dagger\sqrt{N})},\tag{5}$$

where $2\theta\equiv\tanh^{-1}(\overline{q_0}/2L)$. The proof of this relation is simple and is given in Appendix A. It suggests an appropriate transformation for diagonalizing the effective Hamiltonian.

We introduce a set of photonic operators $b_n,b_n^\dagger,n=1,2,\ldots$ for the radiation field by the following unitary transformation:

$$b_n=\sum_{m=1}^{\infty}U_{nm}(-\theta)a_m,\tag{6a}$$

$$a_n=\sum_{m=1}^{\infty}U_{nm}(\theta)b_m,\tag{6b}$$

where the unitary transformation operator $U(\theta)\equiv\exp[\theta(\sqrt{N}A-A^\dagger\sqrt{N})]$ and its matrix elements are defined by $U_{nm}(\theta)\equiv\langle n|\exp[\theta(\sqrt{N}A-A^\dagger\sqrt{N})]|m\rangle$. Note that the unitary operator U satisfies $U^{-1}(\theta)=U^\dagger(\theta)=U(-\theta)$, and hence its matrix elements satisfy $U^*{}_{mn}(\theta)=U_{nm}^\dagger(\theta)=U_{nm}^{-1}(\theta)=U_{nm}(-\theta)$. From the definition of the operators b_n and $b_n^\dagger$, one easily finds that they satisfy the relations $[b_n,b_m]=[b_n^\dagger,b_m^\dagger]=0$, and $[b_n,b_m^\dagger]=\delta_{nm}$ by utilizing the counterparts for the operators $a_n,a_n^\dagger$ and the properties of the unitary operator U. We have in Appendix A calculated the explicit expression of all the matrix elements of the unitary operator U in several different forms. One of these forms reads $U_{nm}(\theta)=0$ for $n=0$ or $m=0$, and

$$U_{nm}(\theta)=(-1)^{m-1}\sqrt{mn}\sum_{k=0}^{\min(n,m)}\times\frac{(n+m-k-1)!}{k!(m-k)!(n-k)!}(\tanh\theta)^{3n+m-2-2k},\tag{7}$$

for $n,m=1,2,\ldots$. Here,

$$\tanh\theta=(\overline{q_0}/2L)/[1+\sqrt{1-(\overline{q_0}/2L)^2}]$$

if one takes $\tanh 2\theta=\overline{q_0}/2L$.

Substituting Eq. (5) into Eq. (4) and using the completeness relation $\Sigma_{k=0}^{\infty}|k\rangle\langle k|=1$ as well as $U_{n0}(\theta)=U_{0l}(-\theta)=0$, $U^*{}_{nk}(\theta)=U_{kn}(-\theta)$, we after a little manipulation arrive at

$$H/f=\omega\sqrt{1-\left(\frac{\overline{q_0}}{2L}\right)^2}\sum_{k,l=1}^{\infty}\left[\sum_{n=1}^{\infty}a_n^\dagger U_{nk}(\theta)\right]\langle k|N|l\rangle\times\left[\sum_{m=1}^{\infty}a_mU_{lm}(-\theta)\right]$$
$$=\omega\sqrt{1-\left(\frac{\overline{q_0}}{2L}\right)^2}\sum_{k,l=1}^{\infty}\left[\sum_{n=1}^{\infty}a_nU_{kn}(-\theta)\right]^\dagger\langle k|N|l\rangle\times\left[\sum_{m=1}^{\infty}a_mU_{lm}(-\theta)\right].\tag{8}$$

It is then straightforward from Eqs. (6) and (8) and $\langle k|N|l\rangle=k\delta_{kl}$ to obtain the diagonal form of the effective Hamiltonian in the fundamental resonance case as follows:

$$H=\frac{\pi}{L}\sqrt{f^2-\left(\frac{q_0}{2L}\right)^2}\sum_{k=1}^{\infty}kb_k^\dagger b_k,\tag{9}$$

where $f=I_0(q_0/L)$, and I_0 is the modified Bessel function of order zero. This equation, together with the photonic operators b_k determined by Eqs. (6) and (7), is the central result of this section.

