Exact quantum states of a general time-dependent quadratic system from classical action

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A generalization of a driven harmonic oscillator with time-dependent mass and frequency, by adding total time-derivative terms to the Lagrangian, is considered. The generalization, which gives a general quadratic Hamiltonian system, does not change the classical equation of motion. Based on the observation by Feynman and Hibbs, the propagators (kernels) of the systems are calculated from the classical action, in terms of solutions of the classical equation of motion: two homogeneous solutions and one particular solution. The kernels are then used to find wave functions that satisfy the Schrödinger equation. One of the wave functions is shown to be that of a Gaussian pure state. In every case considered, we prove that the kernel does not depend on the way of choosing the classical solutions, while the wave functions depend on the choice. The generalization, which gives a rather complicated quadratic Hamiltonian, is simply interpreted as applying unitary transformation to the driven harmonic oscillator system in the Hamiltonian formulation.
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I. INTRODUCTION

Time-dependent quantum-mechanical systems continue to be of great interest. In particular, the system described by an explicit time-dependent quadratic Hamiltonian has attracted considerable attention. One of the typical examples is the harmonic oscillator with time-dependent mass and/or frequencies. Those studies have many applications such as in quantum optics [1], Paul trap [2], and the analyses of quantum fields in curved space-time [3], and they are closely related to the theory of quantum dissipation [4].

In the Hamiltonian formulation of a time-dependent harmonic oscillator, Lewis and Riesenfeld (LR) [5,6] have shown that a quantum-mechanically invariant operator exists whose exact form is determined by an auxiliary function. The invariant operator can then be used to find exact wave functions of quantum states. During the past several years, this LR method has been widely used for the study of general quadratic Hamiltonian systems [7–11]. For the harmonic oscillator with time-dependent mass and frequency, the wave functions and the kernel (propagator) have been found [7,8,11]. Through the Heisenberg-picture approach, Ji *et al.* [9] have refined the derivation, and the wave functions are given. The Heisenberg-picture approach has then been used [10], with successive unitary transformations [2], to find exact wave functions of the general quadratic Hamiltonian system. The auxiliary functions in the LR method were defined through differential equations related to the equation of motion. We also note that the LR method can be applied to more general systems [12].

In this article, we will study the quadratic system in the Lagrangian formulation of Feynman and Hibbs [13]. The Lagrangian we will consider is

$$L=\frac{1}{2}M(t)\dot{x}^2-\frac{1}{2}M(t)w^2(t)x^2+F(t)x+\frac{d}{dt}[M(t)a(t)x^2]+\frac{d}{dt}[b(t)x]+f(t), \tag{1}$$

where $w^2(t)$, $F(t)$, $a(t)$, $b(t)$, and $f(t)$ are arbitrary real functions of t and $M(t)$ is also arbitrary real but always positive. The last three terms in the right hand side of Eq. (1) have no effect in classical dynamics and the classical trajectory $\bar{x}$ of the coordinate x will satisfy the equation

$$\frac{d}{dt}(M\dot{\bar{x}})+M(t)w^2(t)\bar{x}=F(t). \tag{2}$$

The most general solution of this differential equation may be composed of a particular solution and two linearly independent homogeneous solutions. The corresponding *quantum* Hamiltonian may be written as

$$H=\frac{\hat{p}^2}{2M(t)}-a(t)[\hat{p}\hat{x}+\hat{x}\hat{p}]+\frac{1}{2}M(t)c(t)\hat{x}^2-\frac{b(t)}{M(t)}\hat{p}+d(t)\hat{x}+\left(\frac{b^2(t)}{2M(t)}-f(t)\right), \tag{3}$$

where

$$c(t)=w^2+4a^2-2\dot{a}-2\frac{\dot{M}}{M}a,$$
$$d(t)=2ab-\dot{b}-F. \tag{4}$$

In the sense of the differentiation of the operator in Ref. [14], the quantum equation of motion for the operator $\hat{x}$ is again given as

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$$\frac{d}{dt}(M\dot{\hat{x}})+M(t)w^2(t)\hat{x}=F(t).$$

Feynman and Hibbs have shown that the coordinate-dependent part of the kernel is determined from the classical action (Chaps. 3–5 of Ref. [13]). In this article, it will be shown that the remaining part of the kernel for the system will be completely determined from the Schrödinger equation and the initial condition which the kernel should satisfy. In this way the kernel for the system of Eq. (1) will be evaluated. By the method of Ref. [15], the wave functions will then be evaluated from the kernel. This Lagrangian formulation has a clear advantage over the Hamiltonian formulation in showing how the last three terms in the right hand side of Eq. (1) which have no effect on classical dynamics affect the wave functions. The classical action will be evaluated in terms of the two linearly independent homogeneous solutions and one particular solution, and so are the kernel and the wave functions. We will prove that the kernels do not depend on the way of choosing the classical solutions, while the wave functions are *not* unique and depend on the choice of the classical solutions. By comparing with the results on the Gaussian pure states of Ref. [16], it is suggested that choosing different classical solutions might amount to applying unitary transformations to the annihilation operator.

