Diffraction in time in terms of Wigner distributions and tomographic probabilities

Vladimir Man'ko,[1,*] Marcos Moshinsky,[2] and Anju Sharma[2]
[1]Instituto de Ciencias Nucleares, UNAM, Apartado Postal 70-543, 04510 México Distrito Federal, Mexico
[2]Instituto de Física, UNAM, Apartado Postal 20-364, 01000 México Distrito Federal, Mexico

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Long ago in quantum mechanics a discussion appeared about the problem of opening a completely absorbing shutter on which a stream of particles of definite velocity was impinged. The solution of the problem was obtained in a form entirely analogous to the optical one of diffraction by a straight edge. The argument of the Fresnel integrals was time dependent, and thus the first part in the title of this paper. In this paper we reformulate the problem in Wigner distributions and tomographical probabilities. In the former case the probability in phase space is very simple but, as it takes positive and negative values, the interpretation is ambiguous, though it gives a classical limit that agrees entirely with our intuition. In the latter case we can start with our initial conditions in a given reference frame, but obtain our final solution in an arbitrary frame of reference. [S1050-2947(99)04003-2]

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I. INTRODUCTION

Long ago [1] one of us (M.M.) discussed the problem in quantum mechanics of opening, at time $t=0$, a completely absorbing shuttler situated at $x=0$, on which a stream of particles of definite velocity was impinged. In units in which $\hbar$ and the mass m of the particles are unity, the problem reduces to finding a wave function that satisfies the free one-dimensional time-dependent Schrödinger equation, i.e.,

$$i\frac{\partial\psi(x,t)}{\partial t}=-\frac{1}{2}\frac{\partial^2\psi(x,t)}{\partial x^2}, \tag{1.1}$$

with the initial condition

$$\psi(x,0)=\exp(ikx)\,\theta(-x), \tag{1.2}$$

where $\theta(x)$ is the step function given by

$$\theta(x)=\begin{cases}1 & \text{if } x>0\\ 0 & \text{if } x<0.\end{cases} \tag{1.3}$$

The solution of this problem was given in Ref. [1], and later Nussensveig [2] called it $M(x,k,t)$; it can be expressed as [1–3]

$$M(x,k,t)=\frac{1}{2}\exp\left[i\left(kx-\frac{1}{2}k^2t\right)\right]\operatorname{erfc}(e^{-i\pi/4}w)=e^{-i\pi/4}\exp\left[i\left(kx-\frac{1}{2}k^2t\right)\right]\frac{1}{\sqrt{2}}\left\{\left[\frac{1}{2}-C(w)\right]+i\left[\frac{1}{2}-S(w)\right]\right\}, \tag{1.4}$$

where

$$w=\frac{x-kt}{\sqrt{2t}}, \tag{1.5}$$

and the error integral is

$$\operatorname{erfc}(z)=\frac{2}{\sqrt{\pi}}\int_z^\infty e^{-y^2}dy, \tag{1.6}$$

while the Fresnel integrals are defined by

$$C(w)=\sqrt{\frac{2}{\pi}}\int_0^w\cos y^2dy,\quad S(w)=\sqrt{\frac{2}{\pi}}\int_0^w\sin y^2dy. \tag{1.7}$$

Although we have assumed k to be real, as in the units we use it is the velocity or momentum of the impinging particles, all the above expressions remain valid for complex k so long as Im $k<0$. In that case we have the alternative representation [2,3]

$$M(x,k,t)=\frac{i}{2\pi}\int_{-\infty}^{\infty}\frac{\exp\left[i\left(\kappa x-\frac{1}{2}\kappa^2t\right)\right]}{\kappa-k}\,d\kappa, \tag{1.8}$$

which follows from the fact that both sides are solutions of Eq. (1.1) satisfying the initial condition (1.2). The Green function for the one-dimensional free-particle Schrödinger equation has the form

*Permanent address: Lebedev Institute of Physics, Leninsky Prospect 53, 117924 Moscow, Russia.

