Time-of-arrival states

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Although one can show formally that a time-of-arrival operator cannot exist, one can modify the low-momentum behavior of the operator slightly so that it is self-adjoint. We show that such a modification results in the difficulty that the eigenstates are drastically altered. In an eigenstate of the modified time-of-arrival operator, the particle, at the predicted time of arrival, is found far away from the point of arrival with probability 1/2. [S1050-2947(99)03403-4]

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I. INTRODUCTION

In quantum mechanics, observables like position and momentum are represented by operators at a fixed time *t*. However, there is no operator associated with the time it takes for a particle to arrive to a fixed location. One can construct such a time-of-arrival operator [1], but its physical meaning is ambiguous [2–4]. In classical mechanics, one can answer the question as to what time a particle reaches the location $x=0$, but in quantum mechanics, this question does not appear to have an unambiguous answer. In [3] we proved, formally, that in general a time-of-arrival operator cannot exist. This is because one can prove that the existence of a time-of-arrival operator implies the existence of a time operator. As Pauli [5] showed, one cannot have a time operator if the Hamiltonian of the system is bounded from above or below.

There has, however, been renewed interest in the time of arrival, following the suggestion by Grot, Rovelli, and Tate that one can modify the time-of-arrival operator in such away as to make it self-adjoint [6]. The idea is that by modifying the operator in a very small neighborhood around $k=0$, one can formally construct a modified time-of-arrival operator which behaves in much the same way as the unmodified time-of-arrival operator.

In this paper, we examine the behavior of the modified time-of-arrival eigenstates, and show that the modification, no matter how small, radically effects the behavior of the states. We find that the particles in these eigenstates do not arrive with a probability of 1/2 at the predicted time of arrival.

In Sec. II we show why the time-of-arrival operator is not self-adjoint, and explore the possible modifications that can be made in order to make it self-adjoint. We then explore some of the properties of the modified time-of-arrival states. In Sec. III we examine normalizable states which are coherent superpositions of time of arrival eigenstates, and discuss the possibility of localizing these states at the location of arrival at the time-of-arrival. Our results for the ''unmodified'' part of the time-of-arrival state seem to agree with those of Muga, Leavens, and Palao, who have studied these states independently [7]. Our central result is contained in Sec. IV where we show that in an eigenstate of the modified time-of-arrival operator, the particle, at the predicted time of arrival, is found far away from the point of arrival with probability 1/2. We also calculate the average energy of the states, in order to relate them to our proposal [3] that one cannot measure the time of arrival to an accuracy better than $1/\overline{E}_k$ where $\overline{E}_k$ is the average kinetic energy of the particle. We end with concluding remarks in Sec. V.

II. TIME-OF-ARRIVAL OPERATOR

From the correspondence principal, the time-of-arrival operator to the point $x=0$ can be written formally in the k representation as

$$\mathbf{T}(k)=-im\frac{1}{\sqrt{k}}\frac{d}{dk}\frac{1}{\sqrt{k}}=-im\left(\frac{1}{k}\frac{d}{dk}+\frac{d}{dk}\frac{1}{k}\right), \qquad (1)$$

where $\sqrt{k}=i\sqrt{|k|}$ for $k<0$. A set of eigenstates for this operator is given by

$$g_{t_A}(k)=\alpha(k)\frac{1}{\sqrt{2\pi m}}\sqrt{k}e^{(it_Ak^2/2m)}, \qquad (2)$$

where $\alpha=\theta(k)+i\theta(-k)$. However, the operator is not self-adjoint and these eigenstates are not orthogonal:

$$\langle t_A'|t_A\rangle=\frac{1}{2\pi m}\int_0^\infty dk^2\,e^{(ik^2/2m)(t_A-t_A')}$$
$$=\delta(t_A-t_A')-\frac{i}{\pi(t_A-t_A')}. \qquad (3)$$

It is important to recall that a symmetric operator which is not self-adjoint always has complex eigenvalues and eigen-

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functions [8]. If in Eq. (2) we choose t_A complex, having positive imaginary part, then the eigenstate is a square-integrable function (i.e., it is a true eigenstate of the operator) which has complex eigenvalues.