In the next section, we will consider the harmonic oscillator with time-dependent mass and frequency, mainly to expose our method. It will also be shown that the kernel does not depend on the choice of two homogeneous solutions, while the wave functions depend on the choice. In Sec. III, the driven harmonic oscillator will be considered. In Sec. IV, the system of a general quadratic Lagrangian in Eq. (1) will be considered and some previous errors will be corrected. The general system will be shown to be equivalent to the driven harmonic oscillator through a unitary transformation. Section V will be devoted to a summary and discussions. We add an appendix to explain how to determine the time dependent part of the kernel from the Schrödinger equation and the initial condition.

II. THE HARMONIC OSCILLATOR WITH TIME-DEPENDENT MASS AND FREQUENCY

In this section, we will apply our method to the harmonic oscillator without driving force. For this model the Lagrangian is written as

$$L^S=\frac{1}{2}M(t)\dot{x}^2-\frac{1}{2}M(t)w^2(t)x^2. \tag{5}$$

The action (integral) from time t_a to time t_b is written as

$$S=\int_{t_a}^{t_b}L\,dt, \tag{6}$$

which gives the equation of motion for the classical trajectory of the model considered,

$$\frac{d}{dt}(M\dot{\bar{x}})+M(t)w^2(t)\bar{x}=0. \tag{7}$$

To find a simple expression for the classical action that is the integral along the classical trajectory, we can rewrite the action as

$$S^S=\frac{1}{2}M\dot{x}x|_{t_a}^{t_b}-\frac{1}{2}\int_{t_a}^{t_b}x\left(\frac{d}{dt}(M\dot{x})+Mw^2x\right). \tag{8}$$

The classical action is then simply given as

$$S_{\rm cl}^S(a,b)=\frac{1}{2}M(t_b)x_b\dot{\bar{x}}_b-\frac{1}{2}M(t_a)x_a\dot{\bar{x}}_a, \tag{9}$$

where x_a (x_b) and $\dot{\bar{x}}_a$ ($\dot{\bar{x}}_b$) denote fixed end point and $d\bar{x}/dt$ at $t=t_a$ ($t=t_b$), respectively.

Suppose that $u(t)$ and $v(t)$ are two linearly independent real solutions of Eq. (7), so that $\bar{x}$ can be written by a linear combination of them. From the two linearly independent solutions, one can always find $v_s(t)$, a linear combination of the two solutions which satisfies $v_s(t_a)=0$, so that $u(t)$ and $v_s(t)$ are again two linearly independent solutions. One can then easily find that the quantities Ω_s and Ω defined as

$$\Omega_s=M(t)[\dot{v}_s(t)u(t)-\dot{u}(t)v_s(t)],$$

$$\Omega=M(t)[\dot{v}(t)u(t)-\dot{u}(t)v(t)] \tag{10}$$

do not depend on time.

The $\bar{x}(t)$ with two fixed end points x_a, x_b can be written as

$$\bar{x}(t)=x_a\frac{u(t)}{u(t_a)}+\left(x_b-x_a\frac{u(t_b)}{u(t_a)}\right)\frac{v_s(t)}{v_s(t_b)}. \tag{11}$$

Making use of this expression of $\bar{x}$, one can rewrite the classical action in Eq. (9) as

$$\begin{aligned}S_{\rm cl}^S(a,b)=&\frac{x_a^2}{2}M(t_a)\left(-\frac{\dot{u}(t_a)}{u(t_a)}+\frac{u(t_b)}{u(t_a)}\frac{\dot{v}_s(t_a)}{v_s(t_b)}\right)\\&+\frac{x_b^2}{2}M(t_b)\frac{\dot{v}_s(t_b)}{v_s(t_b)}+\frac{x_ax_b}{2}\Bigg[M(t_b)\left(\frac{\dot{u}(t_b)}{u(t_a)}\right.\\&\left.-\frac{u(t_b)\dot{v}_s(t_b)}{u(t_a)v_s(t_b)}\right)-M(t_a)\frac{\dot{v}_s(t_a)}{v_s(t_b)}\Bigg].\end{aligned} \tag{12}$$

The fact that classical dynamics are deterministic implies that $S_{\rm cl}$ is unique, as can be explicitly proved, first, one finds that $S_{\rm cl}^S$ does not depend on the scaling of $u(t)$ or $v_s(t)$ by multiplying constant factors. Second, the classical action in Eq. (12) is invariant under the substitution of $u(t)$ by $u(t)+Cv_s(t)$ with an arbitrary constant C. These two observations lead us to the conclusion that $S_{\rm cl}^S$ does not depend on the particular choice of $u(t)$ or $v_s(t)$ as long as $v_s(t_a)=0$.