We can easily obtain the eigenvalues and eigenkets of the effective Hamiltonian, Eq. (3), in the fundamental resonance case since we have already diagonalized it. Furthermore, we can from the results of this section explicitly study the corresponding dynamics of the system, which will be discussed in the next section. The operators $a_k,a_k^\dagger$ describe the bare photons of the radiation field while $b_k,b_k^\dagger$ describe in some sense the corresponding dressed photons (i.e., the bare photons dressed by the mirror oscillations). Equation (3) indicates that the mirror oscillation causes strong intermode couplings among bare photons, but there exists no interaction among the dressed photons of different modes as governed by Eq. (9). It is also interesting to note that the diagonalized Hamiltonian is identical in form to the one describing the radiation field in a cavity with an effective instantaneous length $q(t)/\sqrt{1-(\overline{q_0}/2L)^2}$. In this sense, the oscillating mirror in the fundamental resonance case has the function of enlarging the effective cavity length and hence decreasing the corresponding eigenfrequencies of the radiation field within the cavity.

IV. DYNAMICS OF THE RADIATION FIELD

How the photon number operators as well as the annihilation and creation operators of the field evolve with time determines completely the dynamics of the radiation field. The time dependence of the dressed operators are easily obtained from the Heisenberg equation $db/dt=-i[b,H]$ and Eq. (9) as follows:

$$b_k(t)=b_ke^{-ik\psi(t)},\quad b_k^\dagger(t)=b_k^\dagger e^{ik\psi(t)},\tag{10}$$

where $\psi(t)\equiv[(\pi/L)\sqrt{f^2-(q_0/2L)^2}]t$, $f\equiv I_0(q_0/L)$, and I_0 is the modified Bessel function of order zero. Note that $n_k^{(b)}(t)\equiv b_k^\dagger(t)b_k(t)$ is constant in time, $n_k^{(b)}(t)=n_k^{(b)}(0)\equiv n_k^{(b)}$.

Our purpose in this section is to find explicitly how both the dressed and bare operators of the radiation field vary with respect to time for given initial bare operators $a_k^\dagger(\equiv a_k^\dagger(0))$ and $a_k(\equiv a_k(0))$. They are easy to obtain from Eqs. (6) and (10). The results are

$$b_k(t)=e^{-ik\psi(t)}\sum_{m=1}^{\infty} U_{km}(-\theta)a_m\,, \tag{11a}$$

$$n_k^{(b)}(t)=\sum_{m,n=1}^{\infty} U_{nk}(\theta)U_{km}(-\theta)a_n^\dagger a_m\,, \tag{11b}$$

$$a_k(t)=\sum_{m=1}^{\infty} G_{km}(\theta,t)a_m\,, \tag{12a}$$

$$n_k^{(a)}(t)\equiv a_k^\dagger(t)a_k(t)=\sum_{m,n=1}^{\infty} G^*{}_{nk}(\theta,t)G_{km}(\theta,t)a_n^\dagger a_m\,, \tag{12b}$$

where

$$G_{km}(\theta,t)=\sum_{n=1}^{\infty} U_{kn}(\theta)U_{nm}(-\theta)\exp[-in\psi(t)].$$

In this form, G_{km} is rather complicated since it involves a triple summation after substituting the expression Eq. (7) for matrix elements U_{kn} and U_{mn}. Its simplification requires some skill and is done in Appendix B. Here we only list its simplified form as follows:

$$G_{km}(\theta,t)=\sqrt{mk}\left(\frac{1-iC}{1+iC}\right)^k\left(\frac{iS}{1+iC}\right)^{k+m-2}\times\sum_{n=0}^{\min(k,m)}(-1)^{m-n}\times\frac{(k+m-n-1)!}{n!(m-n)!(k-n)!}\left(\frac{S^2}{1+C^2}\right)^{k-n}, \tag{13}$$

where

$$S\equiv\sinh(2\theta)\tan\left[\frac{\psi(t)}{2}\right]=\frac{\overline{q}_0}{2L}\,\frac{\tan[\psi(t)/2]}{\sqrt{1-(\overline{q}_0/2L)^2}},$$
$$C\equiv\cosh(2\theta)\tan\left[\frac{\psi(t)}{2}\right]=\frac{\tan[\psi(t)/2]}{\sqrt{1-(\overline{q}_0/2L)^2}}. \tag{14}$$