Trying to make $\mathbf{T}$ self-adjoint by defining boundary conditions at $k=0$ leads to the requirement on square integrable wave functions $u(k)$ and $v(k)$ such that

$$\langle u,\mathbf{T}v\rangle-\langle \mathbf{T}^{*}u,v\rangle = im\left[\lim_{k\to 0^-}\frac{v(k)\overline{u(k)}}{|k|}+\lim_{k\to 0^+}\frac{v(k)\overline{u(k)}}{|k|}\right]=0; \tag{4}$$

i.e., the boundary conditions must be chosen so that $v(k)\overline{u(k)}/k$ is continuous through $k=0$. This continuity condition cannot force $u(k)$ to have the same boundary conditions as $v(k)$ for any choice of boundary condition on $v(k)$. That is, the domains of definition of $\mathbf{T}$ and $\mathbf{T}^{*}$ differ and $\mathbf{T}$ cannot be self-adjoint. The proper eigenstates of T and T^{*}, as well as appropriate boundary conditions will be discussed in a forthcoming paper. The inability to define a self-adjoint operator $\mathbf{T}$ is directly related to the fact that one cannot construct an operator which is conjugate to the Hamiltonian if $\mathbf{H}$ is bounded from above or below [3].

One might try to modify $\mathbf{T}$ in order to make it self-adjoint [6]. Consider the operator

$$\mathbf{T}_{\epsilon}(k)=-im\sqrt{f_{\epsilon}(k)}\frac{d}{dk}\sqrt{f_{\epsilon}(k)}, \tag{5}$$

where $f_{\epsilon}(k)$ is some smooth function which differs from $1/k$ only near $k=0$. Since $u(k)$ and $v(k)$ could diverge at the origin at a rate approaching $1/\sqrt{k}$ and still remain square integrable, if $f_{\epsilon}(k)$ goes to zero at least as fast as k, then $\mathbf{T}_{\epsilon}$ will be self-adjoint and defined over all square-integrable functions. However, as we show in Sec. IV, these eigenstates do not behave as one would expect a time-of-arrival eigenstate to behave.

It can be verified that $\mathbf{T}_{\epsilon}$ has a degenerate set of eigenstates $|t_A,+\rangle$ for $k>0$ and $|t_A,-\rangle$ for $k<0$, given by

$$g^{\pm}_{t_A}(k)=\langle k|t_A,\pm\rangle = \theta(\pm k)\frac{1}{\sqrt{2\pi m}}\frac{1}{\sqrt{f_{\epsilon}(k)}}\exp\left(it_A/m\int_{\pm\epsilon}^{k}\frac{1}{f_{\epsilon}(k')}dk'\right), \tag{6}$$

which are orthonormal as expected. Grot, Rovelli, and Tate [6] choose to work with the states given by

$$f_{\epsilon}(k)=\begin{cases}\dfrac{k}{\epsilon^2}, & |k|<\epsilon.\\[2mm] \dfrac{1}{k}, & |k|>\epsilon.\end{cases} \tag{7}$$

If $\epsilon\to 0$, one might expect $\mathbf{T}_{\epsilon}$ to be a good approximation to the time-of-arrival operator when acting on states that do not have support around $k=0$ [6].

As we showed in [3], when these states are examined in the x representation and if one only considers the contribution to the Fourier transform of $g^{+}_{t_A}(k)$ from $k>\epsilon$ (i.e., the "unmodified" part of the eigenstate), then one finds that at the time of arrival the states are not delta functions $\delta(x)$, but are proportional to $x^{-3/2}$; they have support over all x. However, although the state has long tails out to infinity, the quantity $\int dx'|x'^{-3/2}|^2\sim x^{-2}$ goes to zero as $x\to\infty$. Furthermore, the modulus squared of the eigenstates diverges when integrated around the point of arrival, $x=0$. As a result, one might expect that the normalized state will be localized at the point of arrival at the time of arrival. In Sec. III we show that this is indeed so. However, the full eigenstate is made up both of this "unmodified" piece and a modified piece. The modified part of the eigenstate is not well localized at the time of arrival. The contribution to the Fourier transform of the state $g^{+}_{t_A}(k)$ from $0<k<\epsilon$ is given by

