# Evaluation of hadronic vacuum polarization contribution to muonium hyperfine splitting

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The contribution of hadronic vacuum polarization to the hyperfine splitting of the muonium ground state is evaluated with the account of modern experimental data on the cross section of $e^+e^-\to$annihilation into hadrons. [S1050-2947(99)01903-4]

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In the framework of the quasipotential method, the particle interaction operator for $1\gamma$ and $2\gamma$ processes takes the form [1–3]

$$V=V_{1\gamma}+V_{2\gamma}=V^c+\Delta V,$$

$$V_{1\gamma}=T_{1\gamma},\quad V_{2\gamma}=T_{2\gamma}-T_{1\gamma}\times G^f\times T_{1\gamma},\tag{1}$$

where $V^c$ is the Coulomb potential, $T_{1\gamma}$ and $T_{2\gamma}$ are one- and two-photon scattering amplitudes off the energy shell, and $[G^f]^{-1}=(b^2-\vec{p}^2)/2\mu_R$ is the free two-particle Green function ($\mu_R$ is relativistic reduced mass) [4]. Then the corresponding correction in the energy spectrum may be written as follows:

$$\Delta E=\langle\psi_n^c|\Delta V|\psi_n^c\rangle,\tag{2}$$

where $\psi_n^c$ is the ordinary Coulomb wave function. The hadronic vacuum polarization contribution to the energy spectrum is determined by the Feynman diagrams represented in Fig. 1 by means of Eq. (1). To take into account vacuum polarization in these one-loop diagrams, we must do the following substitution in the photon propagator [2]:

$$\frac{1}{k^2+i\varepsilon}\to\left(\frac{\alpha}{\pi}\right)^2\int_{s_{\rm th}}^{\infty}\frac{\rho(s)ds}{k^2-s+i\varepsilon},\tag{3}$$

where the spectral function $\rho(s)$ is connected with known cross section of $e^+e^-$ annihilation into hadrons $\sigma^h$,

$$\rho(s)=\frac{R(s)}{3s}=\frac{\sigma^h(e^+e^-\to\text{hadrons})}{3s\sigma_{\mu\mu}(e^+e^-\to\mu^+\mu^-)},\tag{4}$$

and $\sigma_{\mu\mu}(e^+e^-\to\mu^+\mu^-)=4\pi\alpha^2/3s$ is the $e^+e^-$ annihilation cross section to muonic pairs. Introducing Feynman's parametrization (parameter λ), we can write the contribution of hadronic vacuum polarization (HVP) to muonium ground-state hyperfine splitting in the form [2]

$$\Delta E^h_{\rm hfs}(\mu e)=\left(\frac{\alpha}{\pi}\right)^2 m_e m_\mu \tilde{E}_F\int_{4m_\pi^2}^{\infty}\frac{R(s)ds}{3s}\times\int_0^1\frac{16-6\lambda-\lambda^2}{m_\mu^2\lambda^2+s(1-\lambda)}d\lambda,\tag{5}$$

where $m_e$, $m_\mu$, and $m_\pi$ are the masses of electron, muon, and charged pion, and $\tilde{E}^F=8\alpha^4m_e^2m_\mu^2/3(m_e+m_\mu)^3$ is the Fermi energy of the muonium. The inner integral in Eq. (5) can be calculated analytically. For the calculation of the external integral, we have used the parametrization of $R(s)$, which accounts for modern experimental data on the annihilation cross section of $e^+e^-$ into hadrons [5–13].

The cross section $\sigma^h$ may be expressed as a sum of terms [2,3]:

$$\sigma^h=\sigma^h_{\pi\pi}+\sigma^h_{\rm res}+\sigma^h_{\rm background}.\tag{6}$$

The main contribution to the cross section $\sigma^h$ is determined by the process $e^++e^-\to\pi^++\pi^-$. The cross section of this reaction is proportional to the modulus squared of the pion form factor $F_\pi$. The corresponding spectral function (4) may be presented in the form

$$\rho_{\pi\pi}(s)=\frac{(s-4m_\pi^2)^{3/2}}{12s^{5/2}}|F_\pi(s)|^2.\tag{7}$$

Contrary to our recent calculation [14], to perform integrations in Eq. (5) for this case we have used the expression of the pion form factor, obtained in [15]:

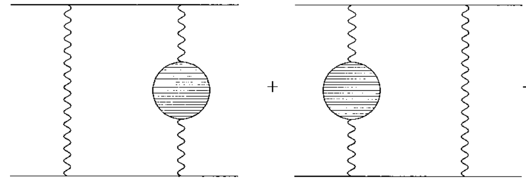


FIG. 1. Feynman diagrams, defining the hadronic vacuum polarization contribution to the hyperfine structure of atom $(\mu e)$.