$$U(x-x',t)=\frac{\exp[i(x-x')^2/2t]}{\sqrt{2\pi it}}, \tag{1.9}$$

as it satisfies Eq. (1.1) for any $t>0$; however, when $t=0$ it becomes the δ function $\delta(x'-x)$. As the initial condition is Eq. (1.2), it is clear [1,2] that the function $M(x,k,t)$ can also be written as

$$M(x,k,t)=\int_{-\infty}^{0}U(x-x',t)\exp(ikx')dx'. \tag{1.10}$$

The expression $|M(x,k,t)|^2$ gives the probability density of finding the particle at point x at time t, when initially it was on the left side of the shutter, i.e., with $x<0$, and had a momentum k. From Eq. (1.4), we see that

$$|M(x,k,t)|^2=\frac{1}{2}\left\{\left[\frac{1}{2}-C(w)\right]^2+\left[\frac{1}{2}-S(w)\right]^2\right\}, \tag{1.11}$$

which is identical to the expression [4] for the intensity of light in Fresnel diffraction by a straight edge. However, the variable w has a very different meaning from the optical problem, as it is now a function of time given by Eq. (1.5). Thus the original paper [1] was given the name ''diffraction in time.''

All we have said above has been very well known for a long time, and has many applications, among which we wish to mention those related to the time-energy uncertainty relations [5] and decay problems [6]. The reason we return to this subject is that now we wish to see its behavior when formulated in terms of Wigner distribution functions [7], and also in relation with the tomographic probability developed recently by one of us (V.M.) and his co-workers [8].

II. DIFFRACTION IN TIME IN WIGNER DISTRIBUTION SPACE

Normally quantum mechanics is discussed in configuration space or, in some cases, in momentum space, but not in both together. Wigner [7] found that this limitation interfered with the application of quantum mechanics to statistical physics, where the description is usually given in phase space. Thus he introduced his concept of Wigner distributions, which allow one to discuss some features of quantum mechanics in phase space.

Our objective will be to formulate the diffraction in time problem, discussed in Sec. I, in terms of Wigner distribution functions. In this way we can visualize the phenomena in phase space and more easily determine its classical limit, and compare it with our intuitive understanding of the behavior of a beam of particles of definite momentum impinging on a shutter when the latter is opened.

In units in which $\hbar$ and the mass m of the particle are unity, and where the configuration space wave function is denoted by $\psi(x)$, and the momentum by p, the Wigner distribution function is defined as [7]

$$W(x,p)\equiv\left(\frac{1}{\pi}\right)\int_{-\infty}^{\infty}\psi^*(x+y)\psi(x-y)\exp(2ipy)dy, \tag{2.1}$$

which has the obvious property that

$$\int_{-\infty}^{\infty}W(x,p)dp=|\psi(x)|^2, \tag{2.2}$$

where the right-hand side is the probability density at the point x, while an integration with respect to x gives us the usual probability density [7] at the momentum value p.

If we now wish to discuss the diffraction in time problem in terms of Wigner distributions, we have to replace $\psi(x)$ in Eq. (2.1) by $M(x,k,t)$ of Eq. (1.4).

While for our analysis k is real, we shall assume for the moment that k is complex with a small negative imaginary part. In this way we can use the expression (1.8) for $M(x,k,t)$ and substituting it into Eq. (2.1) we obtain

$$W(x,p;k,t)=\frac{1}{4\pi^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\exp\left\{-i\left[\kappa(x+y)-\frac{1}{2}\kappa^2t\right]\right\}\exp\left\{i\left[\kappa'(x-y)-\frac{1}{2}\kappa'^2t\right]\right\}}{(\kappa-k^*)(\kappa'-k)}e^{i2py}d\kappa\, d\kappa'\, dy, \tag{2.3}$$

where we now added the momentum k and time t to the Wigner function on the left-hand side, as these variables also appear in $M(x,k,t)$. We also indicate the complex conjugate of k by k^*.