where $\overline{q}_0=q_0/f$, $f=I_0(q_0/L)$, and $2\theta=\tanh^{-1}(\overline{q}_0/2L)$. We have now expressed exactly all the time-varying bare and dressed operators of the quantized field modes explicitly in terms of the initial bare operators of the field. These analytical expressions completely and explicitly describe the dynamics of the electromagnetic fields in an one-dimensional oscillating cavity in the fundamental resonance case. The time evolution of the statistics of the quantized field modes, such as the photon statistics of intermode and intramode correlations, can be obtained easily and explicitly from these expressions once the corresponding initial statistical properties are given.

It can be seen from Eq. (11) that both the bare and dressed sets of photonic operators share a common vacuum state, implying the well-known conclusion that no (bare and dressed) photons can be generated from the vacuum state in the fundamental resonance case [5,8]. This conclusion was rigorously proven previously only up to the first order of the small oscillating amplitude of the moving mirror, while ours is based on the exact solution of the resonant effective Hamiltonian under the rotating-wave approximation. We emphasize that the situation is quite different if the initial state of the field is not the vacuum state. Then, the oscillating mirror causes photon exchanges between different field modes, but the total photon number is conserved if there are no atoms in the cavity. These photon exchanges will significantly alter the transitions as well as photon emission and absorption of an atom placed in such an oscillating cavity, if the atom can resonately interact with some of the quantized field modes of the corresponding unperturbed cavity. Our results here provide a sound basis for such investigations in the fundamental resonance case.

V. CONCLUSIONS AND DISCUSSIONS

In summary, we have investigated the dynamics of a one-dimensional oscillating cavity in the fundamental resonance case by means of an effective resonant Hamiltonian derived by Law [6]. We have developed a method to solve this quantized model and obtained exact analytical expressions of the diagonalized Hamiltonian, the time-varying annihilation and creation as well as photon number operators for the radiation field.

The method presented here manifests its power in solving exactly the effective resonant Hamiltonian describing an oscillating cavity in the fundamental resonance case. It may also be useful in other harmonic resonance cases and hence may finally provide a way to solve analytically a class of such effective resonant Hamiltonians. The exact analytical expressions for the time-varying annihilation, creation and photon number operators for the quantized field modes give explicitly not only all the information of the dynamics of the field modes but also the time evolution of the statistical quantities of the field modes such as various kinds of intermode and intramode correlations in the fundamental resonance case. We have explicitly shown that for the particular choice of the trajectory $q(t)=L\exp[q_0\cos(\Omega t)/L]$ of the moving mirror, no (bare or dressed) photon can be generated out of the vacuum regardless of the oscillation amplitude so long as the rotating-wave approximation is a valid. These results can be utilized to study situations where the initial state of the field is not the vacuum state. They also provide a very convenient basis for studying the resonant photon emission and absorption of an atom placed in such an oscillating cavity. Our results may also be useful in studies of sonoluminescence, high precision optical interferometry, the generation of squeezed light, and quantum nondemolition measurements.

ACKNOWLEDGMENTS

This work was partially supported by the Hong Kong Research Grants Council under Grant No. CUHK 312/96P and a Chinese University Direct Grant (Project No. 2060093). Y.W. was also partially supported by the National Science Foundation of China under Grant Nos. 69688004 and 69788002, and the National Laboratory of MRAMP at Wuhan Institute of Physics and Mathematics, The Chinese Academy of Sciences.

APPENDIX A

In this appendix, we prove Eq. (4) and calculate the explicit forms of the matrix elements of the unitary operator U.

We begin with the proof of Eq. (4). Let $W_\pm=(\sqrt{N}A \pm A^\dagger\sqrt{N})$ and $\tilde{X}=\exp(\theta W_-)X\exp(-\theta W_-)$. We can easily show that $d\tilde{N}/d\theta=[W_-,\tilde{N}]=\tilde{W}_+$ and $d^2\tilde{N}/d\theta^2=[W_-,\tilde{W}_+]=4\tilde{N}$, which combine to give the relation $\tilde{N}=N\cosh 2\theta+\frac{1}{2}W_+\sinh 2\theta$, or $\tilde{N}\sqrt{1-\tanh^2(2\theta)}=N+\frac{1}{2}W_+\tanh 2\theta$. Taking $\tanh 2\theta=(\overline{q_0}/2L)$, we arrive at Eq. (4) and hence complete its proof.