$$_{\epsilon}\tilde{g}^{+}(x)_{t_A}=\frac{\epsilon}{\sqrt{2\pi m}}\int_0^{\epsilon}\frac{dk}{\sqrt{k}}e^{ikx}\exp\left(-it_A\frac{k^2}{2m}e^{(i\epsilon^2 t_A/m)\ln(k/\epsilon)}\right). \tag{8}$$

Because $\mathbf{T}_{\epsilon}$ is no longer the generator of energy translations for $|k|<\epsilon$, $g^{+}_{t_A}(k)$ is not time-translation invariant. For the $t_A=0$ state, (8) can be integrated to give

$$_{\epsilon}\tilde{g}^{+}(x)_{t_A}=\frac{\epsilon}{\sqrt{2xim}}\Phi(\sqrt{i\epsilon x}), \tag{9}$$

where Φ is the probability integral. For large x, $_{\epsilon}\tilde{g}^{+}(x)_{t_A}$ goes as $1/\sqrt{x}$ and the quantity $\int dx'|_{\epsilon}\tilde{g}^{+}_{t_A}(x')|^2\sim\ln x$ diverges as $x\to\infty$. For small x, $_{\epsilon}\tilde{g}^{+}_{t_A}(x)$ is proportional to $e^{-i\epsilon x}$. Its modulus squared vanishes when integrated around a small neighborhood of $x=0$. $_{\epsilon}\tilde{g}^{+}(x)_{t_A}$, then, is not localized around the point of arrival, at the time of arrival. This will also be verified in Sec. III where we examine normalizable states. Although $_{\epsilon}\tilde{g}^{+}(x)_{t_A}$ is not localized around the point of arrival at the time of arrival, one might hope that this part of the state does not contribute significantly in time-of-arrival measurements when $\epsilon\to 0$. However, we will see that for coherent superpositions of these eigenstates, half the norm is made up of the modified piece of the eigenstate.

III. NORMALIZED TIME-OF-ARRIVAL STATES

Since the time-of-arrival states are not normalizable, we will examine the properties of states $|\tau_{\Delta}\rangle$ which are narrow superpositions of the modified time-of-arrival eigenstates. These states are normalizable, although they are no longer orthogonal to each other.[1]

We can now consider coherent superpositions of these eigenstates,

$$|\tau^{\pm}_{\Delta}\rangle=N\int dt_A|t_A,\pm\rangle e^{-(t_A-\tau)^2/\Delta^2}, \tag{10}$$

[1]These coherent states form a positive operator valued measure (POVM). The measurement of time-of-arrival using POVMs has been discussed in [11].

where N is a normalization constant and is given by $N=(2/\pi\Delta^2)^{1/4}$. The spread dt_A in arrival times is of order Δ.

We now examine what the state $\tau(x,t)^+=\langle x|\tau_\Delta^+\rangle$ looks like at the point of arrival as a function of time. In what follows, we will work with the state centered around $\tau=0$ for simplicity. This will not affect any of our conclusions. $\tau^+(x,t)$ is given by

$$\tau^+(x,t)=N\int \langle x|e^{(-i\mathbf{p}^2t/2m)}|t_A,+\rangle e^{(-t_A^2/\Delta^2)}dt_A$$

$$=N\int_0^\epsilon e^{(-t_A^2/\Delta^2)}e^{(-ik^2/2m)t}e^{ikx}g_{t_A}^+(k)dt_A\,dk$$

$$+N\int_\epsilon^\infty e^{(-t_A^2/\Delta^2)}e^{(-ik^2/2m)t}e^{ikx}g_{t_A}^+(k)dt_A\,dk$$