The evaluation of the triple integral (2.3) is done in Appendix A, and it leads to the simple result

$$W(x,p;k,t)=\frac{1}{\pi(k-p)}\sin\{2(pt-x)(k-p)\}\theta(pt-x), \tag{2.4}$$

where θ is the step function defined in Eq. (1.3). Because of the presence of the sine function in Eq. (2.4) we see that the Wigner distribution for the diffraction-in-time problem oscillates between positive and negative values, where the physical significance of the latter is not clear. On the other hand, the presence of θ indicates that the probability density in phase space vanishes when $x>pt$. As in our units $\hbar=m=1$, the momentum p is the same as the velocity, and this result is intuitively expected as the particles in the beam with momentum p could not yet have reached the point x.

What is particularly interesting to us is the classical limit of $W(x,p;k,t)$, which is achieved when the Planck constant $\hbar\rightarrow 0$. We then have to abandon units in which $\hbar$ and m were taken, as 1 and instead use standard cgs ones. The modifications in the form of Eq. (2.3) are trivial, and the resulting distribution function now has the form

$$W(x,p;k,t)=\frac{\sin[g(k-p)]}{\pi(k-p)}\,\theta\left(\frac{pt}{m}-x\right), \tag{2.5}$$

where

$$g\equiv\frac{2}{\hbar}\left(\frac{pt}{m}-x\right). \tag{2.6}$$

If we take the limit $\hbar\to 0$, then $g\to+\infty$ as the step function takes the value 1 only if $(pt/m)-x>0$. We can then use one of the definitions of the δ function [9],

$$\delta(k-p)=\lim_{g\to\infty}\frac{\sin[g(k-p)]}{\pi[k-p]}, \tag{2.7}$$

to write the classical limit of the distribution function as

$$W_{cl}(x,p;k,t)=\delta(k-p)\,\theta\left(\frac{kt}{m}-x\right), \tag{2.8}$$

where we used the presence of the $\delta(k-p)$ in Eq. (2.8) to replace p by k in the step function.

We now see that the classical limit is what we expect, since the only value possible for the momentum of the particle is $p=k$, and since this value is taken only when $x<(kt/m)$, as for $x>(kt/m)$, the particles would not yet have arrived at the point x. Thus the classical limit of the Wigner distribution function for the diffraction-in-time problem confirms our intuition.

III. DIFFRACTION IN TIME IN TERMS OF THE TOMOGRAPHIC PROBABILITIES

In ordinary quantum mechanics the essential concept is the wave function, which in configuration space is denoted by $\psi(x)$. From this concept one derives the probability density $|\psi(x)|^2$ of finding the particle at point x, and also, through appropriate transforms of $\psi(x)$, the probabilities for given values of any other observables.

Recently a change of emphasis was proposed, in which the central concept is the probability itself, but it is defined in a tomographic way [8,10]. This allows us to analyze *through a single concept* the probability either in configuration or momentum space or variables that are linear combinations of both. The tomographic probability density [11] was given in terms of the Wigner distribution through the transform

$$\mathcal{W}(X,\mu,\nu)\equiv\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}W(x,p)e^{-iz(X-x\mu-p\nu)}dz\,dx\,dp, \tag{3.1}$$

where X is the position considered in an ensemble of reference frames [11], which are rotated and scaled with respect to the initial ones through the parameters μ and ν. As an example we have that, when $\mu=1$ and $\nu=0$, X corresponds to the normal position coordinate, but, when $\mu=0$ and $\nu=1$, X is related to the momentum observable.