We now calculate the explicit forms of the matrix elements $U_{nm}(\theta)=\langle n|\exp[\theta(\sqrt{N}A-A^\dagger\sqrt{N})]|m\rangle$. Using $A|m\rangle=\sqrt{m}|m-1\rangle$, $A^\dagger|m\rangle=\sqrt{m+1}|m+1\rangle$ and introducing

$$f_{nm}(\theta)\equiv\sqrt{m}U_{nm}(\theta), \tag{A1}$$

we find

$$\frac{d}{d\theta}f_{nm}=m(f_{nm-1}-f_{nm+1}), \tag{A2}$$

where $f_{n0}=f_{0m}=0$. From this equation and the definition

$$f_n(\theta,x)=\sum_{m=1}^{\infty} f_{nm}(\theta)x^m, \tag{A3}$$

one easily obtains $(f_n)'_\theta=x[(x-x^{-1})f_n]'_x$, where $(f)'_y$ denotes the partial derivative of f with respect to variable y. By defining

$$x\equiv\tanh\xi, \quad R_n\equiv(x-x^{-1})f_n, \tag{A4}$$

we arrive at $(R_n)'_\theta=-(R_n)'_\xi$, which is easily solved to give $R_n(\theta,x)=F_n(\xi-\theta)$. The functional form of F_n is determined by

$$R_n(\theta=0,x)=-\sqrt{n}x^{n-1}(1-x^2)=-\frac{1}{\sqrt{n}}\frac{d}{d\xi}\tanh^n\xi,$$

which is obtained by using $U_{nm}(\theta=0)=\delta_{nm}$ and Eqs. (A1), (A3), and (A4). It is then straightforward to obtain

$$f_n(\theta,x)=\frac{1}{\sqrt{n}}x\frac{d}{dx}\tanh^n(\tanh^{-1}x-\theta). \tag{A5}$$

After some manipulations, we obtain from Eqs. (A1) and (A3–A5) one of the two simple expressions of the needed matrix elements as follows:

$$U_{nm}(\theta)=\frac{1}{\sqrt{nm}(m-1)!}\left[\frac{d^m}{dx^m}\left(\frac{x-\tanh\theta}{1-x\tanh\theta}\right)^n\right]_{x=0}, \tag{A6}$$

where $n,m=1,2,3,\ldots$ and $U_{n0}(\theta)=U_{0m}(\theta)=0$. Using this equation and

$$\frac{1}{(1-x\tanh\theta)^n}=\frac{(\tanh\theta)^{n-1}}{(n-1)!}\frac{d^{n-1}}{dx^{n-1}}\frac{1}{1-x\tanh\theta},$$

it is then easy to obtain another form,

$$U_{nm}(\theta)=(-1)^{m-1}\sqrt{mn}\sum_{k=0}^{\min(n,m)}\times\frac{(n+m-k-1)!}{k!(m-k)!(n-k)!}(\tanh\theta)^{3n+m-2-2k}, \tag{A7}$$

which is the form in Eq. (7) in the main text.

APPENDIX B

We simplify the expression of $G_{km}(\theta,t)=\Sigma_{n=1}^{\infty}U_{kn}(\theta)U_{nm}(-\theta)\exp[-in\psi(t)]$ in this Appendix. In order to reach this goal, we rewrite it as

$$\begin{aligned}G_{km}(\theta,t)&=\sum_{n=1}^{\infty}\langle k|U(\theta)|n\rangle\langle n|U(-\theta)|m\rangle\exp[-in\psi(t)]\\&=\sum_{n=1}^{\infty}\langle k|U(\theta)\exp[-iN\psi(t)]|n\rangle\langle n|U(-\theta)|m\rangle\\&=\langle k|U(\theta)\exp[-iN\psi(t)]U(-\theta)|m\rangle\\&=\langle k|\exp[-i\tilde{N}\psi(t)]|m\rangle,\end{aligned} \tag{B1}$$