Then from the formula (3-51) of Ref. [13], the kernel can be written as

$$K^S(b,a)=\exp\left(\frac{i}{\hbar}[S_{\rm cl}^S(a,b)+D^S(t_a,t_b)]\right), \tag{13}$$

where, as shown in the Appendix, $D(t_a,t_b)$ can be completely determined from the initial condition and the Schrödinger equation. Making use of the formulas in the Appendix, one can find the expression of $K^S(b,a)$ in terms of u and v_s as

$$K^S(b,a)=\sqrt{\frac{M(t_a)}{2\pi i\hbar}\frac{\dot v_s(t_a)}{v_s(t_b)}}\exp\left\{\frac{i}{2\hbar}\left[M(t_a)\times\left(-\frac{\dot u(t_a)}{u(t_a)}+\frac{u(t_b)\dot v_s(t_a)}{u(t_a)v_s(t_b)}\right)x_a^2+M(t_b)\frac{\dot v_s(t_b)}{v_s(t_b)}x_b^2-2x_ax_bM(t_a)\frac{\dot v_s(t_a)}{v_s(t_b)}\right]\right\}. \tag{14}$$

Since the kernel is uniquely determined from the classical action, our argument on the uniqueness of the classical action implies that $K^S(b,a)$ does not depend on the way of choosing $u(t)$ or $v_s(t)$.

To find the wave functions from the kernel following the method of Ref. [15], we define two functions $\rho(t)$ and $z(t)$ as

$$\rho(t)=\sqrt{u^2(t)+v_s^2(t)}, \tag{15}$$

$$z(t)=\frac{u(t)-iv_s(t)}{\rho(t)}. \tag{16}$$

After a little algebra, one can find that the kernel can be written as

$$K^S(b,a)=\frac{1}{\sqrt{\pi\hbar}}\sqrt{\frac{\Omega_s}{\rho(t_a)\rho(t_b)}}\times\exp\left[\frac{x_a^2}{2\hbar}\left(-\frac{\Omega_s}{\rho^2(t_a)}-iM(t_a)\frac{\dot\rho(t_a)}{\rho(t_a)}\right)+\frac{x_b^2}{2\hbar}\left(-\frac{\Omega_s}{\rho^2(t_b)}+iM(t_b)\frac{\dot\rho(t_b)}{\rho(t_b)}\right)\right]\times\sum_{n=0}^{\infty}\frac{z^{n+1/2}(t_b)}{2^n n!}H_n\left(\sqrt{\frac{\Omega_s}{\hbar}}\frac{x_a}{\rho(t_a)}\right)\times H_n\left(\sqrt{\frac{\Omega_s}{\hbar}}\frac{x_b}{\rho(t_b)}\right), \tag{17}$$

where H_n is the nth-order Hermite polynomial. From now on the definition of $\rho(t)$ is modified as $\rho(t)=\sqrt{u^2(t)+v^2(t)}$. From the well-known fact that

$$K(x_b,t_b;x_a,t_a)=\sum_n \psi_n(x_b,t_b)\psi_n^*(x_a,t_a)\quad\text{for}\quad t_b>t_a, \tag{18}$$

one can find the nth-order wave function:

$$\psi_n^S(x,t)=\frac{1}{\sqrt{2^n n!}}\left(\frac{\Omega}{\pi\hbar}\right)^{1/4}\frac{1}{\sqrt{\rho(t)}}\left(\frac{u(t)-iv(t)}{\rho(t)}\right)^{n+1/2}\times e^{(x^2/2\hbar)[-\Omega/\rho^2(t)+iM(t)\dot\rho(t)/\rho(t)]}H_n\left(\sqrt{\frac{\Omega}{\hbar}}\frac{x}{\rho(t)}\right), \tag{19}$$

which satisfies the Schrödinger equation:

$$i\hbar\frac{\partial\psi_n^S}{\partial t}=-\frac{\hbar^2}{2M(t)}\frac{\partial^2\psi_n^S}{\partial x^2}+\frac{M(t)w^2(t)}{2}x^2\psi_n^S. \tag{20}$$

The ψ_n^S *does* depend on the choice of two homogeneous solutions, and any set of two linearly independent solutions can be used to construct the wave functions which satisfy the Schrödinger equation of Eq. (20).

To have a physical interpretation of the fact that different choices of $\{u,v\}$ may give different sets of wave functions $\{\psi_n^S,n=0,1,2,\ldots\}$, we consider the simplest case: the simple harmonic oscillator where $M(t)=m_0$ and $w(t)=w_0$. In this case, if we take $\{u,v\}$ as $\{C\cos w_0t,C\sin w_0t\}$ with arbitrary nonzero constant C, then the ψ_n^S reduces to the usual stationary wave functions whose ground $(n=0)$ state is given as $\tilde\psi_0=(m_0w_0/\hbar\pi)^{1/4}e^{-m_0w_0x^2/2\hbar}$. The choice of $\{u,v\}$ as $\{\cos w_0t,C\sin w_0t\}$ with $C\neq1$, however, gives the wave functions of the probability distribution pulsating as time passes.