In Eq. (3.1) $W(x,p)$ is the Wigner function defined in Eq. (2.1) and, substituting it into Eq. (3.1) the tomographic probability density $\mathcal{W}(X,\mu,\nu)$ is given in terms of the configuration wave function $\psi(x)$ by

$$\mathcal{W}(X,\mu,\nu)=\frac{1}{2\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\psi(x-y)\psi^*(x+y)e^{i2py}e^{-iz(X-x\mu-p\nu)}dz\,dx\,dp\,dy. \tag{3.2}$$

The integration with respect to p gives us the expression

$$\int_{-\infty}^{\infty}dp\,e^{ip(2y+z\nu)}=\pi\delta\left(y+\frac{z\nu}{2}\right), \tag{3.3}$$

and, substituting this into Eq. (3.2), and carrying out the integration with respect to y, we obtain

$$\mathcal{W}(X,\mu,\nu)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\psi\left(x+\frac{z\nu}{2}\right)\psi^*\left(x-\frac{z\nu}{2}\right)e^{-iz(X-\mu x)}dz\,dx. \tag{3.4}$$

Now introducing the variables

$$u=x+\frac{z\nu}{2},\quad r=x-\frac{z\nu}{2}, \tag{3.5}$$

we see that the volume element $dz\,dx$ in Eq. (3.4) becomes $dr\,du/|\nu|$, so in terms of u and r, $\mathcal{W}(X,\mu,\nu)$ becomes [10]

$$\mathcal{W}(X,\mu,\nu)=\frac{1}{2\pi|\nu|}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\psi(u)\psi^*(r)\exp\left\{-i\frac{u-r}{\nu}\left[X-\mu\left(\frac{r+u}{2}\right)\right]\right\}dr\,du=\frac{1}{2\pi|\nu|}|\mathcal{I}(X,\mu,\nu)|^2, \tag{3.6}$$

where

$$\mathcal{I}(X,\mu,\nu)=\int_{-\infty}^{\infty}\psi(u)\exp\left[i\left(\frac{\mu}{2\nu}u^2-u\frac{X}{\nu}\right)\right]du. \quad (3.7)$$

Thus, contrary to the Wigner distribution function, the tomographic probability density is always positive definite.

We now turn to the problem of diffraction in time, which means replacing u by x in Eq. (3.7) and then $\psi(x)$ by $M(x,k,t)$ given in terms of its expression (1.10) containing the Green function of the free-particle motion. The expression $\mathcal{I}(x,\mu,\nu)$ then takes the form

$$\mathcal{I}(X,\mu,\nu)=\int_{-\infty}^{\infty}\int_{-\infty}^{0}\frac{\exp[i(x-x')^2/2t]}{\sqrt{2\pi(it)}}\times e^{ikx'}e^{-iX(x/\nu)}e^{i\mu x^2/2\nu}dx\,dx'. \quad (3.8)$$

This integral is evaluated in a straightforward but laborious way in Appendix B, where its value is given. As we are only interested in its absolute value squared multiplied by $(2\pi|\nu|)^{-1}$ which, from Eq. (3.6) gives us the tomographical probability density, we see that it becomes

$$\mathcal{W}(X,\mu,\nu)=\frac{1}{2|\mu|}\left\{\left[\frac{1}{2}+C(\rho)\right]^2+\left[\frac{1}{2}+S(\rho)\right]^2\right\}, \quad (3.9)$$

where

$$\rho=\frac{k(\mu t+\nu)-X}{\sqrt{2\mu(\mu t+\nu)}}, \quad (3.10)$$

and C and S are the Fresnel integrals defined in Eq. (1.7).

We now proceed to discuss the meaning of the tomographical probability density given in Eq. (3.9). We mentioned above that μ and ν represent a rotation and a scaling of an ensemble of reference frames in phase space with respect to the original one. Thus we can express them as

$$\mu=e^{\tau}\cos\theta, \quad \nu=e^{-\tau}\sin\theta, \quad (3.11)$$

with τ and θ in the intervals $-\infty\leqslant\tau\leqslant\infty$ and $0\leqslant\theta\leqslant2\pi$. These expressions of μ and ν imply that our coordinate and momenta, which we designate by capital X and P, are given in terms of the original ones, which we denote by lower case letters x and p, through the relation [12]

$$\begin{pmatrix}X\\P\end{pmatrix}=\begin{pmatrix}e^{\tau}\cos\theta & e^{-\tau}\sin\theta\\ -e^{\tau}\sin\theta & e^{-\tau}\cos\theta\end{pmatrix}\begin{pmatrix}x\\p\end{pmatrix}, \quad (3.12)$$

which is a linear canonical transformation, as the determinant of the matrix is 1. $\mathcal{W}(X,\mu,\nu)$ of Eq. (3.9), with ρ given by (3.10), then gives the probability density for the diffraction-in-time problem in the configuration coordinate X defined in Eq. (3.12).