where $\tilde{N}=U(\theta)NU(-\theta)$, and uses have been made of the completeness relation $\Sigma_{n=0}^{\infty}|n\rangle\langle n|=1$, $U_{0m}(-\theta)$ and $U(-\theta)=U^{-1}(\theta)$. At the beginning of the Appendix A, we have shown

$$\tilde{N}=N\cosh 2\theta+\tfrac{1}{2}(\sqrt{N}A+A^\dagger\sqrt{N})\sinh 2\theta. \tag{B2}$$

Introducing

$$W_{km}(\tau)=\sqrt{m}\langle k|\exp(\tau\tilde{N})|m\rangle, \tag{B3}$$

where $\tau=-i\psi(t)$, and

$$W_n(\tau,x)=\sum_{m=1}^{\infty}W_{nm}(\tau)x^m, \tag{B4}$$

we can, following the same routine as the one for calculating U_{nm} in Appendix A, obtain

$$(W_n)'_\tau=x\left[\left(C_1+\frac{S_1}{2}(x+x^{-1})\right)W_n\right]'_x,$$

where $(W)'_y$ denotes the partial derivative of W with respect to variable y, $C_1=\cosh 2\theta$ and $S_1=\sinh 2\theta$. By defining

$$\tanh(\xi/2)\equiv S_1 x+C_1, \quad \overline{R}_n\equiv\left[C_1+\frac{S_1}{2}\left(x+\frac{1}{x}\right)\right]W_n, \tag{B5}$$

we arrive at $(\overline{R}_n)'_\tau=-(\overline{R}_n)'_\xi$, which is easily solved to give $\overline{R}_n(\tau,x)=\overline{F}_n(\xi-\tau)$. The functional form of $\overline{F}_n$ is determined by

$$\overline{R}_n(\tau=0,x)=-\frac{1}{\sqrt{n}}\frac{d}{d\xi}x^n,$$

which is obtained by using $W_{nm}(\tau=0)=\sqrt{n}\delta_{nm}$ and Eqs. (B3–B5). Noting that $W_n=-xR_n d\xi/dx$ and $x=[\tanh(\xi/2)-C_1]/S_1$, $\tau=-i\psi(t)$ and $\tanh(iy)=i\tan(y)$, it is then straightforward from Eqs. (B1,B3–B5) to obtain

$$G_{km}(\theta,t)=\frac{1}{\sqrt{km}(m-1)!}\left[\frac{d^m}{dx^m}\left(\frac{x(1-iC)-iS}{1+iC+iSx}\right)^k\right]_{x=0}, \tag{B6}$$

where $k,m=1,2,3,\ldots$, $G_{k0}(\theta,t)=G_{0m}(\theta,t)=0$, and

$$S=\sinh(2\theta)\tan\left[\frac{\psi(t)}{2}\right]=\frac{\overline{q}_0}{2L}\frac{\tan[\psi(t)/2]}{\sqrt{1-(\overline{q}_0/2L)^2}},$$

$$C=\cosh(2\theta)\tan\left[\frac{\psi(t)}{2}\right]=\frac{\tan[\psi(t)/2]}{\sqrt{1-(\overline{q}_0/2L)^2}}. \tag{B7}$$

Note that Eq. (B6) gives $G_{km}(\theta=0,t)=\delta_{km}\exp[-i\psi(t)m]$, $k,m=1,2,3,\ldots$, which is identical to the result directly calculated from its definition as it should be.