In the general case, by defining γ as $\gamma_1+i\gamma_2$, where

$$\gamma_1=\frac{\Omega}{\hbar\rho^2}\quad\text{and}\quad\gamma_2=-\frac{M\dot\rho}{\hbar\rho},$$

we can rewrite the ψ_0^S as

$$\psi_0^S=\left(\frac{\gamma_1}{\pi}\right)^{1/4}\exp[i\delta_0(t)]\exp\left(\frac{1}{2}\gamma x^2\right), \tag{21}$$

with a real function δ_0 of t. Therefore, ψ_0^S is one of the wave functions of the Gaussian pure states extensively studied in Ref. [16]. There, it has been shown that any Gaussian pure state is the eigenstate of a certain linear combination of creation and annihilation operators. If we choose different classical solutions, then we could have different γ. The studies of Ref. [16] suggest that choosing different classical solutions might amount to applying unitary transformations to the annihilation operator of the representation system.

III. DRIVEN HARMONIC OSCILLATOR

In this section we will consider the system described by the Lagrangian:

$$L^F=\frac{1}{2}M(t)\dot x^2-\frac{1}{2}M(t)w^2(t)x^2+F(t)x. \tag{22}$$

Let us denote the particular solution of Eq. (2) as $x_p(t)$, so that $x_p(t)$ satisfies the equation

$$\frac{d}{dt}(M\dot x_p)+M(t)w^2(t)x_p=F(t).$$

Then one can rewrite the Lagrangian as

$$L^F=\frac{1}{2}\frac{d}{dt}[M(t)(x-x_p)(\dot{x}-\dot{x}_p)]+\frac{d}{dt}[M(t)\dot{x}_p(x-x_p)]$$
$$-\frac{1}{2}(x-x_p)\left(\frac{d}{dt}[M(\dot{x}-\dot{x}_p)]+Mw^2(x-x_p)\right)$$
$$+\frac{1}{2}\frac{d}{dt}[M\dot{x}_p x_p]+\frac{d}{dt}Y_{x_p}(t), \tag{23}$$

where $Y(t)$ is defined as

$$Y_{x_p}(t)=\int_{t_0}^{t}\frac{1}{2}x_p(t')F(t')dt', \tag{24}$$

with arbitrary constant t_0. The classical action $S_{\rm cl}^F(a,b)$ from time t_a to t_b can be written as

$$\tilde{S}_{\rm cl}^F(a,b;x_p)=S_{\rm cl}^F(a,b)-\Delta S_1(x_p(t),Y_{x_p}(t))\Big|_{t_a}^{t_b}$$
$$=\left(\frac{1}{2}M(t)(\overline{x}-x_p)(\overline{\dot{x}}-\dot{x}_p)+M(t)\dot{x}_p(\overline{x}-x_p)\right)\Bigg|_{t_a}^{t_b}. \tag{25}$$

By adding a homogeneous solution $Cu(t)+Dv(t)$ to the particular $x_p(t)$ with arbitrary constants C and D, one can have a new particular solution $x_p'(t)$. By rewriting the classical action as

$$S_{\rm cl}^F(a,b)=\int_{t_a}^{t_b}\left(\frac{1}{2}\frac{d}{dt}(M\overline{\dot{x}}\overline{x})-\frac{1}{2}\frac{d}{dt}(M\overline{\dot{x}}x_p)\right.$$
$$\left.+\frac{1}{2}\frac{d}{dt}(M\dot{x}_p\overline{x})+\frac{1}{2}x_pF\right)dt, \tag{26}$$

one can easily find that the classical action does *not* depend on the choice of particular solution. That is,

$$S_{\rm cl}^F(a,b)=\tilde{S}_{\rm cl}^F(a,b;x_p)+\Delta S_1(x_p(t),Y_{x_p}(t))\Big|_{t_a}^{t_b}$$
$$=\tilde{S}_{\rm cl}^F(a,b;x_p')+\Delta S_1(x_p'(t),Y_{x_p'}(t))\Big|_{t_a}^{t_b}. \tag{27}$$

Through the methods of the preceding section, one can find the end-point-dependent part of the classical action:

$$\tilde{S}_{\rm cl}^F(a,b;x_p)=\frac{M(t_a)[x_a-x_p(t_a)]^2}{2}\left(-\frac{\dot{u}(t_a)}{u(t_a)}+\frac{u(t_b)\dot{v}_s(t_a)}{u(t_a)v_s(t_b)}\right)$$
$$+\frac{M(t_b)[x_b-x_p(t_b)]^2}{2}\frac{\dot{v}_s(t_b)}{v_s(t_b)}-[x_a-x_p(t_a)]$$
$$\times[x_b-x_p(t_b)]\frac{M(t_a)\dot{v}_s(t_a)}{v_s(t_b)}+M(t_b)\dot{x}_p(t_b)x_b$$
$$-M(t_a)\dot{x}_p(t_a)x_a. \tag{28}$$

The kernel can be written as [13]

$$K^F(a,b)=\exp\left(\frac{i}{\hbar}[S_{\rm cl}^F(a,b)+D^F(t_a,t_b)]\right)$$
$$=\exp\left(\frac{i}{\hbar}[\tilde{S}_{\rm cl}^F(a,b)+\tilde{D}^F(t_a,t_b)]\right). \tag{29}$$