 

$$F_\pi(s)=\frac{m_\rho}{m_\rho^2-s-im_\rho\Gamma_\rho(s)}\exp\left\{-\frac{s}{96\pi^2f_\pi^2}\mathrm{Re}\left[A\left(\frac{m_\pi^2}{s},\frac{m_\pi^2}{m_\rho^2}\right)+\frac{1}{2}A\left(\frac{m_K^2}{s},\frac{m_K^2}{m_\rho^2}\right)\right]\right\}, \tag{8}$$

where the function

$$A\left(\frac{m_P^2}{s},\frac{m_P^2}{\mu^2}\right)=\ln\left(\frac{m_P^2}{\mu^2}\right)+\frac{8m_P^2}{s}-\frac{5}{3}+\sigma_P^3\ln\left(\frac{\sigma_P+1}{\sigma_P-1}\right),$$

$$\sigma_P=\sqrt{1-4m_P^2/s}, \tag{9}$$

$$\Gamma_\rho(s)=-\frac{m_\rho s}{96\pi^2f_\pi^2}\mathrm{Im}\left[A\left(\frac{m_\pi^2}{s},\frac{m_\pi^2}{m_\rho^2}\right)+\frac{1}{2}A\left(\frac{m_K^2}{s},\frac{m_K^2}{m_\rho^2}\right)\right], \tag{10}$$

where $f_\pi=92.42\pm0.07\pm0.25$ MeV [16], and $m_K$ and $m_\rho$ are the masses of $K^\pm$ and $\rho$ mesons. This pion form factor gives a good description of the experimental data up to energies of the order of 1 GeV in terms of $m_\pi$, $m_K$, $m_\rho$, and $f_\pi$. To obtain the resonance contribution to $\sigma^h$, the Breit-Wigner representation has been exploited. The contribution of $\sigma^h_{\mathrm{background}}$ was found by fitting of the experimental points [5–13]. For a description of the experimental data, we have used the polynomial fit $R(s)=\Sigma_{k=0}^n c_k(\sqrt{s})^k$ of high order on $n$. The coefficients $c_k$ were determined by means of the least-squares method. The region of $\sqrt{s}>60$ GeV was described by the QCD formula, following from the assumption of asymptotic freedom with six quarks. The results of our calculations are presented in Table I. The total contribution of HVP to hyperfine muonium splitting is equal to

$$\Delta E^h_{\mathrm{hfs}}(\mu e)=(3.5988\pm0.1045)\left(\frac{\alpha}{\pi}\right)^2\frac{m_e m_\mu}{m_\pi^2}\tilde{E}^F$$
$$=0.2397\pm0.0070\ \ \mathrm{kHz}. \tag{11}$$

This value is in accordance with the result obtained in [3]: $\Delta E^h_{\mathrm{hfs}}=0.250\pm0.016$ kHz. The more than twice higher precision of Eq. (11) is explained by the account of new experimental data and by more accurate data treatment [11]. The new value (11) allows one to increase the accuracy of testing the radiative corrections to the ground-state hyperfine splitting in muonium. It is important because the precision of hyperfine muonium splitting measurements in the new Los Alamos experiment is of the order of $10^{-2}$ kHz. Much theoretical effort was spent in the last years in order to calculate the new contributions to the muonium hyperfine splitting of a high degree in $\alpha$ and $m_e/m_\mu$ [17,18,19]. The theoretical value $\Delta E^{\mathrm{th}}_{\mathrm{hfs}}=4\,463\,302.69$ (1.34) (0.21) (0.16) kHz [17], taking into account Eq. (11), is in good agreement with the experimental result $\Delta E^{\mathrm{expt}}_{\mathrm{hfs}}=4\,463\,302.88$ (16) kHz [1]. The first and second errors in $\Delta E^{\mathrm{th}}_{\mathrm{hfs}}$ reflect the uncertainties in the measurements of $\alpha^{-1}$ [20] and $m_\mu/m_e$. The third error is purely theoretical and dominated by the coefficients in the $\alpha(Z\alpha)^2$ and $\alpha(Z\alpha)^3\ln(Z\alpha)$ corrections [18].