Equation (B6) can be put into another form,

$$G_{km}(\theta,t)=\left(\frac{1-iC}{1+iC}\right)^k\frac{1}{\sqrt{km}(m-1)!}\left[\frac{d^m}{dx^m}\left(\frac{x+gx}{1+gx}\right)^k\right]_{x=0}, \tag{B8}$$

where $g=iS/(1+iC)$. Using this equation and

$$\frac{1}{(1+gx)^k}=\frac{(-g)^{k-1}}{(k-1)!}\frac{d^{k-1}}{dx^{k-1}}\frac{1}{1+gx},$$

it is then easy to obtain

$$G_{km}(\theta,t)=\sqrt{mk}\left(\frac{1-iC}{1+iC}\right)^k\left(\frac{iS}{1+iC}\right)^{k+m-2}\times\sum_{n=0}^{\min(k,m)}(-1)^{m-n}\frac{(k+m-n-1)!}{n!(m-n)!(k-n)!}\times\left(\frac{S^2}{1+C^2}\right)^{k-n}, \tag{B9}$$

where S,C are given by Eq. (B7). Equations (B9) and (B7) are, respectively, Eqs. (13) and (14) in the main text.

[1] G. T. Moore, J. Math. Phys. **11**, 2679 (1970).

[2] P. W. Milonni, *The Quantum Vacuum* (Academic Press, Boston, 1994); N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, 1982).

[3] M. T. Jaekel and S. Reynaud, J. Phys. I **2**, 149 (1992); V. V. Dodonov, A. B. Klimov, and D. E. Nikonov, J. Math. Phys. **34**, 2742 (1993); C. K. Law, Phys. Rev. Lett. **73**, 1931 (1994).

[4] S. Sarkar, Quantum Opt. **4**, 345 (1992); S. A. Fulling and P. C. W. Davies, Proc. R. Soc. London, Ser. A **348**, 393 (1976); M. Castagnino and R. Ferraro, *ibid.* **154**, 1 (1984); G. Barton and A. Calogeracos, *ibid.* **238**, 227 (1995); A. Calogeracos and G. Barton, *ibid.* **238**, 268 (1995); E. Sassaroli, Y. N. Srivastava, and A. Widom, Phys. Rev. A **50**, 1027 (1994); G. M. Salamone and G. Barton, *ibid.* **51**, 3506 (1995); G. Calucci, J. Phys. A **25**, 3873 (1992); P. A. Maia Neto, *ibid.* **27**, 2167 (1995); L. S. Levitov, Europhys. Lett. **8**, 499 (1989); V. B. Braginsky and F. Ya. Khalili, Phys. Lett. A **161**, 197 (1991); V. V. Dodonov, A. B. Klimov, and V. I. Man'ko, *ibid.* **142**, 511 (1989); R. Jauregui and C. Villarreal, Phys. Rev. A **54**, 3480 (1996).

[5] V. V. Dodonov and A. B. Klimov, Phys. Rev. A **53**, 2664 (1996); Phys. Lett. A **167**, 309 (1992); *ibid.* **149**, 225 (1990).

[6] C. K. Law, Phys. Rev. A **49**, 433 (1994).

[7] M. Razavy and J. Terning, Phys. Rev. D **31**, 307 (1985); G. Barton and C. Eberlein, Ann. Phys. (N.Y.) **227**, 393 (1993); G. Calucci, J. Phys. A **25**, 3873 (1992); M. T. Jaekel and S. Reynaud, J. Phys. I **3**, 1 (1993); C. K. Law, Phys. Rev. A **51**, 2537 (1995).

[8] C. Eberlein, Phys. Rev. A **53**, 2772 (1996); J. Schwinger, Proc. Natl. Acad. Sci. USA **89**, 4091 (1992); **90**, 7285 (1993); **91**, 6473 (1994); B. P. Barber *et al.*, Phys. Rep. **281**, 65 (1997).

[9] C. M. Caves, Phys. Rev. Lett. **45**, 75 (1980); R. Loudon, *ibid.* **47**, 815 (1981); P. Meystre, E. M. Wright, J. D. McCullen, and E. Vignes, J. Opt. Soc. Am. B **2**, 1830 (1985). A. F. Pace, M. J. Collett, and D. F. Walls, Phys. Rev. A **47**, 3173 (1993).

[10] S. Mancini and P. Tombesi, Phys. Rev. A **49**, 4055 (1994); C. Fabre *et al.*, *ibid.* **49**, 1337 (1994).

[11] K. Jacobs, P. Tombesi, M. J. Collett, and D. F. Walls, Phys. Rev. A **49**, 1961 (1994).

[12] D. A. R. Dalvit and F. D. Mazzitelli, Phys. Rev. A **57**, 2113 (1998).