Since the $S_{\rm cl}^F$ does not depend on the choice of the classical solutions within the given restriction and D^F is uniquely determined from the $S_{\rm cl}^F$, the kernel is again unique. For the explicit evaluation, we require the particular solution to satisfy $x_p(t_a)=0$. In the notations of the Appendix, B and β are then given as

$$B=\frac{M(t_b)}{2}\frac{\dot{v}_s(t_b)}{v_s(t_b)}, \tag{30}$$

$$\beta=-M(t_b)x_p(t_b)\frac{\dot{v}_s(t_b)}{v_s(t_b)}-M(t_b)\dot{x}_p(t_b), \tag{31}$$

and the kernel is written as

$$K^F(b,a)=\sqrt{\frac{M(t_a)}{2\pi i\hbar}\frac{\dot{v}_s(t_a)}{v_s(t_b)}}\exp\left\{\frac{i}{2\hbar}\left[x_a^2M(t_a)\left(-\frac{\dot{u}(t_a)}{u(t_a)}+\frac{u(t_b)\dot{v}_s(t_a)}{u(t_a)v_s(t_b)}\right)+[x_b-x_p(t_b)]^2M(t_b)\frac{\dot{v}_s(t_b)}{v_s(t_b)}-2x_a[x_b\right.\right.$$
$$-x_p(t_b)]M(t_a)\frac{\dot{v}_s(t_a)}{v_s(t_b)}+2M(t_b)\dot{x}_p(t_b)x_b-2M(t_a)\dot{x}_p(t_a)x_a-M(t_b)\frac{\dot{v}_s(t_b)}{v_s(t_b)}x_p^2(t_b)-\int_{t_a}^{t_b}\frac{M(t)}{v_s^2(t)}[x_p(t)\dot{v}_s(t)$$
$$\left.\left.-\dot{x}_p(t)v_s(t)]^2dt\right]\right\}. \tag{32}$$

From the expression of the kernel in Eq. (32), as in the preceding section, one can find the nth-order wave function as

$$\psi_n^F(x,t)=\frac{1}{\sqrt{2^n n!}}\left(\frac{\Omega}{\pi\hbar}\right)^{1/4}\frac{1}{\sqrt{\rho(t)}}\left(\frac{u(t)-iv(t)}{\rho(t)}\right)^{n+1/2}\exp\left[\frac{[x-x_p(t)]^2}{2\hbar}\left(-\frac{\Omega}{\rho^2(t)}+iM(t)\frac{\dot{\rho}(t)}{\rho(t)}\right)\right]$$
$$\times H_n\left(\sqrt{\frac{\Omega}{\hbar}}\frac{x-x_p(t)}{\rho(t)}\right)\exp\left\{\frac{i}{\hbar}\left[M(t)\dot{x}_p(t)x-\frac{M(t)}{2}\frac{\dot{v}(t)}{v(t)}x_p^2(t)-\frac{1}{2}\int_{t_0}^{t}M(z)\left(x_p(z)\frac{\dot{v}(z)}{v(z)}-\dot{x}_p(z)\right)^2dz\right]\right\}. \tag{33}$$

In Eq. (33), $\{u,v\}$ is the set of arbitrary linear-independent homogeneous solutions, and x_p is an arbitrary particular solution. The wave functions, again, depend on the way of choosing classical solutions and one can explicitly find that these wave functions indeed satisfy the Schrödinger equation:

$$i\hbar\frac{\partial\psi_n^F}{\partial t}=-\frac{\hbar^2}{2M(t)}\frac{\partial^2\psi_n^F}{\partial x^2}+\frac{M(t)w^2(t)}{2}x^2\psi_n^F-F(t)x\psi_n^F =H_F\psi_n^F. \tag{34}$$

IV. THE GENERAL QUADRATIC SYSTEM

In this section we will consider the general quadratic system described by the Lagrangian of Eq. (1). As in the previous sections, one can find the end-point-dependent part of the classical action:

$$\begin{aligned}\tilde{S}_{\rm cl}^G(a,b;x_p)=&\frac{M(t_a)[x_a-x_p(t_a)]^2}{2}\left(-\frac{\dot u(t_a)}{u(t_a)}+\frac{u(t_b)\dot v_s(t_a)}{u(t_a)v_s(t_b)}\right)\\&+\frac{M(t_b)[x_b-x_p(t_b)]^2}{2}\frac{\dot v_s(t_b)}{v_s(t_b)}-[x_a-x_p(t_a)]\\&\times[x_b-x_p(t_b)]\frac{M(t_a)\dot v_s(t_a)}{v_s(t_b)}+M(t_b)\dot x_p(t_b)x_b\\&-M(t_a)\dot x_p(t_a)x_a+M(t_b)a(t_b)x_b^2\\&-M(t_a)a(t_a)x_a^2+b(t_b)x_b-b(t_a)x_a.\end{aligned} \tag{35}$$