If we want to return to our original configuration space, we see from Eq. (3.12) that there we must take $\tau=\theta=0$, which implies $X=x, \mu=1$, and $\nu=0$.

In that case ρ of Eq. (3.10) becomes

$$\rho=\frac{kt-x}{\sqrt{2t}}=-w, \quad (3.13)$$

where w was defined in Eq. (1.5). As the Fresnel integrals are odd functions of the argument, we have from Eq. (3.13) that

$$C(\rho)=-C(w), \quad S(\rho)=-S(w), \quad (3.14)$$

and thus the particular tomographic density $\mathcal{W}(x,1,0)$ becomes

$$\mathcal{W}(x,1,0)=\tfrac{1}{2}\{[\tfrac{1}{2}-C(w)]^2+[\tfrac{1}{2}-S(w)]^2\}, \quad (3.15)$$

which is identical to expression (1.11), as we should expect.

Thus we see that the analysis of diffraction-in-time phenomena, in terms of the tomographic probabilities, allows us to study the problem in a wide ensemble of reference frames in phase space, as indicated in Eqs. (3.8)–(3.11). This ensemble of course includes the original phase space (x,p) in which the result is given by Eq. (3.15) which agrees exactly with the initial analysis of the problem [1].

IV. CONCLUSION

In the present paper the problem of diffraction in time was visualized from three different viewpoints. The first was the original one [1], in which both the initial conditions and the solution of the problem were analyzed in the same frame of reference. Solution (1.4) was given in terms of the Fresnel integrals, and using the Cornu spiral we showed that the usual diffraction pattern appeared as a function of time.

In the second approach we translated our solution to the Wigner distribution space. The final expression for the probability density in phase space turned out to be very simple but, unfortunately, it could take both positive and negative values, which made its interpretation ambiguous.

Fortunately it was possible to consider its classical limit by taking $\hbar\rightarrow0$, and the resulting expression (2.8) agreed entirely with our intuitive view, i.e., the probability in phase space was only different from zero when $p=k$ and $x<(pt/m)$, where all the observables are in cgs units. This shows that diffraction in time is a purely quantum phenomena, as it disappears in the classical limit.

The third approach implied formulating our solution in terms of tomographic probabilities. The latter were introduced recently [8,10,11] to allow us to express the solutions in any reference frame that is rotated and scaled with respect to original one. In effect, it implies carrying out a canonical transformation on the original solution of the diffraction-in-time problem. The tomographic probability solution (3.9) is again expressed in terms of Fresnel integrals but of an argument quite different from the one appearing in Eqs. (1.4) and (1.5). If the canonical transformation is the unit 1, i.e., $X=x$ and $P=p$, then the tomographic probability reduces to the solution (1.4), providing us with a check of the analysis developed in Sec. III.

We finally wish to indicate that the diffraction-in-time phenomena derived theoretically by Moshinsky [1] was, in a somewhat changed form, measured experimentally by Szriftigiser, Guéry-Odelin, Arndt, and Dalibard [13]. Possibly a similar fate, in the distant future, awaits the reformulation of the phenomena presented in this paper.