$$\equiv{}_\epsilon\tau^+(x,t)+{}_0\tau^+(x,t). \tag{11}$$

As argued in the previous section, the second term should act like a time-of-arrival state. The first term is due to the modification of **T** and has nothing to do with the time of arrival. We will first show that the second term can indeed be localized at the point of arrival, $x=0$, at the time of arrival, $t=t_A$. We will do this by expanding it around $x=0$ in a Taylor series. After taking the limit $\epsilon\to 0$, its nth derivative at $x=0$ is given by

$$\frac{d^n}{dx^n}{}_0\tau^+(x,t)|_{x=0}$$

$$=\frac{N}{\sqrt{2\pi m}}\int\int_\epsilon^\infty e^{(-t_A^2/\Delta^2)}\theta(k)\sqrt{k}(ik)^n$$

$$\times e^{(ik^2/2m)(t_A-t)}dt_A\,dk$$

$$=\frac{N\Delta}{\sqrt{2m}}i^n\int_0^\infty e^{(-k^4\Delta^2/16m^2)}e^{(-ik^2t/2m)}k^{1/2+n}dk$$

$$=\frac{2^{3/8+3n/4}i^n}{\pi^{1/4}}\Gamma\left(\frac{3}{4}+\frac{n}{2}\right)\left(\frac{m}{\Delta}\right)^{1/4+n/2}$$

$$\times e^{(-t^2/2\Delta^2)}D_{-3/4-n/2}\left(\frac{it\sqrt{2}}{\Delta}\right), \tag{12}$$

where $D_p(z)$ are the parabolic-cylinder functions. For any finite t, we can choose Δ small enough so that the argument of $D_p(z)$ is large and can be expanded. We can now write ${}_0\tau^+(0,t)$ as a Taylor expansion around $x=0$,

$${}_0\tau^+(x,t)\simeq\sqrt{\Delta}\left(\frac{m}{t^3}\right)^{1/4}\sum_{n=0}^\infty a_n\left(\sqrt{\frac{m}{t}}x\right)^n, \tag{13}$$

where a_n is a numerical constant given by

$$a_n=i^{-3/4+n/2}2^{(n-1)/2}\pi^{-1/4}\Gamma\left(\frac{3}{4}+\frac{n}{2}\right). \tag{14}$$

We can now see that for any finite t the amplitude for finding the particle around $x=0$ goes to zero as Δ goes to zero. The probability of being found at the point of arrival at a time other than the time of arrival can be made arbitrarily small. On the other hand, at the time of arrival $t=0$, we will now show that the state ${}_0\tau^+(x,t)$ can be as localized as one wishes around $x=0$.

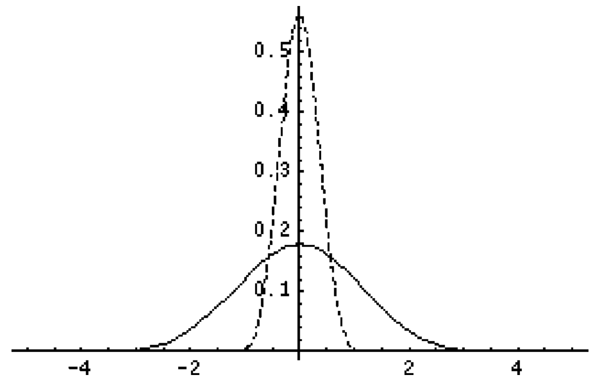

FIG. 1. $|{}_0\tau^+(x,\tau)|^2$ vs x, with $\Delta=m$ (solid line) and $\Delta=m/10$ (dashed line). As Δ gets smaller, the probability function gets more and more peaked around the origin.