TABLE I. Contributions to muonium hyperfine splitting.

| Final state | Energy range $\sqrt{s}$ (GeV) | $\Delta E^h_{\mathrm{hfs}}(\mu e)\Big/\left(\frac{\alpha}{\pi}\right)^2\frac{m_e m_\mu}{m_\pi^2}\tilde{E}^F$ |
|---|---|---|
| $\rho,\omega\to2\pi$ | (0.28, 0.81) | $2.0518\pm0.0134$ |
| $\omega\to3\pi$ | (0.42, 0.81) | $0.1869\pm0.0061$ |
| $\phi$ | | $0.2122\pm0.0077$ |
| $J/\Psi$ | | $0.0538\pm0.0059$ |
| $\Upsilon$ | | $0.0006\pm0.0001$ |
| Hadrons | (0.81, 1.4) | $0.5562\pm0.0260$ |
| Hadrons | (1.4, 2.2) | $0.2406\pm0.0199$ |
| Hadrons | (2.2, 3.1) | $0.1252\pm0.0110$ |
| Hadrons | (3.1, 5.0) | $0.0906\pm0.0122$ |
| Hadrons | (5.0, 10.0) | $0.0560\pm0.0015$ |
| Hadrons | (10.0, 40.0) | $0.0227\pm0.0007$ |
| Hadrons | (40.0, 60.0) | $0.0012\pm0.0001$ |
| Hadrons | $\sqrt{s}\geqslant60.0$ | 0.0010 |
| | Summary contribution | $3.5988\pm0.1045$ |

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[1] V. V. Dvoeglazov, R. N. Faustov, and Yu. N. Tukhtyaev, Part. Nuclei **25**, 144 (1994).
[2] J. R. Sapirstein *et al.*, Phys. Rev. D **29**, 2290 (1984).
[3] A. Karimkhodzaev and R. N. Faustov, Yad. Fiz. **53**, 1012 (1991) [Sov. J. Nucl. Phys. **53**, 626 (1991)].
[4] A. P. Martynenko and R. N. Faustov, Theor. Math. Phys. **64**, 765 (1985).
[5] S. I. Dolinsky *et al.*, Phys. Rep., Phys. Lett. **202C**, 99 (1991).
[6] D. Bisello *et al.*, Phys. Lett. B **220**, 321 (1989).
[7] I. Adachi *et al.*, Phys. Lett. B **234**, 525 (1990).
[8] C. Edwards *et al.*, Report No. SLAC-PUB-5160, 1990.
[9] A. E. Blinov *et al.*, Z. Phys. C **70**, 31 (1996).
[10] R. R. Akhmetshin *et al.*, Phys. Lett. B **364**, 199 (1995).
[11] S. Eidelman and F. Jegerlehner, Z. Phys. C **67**, 585 (1995).
[12] L. Martinovic and S. Dubnicka, Phys. Rev. D **42**, 884 (1990).
[13] M. L. Swartz, Phys. Rev. D **53**, 5268 (1996).
[14] A. P. Martynenko and R. N. Faustov, Phys. At. Nucl. **61**, 471 (1998).
[15] F. Guerrero and A. Pich, Phys. Lett. B **412**, 382 (1997).
[16] Particle Data Group, R. M. Barnett *et al.*, Review of Particle Properties, Phys. Rev. D **54**, 1 (1996).
[17] T. Kinoshita and M. Nio, Phys. Rev. D **53**, 4909 (1996).
[18] T. Kinoshita and M. Nio, Phys. Rev. D **55**, 7267 (1997).
[19] M. I. Eides, H. Grotch, and V. A. Shelyuto, Phys. Rev. D **58**, 013008 (1998).
[20] T. Kinoshita, Rep. Prog. Phys. **59**, 1459 (1996).