APPENDIX A: DETERMINATION OF THE WIGNER FUNCTION $W(x,p;k,t)$

We start with the expression (2.3) for $W(x,p;k,t)$, and rewrite it as

$$W(x,p;k,t)=\frac{1}{4\pi^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty}\exp\left[2iy\left(p-\frac{\kappa+\kappa'}{2}\right)\right]dy\right\}\left\{\frac{\exp\left[-i\left(\kappa x-\frac{1}{2}\kappa^2 t\right)\right]}{(\kappa-k^*)}\frac{\exp\left[i\left(\kappa' x-\frac{1}{2}\kappa'^2 t\right)\right]}{(\kappa'-k)}\right\}d\kappa\, d\kappa'. \tag{A1}$$

The first integral obviously gives the δ function $2\pi\delta(2p-\kappa-\kappa')$, and so, introducing it in Eq. (A1) and integrating with respect to κ', we obtain

$$W(x,p;k,t)=-\frac{1}{2\pi^2}\exp[2ip(x-pt)]\int_{-\infty}^{\infty}\frac{\exp[-2i\kappa(x-pt)]}{(\kappa-k^*)(\kappa+k-2p)}d\kappa. \tag{A2}$$

We now note, as indicated in the text before Eq. (2.3) that we start by assuming that k has as a small negative imaginary part, so that

$$k\rightarrow k-i\epsilon,\quad k^*\rightarrow k+i\epsilon,\quad -k+2p\rightarrow -k+2p+i\epsilon. \tag{A3}$$

Thus both singularities in the integral in Eq. (A2) are in the upper half of the κ plane.

We can close the contour in Eq. (A2) by a large circle in the upper half of the complex κ plane if $x-pt<0$, thus obtaining the residues of the integrals at the points $k+i\epsilon$ and $-k+2p+i\epsilon$. On the other hand, if $x-pt>0$, we have to close the contour by a large circle in the lower half plane, and, as the function is analytic inside the contour, the integral vanishes. Then, passing to the limit when $\epsilon\rightarrow 0$, as required for our problem where k is real, we obtain, after carrying some of the multiplications, that

$$W(x,p;k,t)=\frac{\theta(pt-x)}{2\pi i(k-p)}\{\exp[2i(k-p)(x-pt)] - \exp[-2i(k-p)(x-pt)]\}. \tag{A4}$$

where θ is the step function (1.3). As the curly bracket divided by $2i$ is a sine function, we then obtain expression (2.4).

APPENDIX B: DETERMINATION OF THE TOMOGRAPHIC PROBABILITY $\mathcal{W}(X,\mu,\nu)$

The tomographic probability is proportional to the absolute square of $\mathcal{I}(X,\mu,\nu)$ where the latter is given by Eq. (3.8), and we rewrite it in the form

$$\mathcal{I}(X,\mu,\nu)=\int_{-\infty}^{0}\left\{\frac{\exp[ikx'+i(x'^2/2t)]}{\sqrt{2\pi it}}\int_{-\infty}^{\infty}\exp[i(ax^2-bx)]dx\right\}dx', \tag{B1}$$

where

$$a\equiv\frac{\mu}{2\nu}+\frac{1}{2t},\quad b\equiv\frac{X}{\nu}+\frac{x'}{t}, \tag{B2}$$

We can rewrite the expressions in the last round bracket in Eq. (B1) as

$$ax^2-bx=\left(\sqrt{a}x-\frac{b}{2\sqrt{a}}\right)^2-\left(\frac{b^2}{4a}\right). \tag{B3}$$

As $b^2/4a$ depends on x' but not on x, we first evaluate the integral

$$\int_{-\infty}^{\infty}\exp\left[i\left(\sqrt{a}x-\frac{b}{2\sqrt{a}}\right)^2\right]dx=\sqrt{\frac{\pi}{a}}e^{i\pi/4} \tag{B4}$$

to obtain

$$\mathcal{I}(X,\mu,\nu)=\frac{e^{i\pi/4}}{\sqrt{i\left(\frac{\mu t}{\nu}+1\right)}}\int_{-\infty}^{0}\exp[ikx'+i(x'^2/2t)]\exp\left[\frac{-i\left(\frac{x'}{t}+\frac{X}{\nu}\right)^2}{2\left(\frac{\mu}{\nu}+\frac{1}{t}\right)}\right]dx'$$