From Eq. (12), we expand ${}_0\tau^+(x,0)$ as a Taylor series,

$${}_0\tau^+(x,0)=\left(\frac{m}{\Delta}\right)^{1/4}\sum_{n=0}^\infty b_n\left(\sqrt{\frac{m}{\Delta}}x\right)^n, \tag{15}$$

where

$$b_n=i^n2^{n-3/4}\pi^{-1/4}\Gamma\left(\frac{3}{8}+\frac{n}{4}\right). \tag{16}$$

We see then that ${}_0\tau^+(x,0)$ is a function of $(\sqrt{m/\Delta})x$ [with a constant of $(m/\Delta)^{1/4}$ out front]. As a result, the probability of finding the particle in a neighborhood δ of x is given by

$$\int_{-\delta}^{\delta}\left|{}_0\tau^+\left(\sqrt{\frac{m}{\Delta}}x,0\right)\right|^2dx=\sqrt{\frac{\Delta}{m}}\int_{-\delta\sqrt{m/\Delta}}^{\delta m/\Delta}|{}_0\tau^+(u,0)|^2du. \tag{17}$$

Since $|{}_0\tau^+(u,0)|^2$ is proportional to m/Δ and is square integrable, we see that for any δ one need only make Δ small enough in order to localize the entire particle in the region of integration. ${}_0\tau^+(x,t)$ is localized in a neighborhood δ around the point of arrival at the time of arrival as $\Delta\to 0$. The state is localized in a region δ of order $\sqrt{\Delta/m}$. This is what one would expect from physical grounds, since we have

$$dx\sim dt_A\frac{\langle k\rangle}{m}\sim\sqrt{\frac{\Delta}{m}}. \tag{18}$$

($\langle k\rangle$ is calculated in the following section and is proportional to $\sqrt{m/\Delta}$.) The probability distribution of ${}_0\tau^+(x,t)$ at $t=\tau$ is shown in Fig. 1. This behavior of the unmodified piece of the time-of-arrival state, ${}_0\tau^+(x,t)$, as a function of time appears to agree with the results of Muga and Leavens, who have studied these coherent states independently [7].

The modified part of the time-of-arrival state, ${}_\epsilon\tau^+(x,0)$, is not found near the origin at $t=t_A=0$. We find

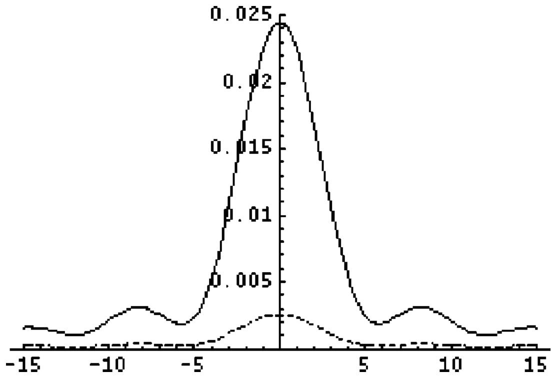

FIG. 2. $(1/\epsilon)|_\epsilon\tau^+(x,\tau)|^2$ vs ϵx, with $\Delta\epsilon^2=m/10$ (solid line) and $\Delta\epsilon^2=m/100$ (dashed line). As Δ or ϵ gets smaller, the probability function drops near the origin, and grows longer tails which are exponentially far away.

$$
{}_\epsilon\tau^+(x,0)=N\frac{\epsilon}{\sqrt{2\pi m}}\int_{-\infty}^{\infty}\int_0^{\epsilon}\exp\left(-\frac{t_A^2}{\Delta^2}\right)\frac{1}{\sqrt{k}}\times\exp\left(\frac{i\epsilon^2 t_A}{m}\ln\frac{k}{\epsilon}\right)e^{ikx}dk\,dt_A
$$
$$
=N\frac{\epsilon^{3/2}}{\sqrt{2\pi m}}\int_{-\infty}^{\infty}\exp\left(-\frac{t_A^2}{\Delta^2}\right)\gamma\left(\frac{i\epsilon^2 t_A}{m}+\frac{1}{2},-i\epsilon x\right)\times(-i\epsilon x)^{-1/2-(i\epsilon^2 t_a/m)}dt_A. \quad (19)
$$