The only way $\tilde S_{\rm cl}^G$ is different from $\tilde S_{\rm cl}^F$ is the last four terms on the right-hand side of Eq. (35). Again, by requiring $x_p(t_a)=0$, one can evaluate the kernel in terms of classical solutions;

$$\begin{aligned}K^G(b,a)=&\sqrt{\frac{M(t_a)}{2\pi i\hbar}\frac{\dot v_s(t_a)}{v_s(t_b)}}\exp\left\{\frac{i}{2\hbar}\left[x_a^2M(t_a)\left(-\frac{\dot u(t_a)}{u(t_a)}+\frac{u(t_b)\dot v_s(t_a)}{u(t_a)v_s(t_b)}\right)+[x_b-x_p(t_b)]^2M(t_b)\frac{\dot v_s(t_b)}{v_s(t_b)}\right.\right.\\&-2x_a[x_b-x_p(t_b)]M(t_a)\frac{\dot v_s(t_a)}{v_s(t_b)}+2M(t_b)\dot x_p(t_b)x_b-2M(t_a)\dot x_p(t_a)x_a\\&+2M(t_b)a(t_b)x_b^2-2M(t_a)a(t_a)x_a^2+2b(t_b)x_b-2b(t_a)x_a-M(t_b)\frac{\dot v_s(t_b)}{v_s(t_b)}x_p^2(t_b)\\&\left.\left.-\int_{t_a}^{t_b}\left(-2f(t)+\frac{M(t)}{v_s^2(t)}[x_p(t)\dot v_s(t)-\dot x_p(t)v_s(t)]^2\right)dt\right]\right\},\end{aligned} \tag{36}$$

whose difference from $K^F(b,a)$ is just from the above-mentioned four terms and an integral of f. As in the previous sections, one can prove that this kernel does not depend on the method of choosing classical solutions.

The nth-order wave function ψ_n can be found from the kernel as

$$\begin{aligned}\psi_n^G(x,t)=&\frac{1}{\sqrt{2^n n!}}\left(\frac{\Omega}{\pi\hbar}\right)^{1/4}\frac{1}{\sqrt{\rho(t)}}\left(\frac{u(t)-iv(t)}{\rho(t)}\right)^{n+1/2}\\&\times\exp\left(\frac{i}{\hbar}\{M(t)a(t)x^2+[M(t)\dot x_p(t)+b(t)]x\}\right)\\&\times\exp\left[\frac{[x-x_p(t)]^2}{2\hbar}\left(-\frac{\Omega}{\rho^2(t)}+iM(t)\frac{\dot\rho(t)}{\rho(t)}\right)\right]\\&\times H_n\left(\sqrt{\frac{\Omega}{\hbar}}\frac{x-x_p(t)}{\rho(t)}\right)\exp\left(\frac{i}{\hbar}\left\{-\frac{M(t)}{2}\frac{\dot v(t)}{v(t)}x_p^2(t)\right.\right.\\&\left.\left.+\int_{t_0}^{t}\left[f(z)-\frac{M(z)}{2}\left(x_p(z)\frac{\dot v(z)}{v(z)}-\dot x_p(z)\right)^2\right]dz\right\}\right).\end{aligned} \tag{37}$$

Again, $\{u,v\}$ is the set of arbitrary linear-independent homogeneous solutions, and x_p is an arbitrary particular solution. One can explicitly apply the Schrödinger equation to these wave functions, to find that they indeed satisfy the equation.

In the Lagrangian of Eq. (1), the conjugate momentum p of the coordinate x is written as $p=M\dot x+2Max+b$. One may interpret x_p as the classical coordinate, and the conjugate momentum is then written as

$$p_p=M\dot x_p+2Max_p+b. \tag{38}$$

As in Sec. II, we define γ' as $\gamma_1'+i\gamma_2'$ where

$$\gamma_1'=\gamma_1\quad\text{and}\quad\gamma_2'=-\frac{M}{\hbar}\left(2a+\frac{\dot\rho}{\rho}\right).$$

Then, the wave function ψ_n^G can be simply written as

$$\begin{aligned}\psi_n^G(x,t)=&\frac{1}{\sqrt{2^n n!}}\left(\frac{\gamma_1}{\pi}\right)^{1/4}e^{i\delta(t)}\exp\left(-\frac{\gamma'}{2}(x-x_p)^2\right.\\&\left.+\frac{i}{\hbar}xp_p\right)H_n(\sqrt{\gamma_1}(x-x_p)),\end{aligned} \tag{39}$$

where $\delta(t)$ is a real function of t. The wave functions agree with those in Ref. [11] except for the fact that $\delta(t)$ is real, which is necessary for the conservation of total probability $\int \psi_n^{G*}\psi_n^G dx$. The expression of ψ_n^G in Eq. (39) shows that ψ_0^G is a wave function of a Gaussian pure state [16], so the discussions of Sec. II are still valid in the general case.