$$=\frac{e^{i\pi/4}\exp\left[\frac{-iX^2}{2\nu^2}\left(\frac{\mu}{\nu}+\frac{1}{t}\right)^{-1}\right]}{\sqrt{i\left(\frac{\mu t}{\nu}+1\right)}}\int_{-\infty}^{0}\exp[i(\alpha x'^2+\beta x')]dx', \tag{B5}$$

where

$$\alpha=\frac{(\mu/\nu)}{2\left(\frac{\mu t}{\nu}+1\right)},\quad \beta=\frac{k\left(\frac{\mu t}{\nu}+1\right)-\frac{X}{\nu}}{\left(\frac{\mu t}{\nu}+1\right)}, \tag{B6}$$

Using again relation (B3), we obtain

$$\alpha x'^2+\beta x'=\left(\sqrt{\alpha}x'+\frac{\beta}{2\sqrt{\alpha}}\right)^2-\frac{\beta^2}{4\alpha}, \tag{B7}$$

and, as $\beta^2/4\alpha$ is independent of x', we need to consider first the integral

$$\int_{-\infty}^{0}e^{i[\sqrt{\alpha}x'+(\beta/2\sqrt{\alpha})]^2}dx'=\int_{-\infty}^{\beta/2\sqrt{\alpha}}e^{iy^2}\frac{dy}{\sqrt{\alpha}}=\int_{-\infty}^{0}e^{iy^2}\frac{dy}{\sqrt{\alpha}}+\frac{1}{\sqrt{\alpha}}\int_{0}^{\beta/2\sqrt{\alpha}}(\cos y^2+i\sin y^2)dy$$

$$=\frac{\sqrt{\pi}}{2\sqrt{\alpha}}\frac{(1+i)}{\sqrt{2}}+\frac{1}{\sqrt{\alpha}}\frac{\sqrt{\pi}}{\sqrt{2}}\left[C\left(\frac{\beta}{2\sqrt{\alpha}}\right)+iS\left(\frac{\beta}{2\sqrt{\alpha}}\right)\right] \tag{B8}$$

where C and S are the Fresnel integrals of Eq. (1.7), and α and β are given by Eq. (B6).

Using Eq. (B7) to introduce Eq. (B8) in Eq. (B5), we obtain

$$\mathcal{I}(X,\mu,\nu)=\frac{\sqrt{\pi}e^{i\pi/4}}{\sqrt{2i\left(\frac{\mu t}{\nu}+1\right)\alpha}}\exp(-i\beta^2/4\alpha)\exp\left\{-i(X^2/2\nu^2)\left(\frac{\mu}{\nu}+\frac{1}{t}\right)^{-1}\right\}\left\{\left[\frac{1}{2}+C\left(\frac{\beta}{2\sqrt{\alpha}}\right)\right]+i\left[\frac{1}{2}+S\left(\frac{\beta}{2\sqrt{\alpha}}\right)\right]\right\}. \tag{B9}$$

Finally replacing α and β by their values [Eq. (B6)], we obtain

$$\mathcal{I}(X,\mu,\nu)=\frac{\sqrt{\pi}e^{i\pi/4}}{\sqrt{i(\mu/\nu)}}\exp\left[-i(X^2/2\nu^2)\left(\frac{\mu}{\nu}+\frac{1}{t}\right)^{-1}\right]\exp(-i\rho^2)\left\{\left[\frac{1}{2}+C(\rho)\right]+i\left[\frac{1}{2}+S(\rho)\right]\right\}, \tag{B10}$$

where

$$\rho=\frac{k(\mu t+\nu)-X}{\sqrt{2\mu(\mu t+\nu)}}. \tag{B11}$$

When taking the absolute square value of $\mathcal{I}(X,\mu,\nu)$, mainly the curly bracket remains, and thus we obtain Eq. (3.9), whose properties are discussed in the main text.

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