If $i\epsilon x$ is not large, we can use the fact that, for Δ and ϵ very small, $i\epsilon^2 t_A/m\ll 1/2$ so that we have

$$
{}_\epsilon\tau^+(x,0)\simeq(2\pi)^{1/4}\sqrt{\frac{\epsilon^3\Delta}{2m}}\frac{\Phi(\sqrt{-i\epsilon x})}{\sqrt{-i\epsilon x}}. \quad (20)
$$

Note the similarity between this state (the form above is not valid for large x) and that of the modified part of the eigenstate (9). We are interested in the case where $\epsilon^2\Delta/m$ goes to zero, in which case ${}_\epsilon\tau^+(x,0)$ vanishes near the origin. For large ϵx, it goes as $\sqrt{(\epsilon^2\Delta/xm)}$. From Eq. (19) we can also see that, if $\epsilon x>e^{(m/\epsilon^2\Delta)}$, then the last factor in the integrand oscillates rapidly and the integral falls rapidly for larger x. Thus, as we make $(\epsilon^2\Delta/m)$ smaller, the value of the modulus squared decrease around $x=0$, but the tails, which extend out to $e^{m/\epsilon^2\Delta}/\epsilon$, get longer. $\int^x|_\epsilon\tau^+(x,0)|^2$ goes as $(\epsilon^2\Delta/m)\ln x$ up to $\epsilon x\sim e^{m/\epsilon^2\Delta}$.

As $(\epsilon^2\Delta/m)\to 0$, the particle is always found in the far away tail. The state ${}_\epsilon\tau^+(x,0)$ is not found near the point of arrival at the time of arrival. Its probability distribution at $t=t_A=0$ is shown in Fig. 2.

IV. CONTRIBUTION TO THE NORM DUE TO MODIFICATION OF T

We now show that the modified part of $|\tau_\Delta^+\rangle$ contains at least half the norm, no matter how small ϵ is made. The norm of the state $|\tau_\Delta^+\rangle$ can be written as

$$
\int|\langle k|\tau_\Delta^+\rangle|^2dk=N^2\int_0^{\epsilon}|e^{-(t_A^2/\Delta^2)}g_{t_A}^+(k)dt_A|^2dk+N^2\int_{\epsilon}^{\infty}|e^{-(t_A^2/\Delta^2)}g_{t_A}^+(k)dt_A|^2dk\equiv N_\epsilon^2+N_0^2, \quad (21)
$$

where N_ϵ^2 is the norm of the modified part of the time-of-arrival state and N_0^2 is the norm of the unmodified part. The second term can be integrated to give

$$
N_0^2=\frac{N^2}{2\pi m}\int\int_{\epsilon}^{\infty}\exp\left(-\frac{t_A^2+t_A'^2}{\Delta^2}\right)\times\exp\left(i\frac{k^2-\epsilon^2}{2m}(t_A-t_A')\right)dt_A dt_A' dk
$$
$$
=\frac{N^2\Delta^2\pi}{m}\int_0^{\infty}d\tilde{k}\,\tilde{k}\exp\left(\frac{-\tilde{k}^4\Delta^2}{8m^2}\right)=\frac{1}{2}, \quad (22)
$$

where without loss of generality we are looking at the state centered around $\tau=0$ at $t=0$.

The unmodified piece can contain only half the norm. The rest is found in the modified piece:

$$
N_\epsilon^2=\frac{N^2}{2\pi m}\int_0^{\epsilon}dk\int dt_A dt_A'\frac{\epsilon^2}{k}\exp\left(\frac{-t_A^2-t_A'^2}{\Delta^2}\right)\times\exp\left(i\epsilon^2\ln\frac{k}{\epsilon}\frac{t_A'-t_A}{m}\right)
$$
$$
=\frac{N^2\Delta^2}{2m}\int_0^{\epsilon}dk\,\exp\left(\frac{-\epsilon^4\Delta^2\ln^2 k/\epsilon}{2m^2}\right)\frac{\epsilon^2}{k}=\frac{1}{2}. \quad (23)
$$