With the wave functions, one can calculate the expectation values of operators, and the uncertainty relations read

$$
\begin{aligned}
&{}_m\langle(\Delta x)^2\rangle_{m\,m}\langle(\Delta p)^2\rangle_m \\
&\quad=({}_m\langle x^2\rangle_m-{}_m\langle x\rangle_m^2)({}_m\langle p^2\rangle_m-{}_m\langle p\rangle_m^2) \\
&\quad=\left(m+\frac{1}{2}\right)^2\hbar^2\left(1+\frac{1}{\Omega^2}(2Ma\rho^2+M\rho\dot{\rho})^2\right),
\end{aligned}
\tag{40}
$$

$$
\begin{aligned}
&{}_{m+1}\langle(\Delta x)^2\rangle_{m\,m+1}\langle(\Delta p)^2\rangle_m \\
&\quad=\frac{1}{\sqrt{2}}\left(\frac{(m+1)\hbar}{\Omega}\right)^{3/2}(u+iv)^3\left(\frac{2\sqrt{2\Omega}x_p}{\sqrt{(m+1)\hbar}(u+iv)}-1\right) \\
&\qquad\times\left(2Ma+M\frac{\dot{\rho}}{\rho}+i\frac{\Omega}{\rho^2}\right)\left[p_p-\frac{1}{2}\sqrt{\frac{(m+1)\hbar}{2\Omega}}(u+iv)\right. \\
&\qquad\left.\times\left(2Ma+M\frac{\dot{\rho}}{\rho}+i\frac{\Omega}{\rho^2}\right)\right],
\end{aligned}
\tag{41}
$$

$$
\begin{aligned}
&{}_{m+2}\langle(\Delta x)^2\rangle_{m\,m+2}\langle(\Delta p)^2\rangle_m \\
&\quad=(m+2)(m+1)\left(\frac{\hbar}{2\Omega}\right)^2(u+iv)^4\left(2Ma+M\frac{\dot{\rho}}{\rho}+i\frac{\Omega}{\rho^2}\right)^2,
\end{aligned}
\tag{42}
$$

with the notation ${}_n\langle O\rangle_m=\int_{-\infty}^{\infty}\psi_n^{G*}(x,t)O\psi_m^G(x,t)$. If we take $u=\rho\cos\theta$ and $v=\rho\sin\theta$, then $\Omega=M\rho^2\dot{\theta}$ and the functions $\rho(t)$, $\theta(t)$ should satisfy

$$\ddot{\theta}+2\frac{\dot{\rho}}{\rho}\dot{\theta}+\frac{\dot{M}}{M}\dot{\theta}=0,$$

$$\ddot{\rho}+\frac{\dot{M}}{M}\dot{\rho}-\rho\dot{\theta}^2+w^2\rho=0.$$

With these notations, the uncertainty relations in Eqs. (40)–(42) are written as

$$
{}_m\langle(\Delta x)^2\rangle_{m\,m}\langle(\Delta p)^2\rangle_m=\left(m+\frac{1}{2}\right)^2\hbar^2\left[1+\frac{1}{\dot{\theta}^2}\left(2a+\frac{\dot{\rho}}{\rho}\right)^2\right],
\tag{43}
$$

$$
\begin{aligned}
&{}_{m+1}\langle(\Delta x)^2\rangle_{m\,m+1}\langle(\Delta p)^2\rangle_m \\
&\quad=\frac{(m+1)^2}{4}\hbar^2e^{4i\theta}\frac{1}{\dot{\theta}^2}\left(1-\frac{2\sqrt{2M\dot{\theta}}}{\sqrt{(m+1)\hbar}}x_pe^{-i\theta}\right) \\
&\qquad\times\left(2a+\frac{\dot{\rho}}{\rho}+i\dot{\theta}\right) \\
&\qquad\times\left(2a+\frac{\dot{\rho}}{\rho}+i\dot{\theta}-\frac{2\sqrt{2\dot{\theta}}}{\sqrt{(m+1)M\hbar}}p_pe^{-i\theta}\right),
\end{aligned}
\tag{44}
$$

$$
\begin{aligned}
&{}_{m+2}\langle(\Delta x)^2\rangle_{m\,m+2}\langle(\Delta p)^2\rangle_m \\
&\quad=\frac{(m+1)(m+2)}{4}\hbar^2e^{4i\theta}\frac{1}{\dot{\theta}^2}\left(2a+\frac{\dot{\rho}}{\rho}+i\dot{\theta}\right)^2,
\end{aligned}
\tag{45}
$$

respectively. The uncertainty relations of Eqs. (43) and (45) *exactly* agree with those of Ref. [11], but the uncertainty relation of Eq. (44) differs from the corresponding one there.

The terms which do not affect the classical dynamics of the model in Eq. (1) can be written as $L-L^F$. The effects of those terms on the wave functions could simply be represented by writing ψ_n^G as

$$
\psi_n^G(x,t)=\exp\left[\frac{i}{\hbar}\int^t(L-L^F)\left(x,\frac{dx}{dz},z\right)dz\right]\psi_n^F(x,t).
\tag{46}
$$

This relation suggests that ψ_n^G can be obtained from ψ_n^F by acting unitary operator U;

$$
U=\exp\left[\frac{i}{\hbar}\left(M(t)a(t)x^2+b(t)x+\int^t f(z)dz\right)\right].
\tag{47}
$$

By defining operator O_F, O_G as

$$
O_F=-ih\frac{\partial}{\partial t}+H_F,\quad O_G=-ih\frac{\partial}{\partial t}+H,
\tag{48}
$$

one may find the relation

$$
UO_FU^{\dagger}=O_G,
\tag{49}
$$

which proves that the Schrödinger equation of the general quadratic system is equivalent to that of the driven harmonic oscillator through the unitary transformation.

V. SUMMARY

The Feynman and Hibbs formulation (or an observation) on the quadratic Lagragian system gives a good explanation of the fact that the quantum wave function can be written in terms of solutions of the classical equation of motion. By developing the observation, we find the kernel and wave functions of the general quadratic system in terms of classical solutions. Furthermore, the kernel is shown to be

be independent of the choice of classical solutions. These results are then used to show that the general quadratic system is equivalent to the driven harmonic oscillator through a unitary transformation. This fact [17] shows that unitary transformation (or, canonical transformation, its classical correspondent) could make the problem simpler or more complicated, and could change the uncertainty relations as in Eqs. (40)–(42).

APPENDIX

In this appendix, it will be shown that, if

$$K(a,b)=\exp\left(\frac{i}{\hbar}(Ax_a^2+Bx_b^2+hx_ax_b+\alpha x_a+\beta x_b+s)\right), \tag{A1}$$

where A, B, h, α, and β are already known functions of t_a,t_b, then the function $s(t_a,t_b)$ is uniquely determined from the initial condition and Schrödinger equation. For the system of the Hamiltonian given in Eq. (3), the kernel should satisfy the Schrödinger equation:

$$\begin{aligned}i\hbar\frac{\partial}{\partial t_b}K=\Bigg[&\frac{1}{2M(t_b)}\left(\frac{\hbar}{i}\frac{\partial}{\partial x_b}\right)^2-2a(t_b)\frac{\hbar}{i}x_b\frac{\partial}{\partial x_b}\\&+\frac{M(t_b)c(t_b)}{2}x_b^2-\frac{b(t_b)}{M(t_b)}\frac{\hbar}{i}\frac{\partial}{\partial x_b}+d(t_b)x_b\\&+\left(\frac{b^2(t_b)}{2M(t_b)}-f(t_b)+i\hbar a(t_b)\right)\Bigg]K,\end{aligned} \tag{A2}$$

which gives the following differential equations:

$$\frac{\partial A}{\partial t_b}=-\frac{h^2}{2M(t_b)}, \tag{A3}$$

$$\frac{\partial B}{\partial t_b}=-\frac{2B^2}{M(t_b)}+4a(t_b)B-\frac{M(t_b)c(t_b)}{2}, \tag{A4}$$

$$\frac{\partial h}{\partial t_b}=-\frac{2Bh}{M(t_b)}+2a(t_b)h, \tag{A5}$$

$$\frac{\partial \alpha}{\partial t_b}=-\frac{h\beta}{M(t_b)}+\frac{b(t_b)}{M(t_b)}h, \tag{A6}$$

$$\frac{\partial \beta}{\partial t_b}=-\frac{2B\beta}{M(t_b)}+2a(t_b)\beta+2\frac{b(t_b)}{M(t_b)}B-d(t_b), \tag{A7}$$

$$\begin{aligned}\frac{\partial s}{\partial t_b}=&-\frac{\hbar}{i}\frac{B}{M(t_b)}-\frac{\beta^2}{2M(t_b)}+\frac{b(t_b)}{M(t_b)}\beta\\&-\frac{b^2(t_b)}{2M(t_b)}+f(t_b)-i\hbar a(t_b).\end{aligned} \tag{A8}$$

With the explicit expressions of A, B, h, α, and β, one may check that Eqs. (A3)–(A7) are satisfied. For example, in the general quadratic system considered in Sec. IV, B is given as $B(t_a,t_b)=[M(t_b)/2][\dot{v}_s(t_b)/v_s(t_b)]+M(t_b)a(t_b)$, which satisfies Eq. (A4).

$s(t_a,t_b)$ can be determined from Eq. (A8) up to a function $g(t_a,t_0)$;

$$\begin{aligned}s(t_a,t_b)=g(t_a,t_0)&-\frac{\hbar}{i}\int_{t_0}^{t_b}\frac{B}{M(t)}dt-\int_{t_a}^{t_b}\left(\frac{\beta^2}{2M(t)}-\frac{b(t)}{M(t)}\beta\right.\\&\left.+\frac{b^2(t)}{2M(t)}-f(t)+i\hbar a(t)\right)dt.\end{aligned} \tag{A9}$$

A wave function ψ satisfies the integral equation

$$\psi(x_b,t_b)=\int_{-\infty}^{\infty}K(x_b,t_b;x_a,t_a)\psi(x_a,t_a)dx_a. \tag{A10}$$

In the limit of $t_b\to t_a$, the classical action approaches

$$\frac{M(t_a)}{2(t_b-t_a)}(x_b-x_a)^2.$$

In order that the relation of Eq. (A10) be satisfied in the limit, the kernel should satisfy the relation

$$\begin{aligned}K(b,a)\to&\sqrt{\frac{M(t_a)}{2\pi i\hbar(t_b-t_a)}}\exp\left(\frac{iM(t_a)}{2\hbar(t_b-t_a)}\right.\\&\left.\times(x_a-x_b)^2\right)\ \text{ as }\ t_b\to t_a.\end{aligned} \tag{A11}$$

This initial condition determines the g uniquely.

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