The reason for this is that, essentially, the modification $1/k\to f_\epsilon(k)$ involves expanding the region $0<k<\epsilon$ into the entire negative k axis. That is, we see from Eq. (3) that in order to make the eigenstates orthogonal, one needs the integration variable to go from $-\infty$ to ∞ and this involves making the modification

$$
k^2\to z^{\pm}=\int_{\pm\epsilon}^{k}\frac{dk'}{f_\epsilon(k')}. \quad (24)
$$

The orthogonality condition then becomes

$$
\langle t_A',\pm|t_A,\pm\rangle=\int_{-\infty}^{\infty}dz^{\pm}\frac{1}{2\pi m}e^{i(t_A-t_A')z^{\pm}/m}=\delta(t_A-t_A'). \quad (25)
$$

No matter how small we make ϵ, half the norm comes from the contribution $z^{\pm}<0$ which is the modified part of the eigenstate. As a result, if one makes a measurement of the time of arrival, then one finds that half the time the particle is not found at the point of arrival at the predicted time of arrival. Modified time-of-arrival states do not always arrive on time.

From Eq. (23), one can also see that if $f_\epsilon(k)$ goes to zero faster than k, then N_ϵ will diverge as Δ or ϵ go to zero. If $f_\epsilon(k)=k^{1+\delta}$, then we find

$$N_\epsilon=\frac{1}{2}e^{(\delta^2 m^2/2\epsilon^4\Delta^2)}\left[1-\Phi\left(\frac{-\delta\epsilon^2\Delta\sqrt{2}}{m}\right)\right]. \qquad (26)$$

As ϵ or Δ go to zero, N_ϵ diverges, and if we renormalize the state, the entire norm will be made up of the modified part of the eigenstate.

It is also of interest to calculate the average value of the kinetic energy for these states, since in [3] we found that if one measures the time of arrival with a clock, then the accuracy of the clock cannot be greater than $1/\bar{E}_k$. In calculating the average energy, the modified piece will not matter since k^2 goes to zero at $k=0$ faster than $1/\sqrt{k}$ diverges. We find

$$\langle\tau_\Delta^+|\mathbf{H}_k|\tau_\Delta^+\rangle=\int dk\frac{k^2}{2m}\langle\tau_\Delta^+|k\rangle\langle k|\tau_\Delta^+\rangle$$
$$=\frac{N^2}{\pi(2m)^2}\int_0^\infty k^3 e^{i(t_A-t_A')k^2/2m}$$
$$\times e^{-(t_A^2+t_A'^2)/\Delta^2}dt_A\,dt_A'\,dk=\frac{4}{\Delta\sqrt{2\pi}}. \qquad (27)$$

We see therefore that the kinematic spread in arrival times of these states is proportional to $1/\bar{E}_k$. Since the probability of triggering the model clocks discussed in [3] decays as $\sqrt{E_k\delta t_A}$, where δt_A is the accuracy of the clock, we find that the states $|\tau_\Delta^+\rangle$ will not always trigger a clock whose accuracy is $\delta t_A=\Delta$.

V. CONCLUSION

We have seen that if one modifies the time-of-arrival operator so as to make it self-adjoint, then its eigenstates no longer behave as one expects time of arrival states to behave. Half the time, a particle which is in a time-of-arrival state will not arrive at the predicted time-of-arrival. The modification also results in the fact that the states are no longer time-translation invariant.

For wave functions which do not have support at $k=0$, measurements can be carried out in such a way that the modification will not effect the results of the measurement [3]. Nonetheless, after the measurement, the particle will not arrive on time with a probability of 1/2. One cannot use $\mathbf{T}_\epsilon$ to prepare a system in a state which arrives at a certain time.

Previously, we have argued that time-of-arrival measurements should be thought of as continuous measurement processes, and that there is an inherent inaccuracy in time-of-arrival measurements, given by $\delta t_A>1/\bar{E}_k$ [3,8]. This current paper supports the claim that the time of arrival is not a well defined observable in quantum mechanics [12].

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