Entanglement processing and statistical inference: The Jaynes principle can produce fake entanglement

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We show, by explicit examples, that the Jaynes principle when applied to compound quantum systems may produce the *entangled* maximum entropy states compatible with data coming from *nonentangled* (separable) states. It means that the Jaynes statistical inference scheme may lead to a wrong conclusion about entanglement, which is a crucial parameter in quantum information theory. We suggest that in all the processes where entanglement is needed, the proper inference scheme should involve minimization of entanglement. Examples illustrating the proposed scheme are provided. [S1050-2947(99)02303-3]

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I. INTRODUCTION

It is well known that quantum mechanics allows us to reconstruct completely a state of the quantum-mechanical system from mean values of a complete system of observables measured on the ensemble of identically prepared systems [1]. By a complete set of observables [1], one means the maximal set of linearly independent observables where the trivial observable represented by an identity operator is excluded. In practice we often deal with situations when the state of the system is unknown and only mean values a_i ($i=1,\ldots,p$) of some *incomplete* set of observables $\{A_i\}_{i=1}^p$ are available from experiments, i.e.,

$$\mathrm{Tr}\,\varrho A_i \equiv \langle A_i \rangle = a_i, \quad i=1,\ldots,p. \tag{1}$$

Then, of course, there can be many states, which are in agreement with the measured data. It involves the problem of estimation of the state on the basis of the exact mean values of given observables. According to the maximum entropy principle [2–4], we have to choose from a set of states ϱ that fulfill the constraint (1), the most probable (or representative) state ϱ_J, which maximizes the von Neumann entropy,

$$S(\varrho) = -\mathrm{Tr}\,\varrho \ln \varrho. \tag{2}$$

Then, the representative state ϱ_J is given by [3]

$$\varrho_J = Z(\boldsymbol{\lambda})^{-1} \exp\left(-\sum_{i=1}^{p} \lambda_i A_i\right), \tag{3}$$

where $Z(\boldsymbol{\lambda}) = \mathrm{Tr}\exp(-\Sigma_{i=1}^p \lambda_k A_k)$ is the partition function and the vector $\boldsymbol{\lambda}(\boldsymbol{a}) = (\lambda_1,\ldots,\lambda_p)$ is uniquely determined by the vector $\boldsymbol{a} = (a_1,\ldots,a_p)$,

$$-\frac{\partial \ln Z(\boldsymbol{\lambda})}{\partial \lambda_i} = a_i, \quad i=1,\ldots,p. \tag{4}$$

The above maximum-entropy principle (or Jaynes principle) was applied for partial reconstruction of pure and mixed states of many different systems [5]. In particular, it allowed us to interpret quantum statistical mechanics as a special type of statistical inference [3] based on the entropic criterion.

The Jaynes principle is the most rational inference scheme in the sense that it does not permit us to draw any conclusions unwarranted by the experimental data. However, this argument making the principle plausible does not actually prove it [6]. In this context one can ask the question: Is the entropic criterion universal? Surprisingly, as we will show in this paper, there *are* situations where the Jaynes principle fails. This concerns compound quantum systems that have recently attracted much attention due to the new phenomena such as quantum teleportation [7], quantum dense coding [8], or quantum cryptography [9].

In all the above effects the most important characteristics of state is entanglement (or inseparability) [10]. Suppose we need the entanglement to deal with one of these effects having, however, the compound system in an unknown state and some *incomplete* data of type (1). Then, usually, to proceed further, we must somehow estimate the state of the system from the data. But what scheme of inferring can be used in this case? The fact that we need the entanglement for our purposes imposes a basic condition on possible inference schemes. Namely they certainly should not give us an inseparable estimated state if only theoretically there exists a separable state compatible with the measured data. Otherwise, it may happen that we get into trouble trying to use the entanglement we inferred to be present, while in fact, there is no entanglement at all. In this paper we show that the Jaynes principle when applied to composite systems may produce fake entanglement, i.e., the maximum entropy state compatible with incomplete data may be entangled even if the data come from a nonentangled (separable) state. It suggests that

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in all the processes involving entanglement, one must replace the Jaynes scheme by the one governed by minimization of entanglement. In Sec. II we provide explicit examples of data for which there exists a *separable* state compatible with them, while the state obtained by means of the Jaynes principle is *inseparable* (entangled). In Sec. III we propose a scheme based on minimization of entanglement and compare it with the Jaynes scheme. In the last section we discuss the results, in particular, in the context of the thermodynamical analogies in quantum information theory.

II. JAYNES PRINCIPLE VERSUS ENTANGLEMENT: COUNTEREXAMPLES

As the first example we will take the Bell-CHSH observable [12],

$$B=\sqrt{2}(\sigma_x\otimes\sigma_x+\sigma_z\otimes\sigma_z)=2\sqrt{2}(|\Phi^+\rangle\langle\Phi^+|-|\Psi^-\rangle\langle\Psi^-|) \tag{5}$$

with the mean value

$$\langle B\rangle=b,\quad 0\leqslant b\leqslant 2\sqrt{2} \tag{6}$$

(i.e., we have only one constraint). Here, the observable is expressed in terms of the so-called Bell basis [14] given by

$$\psi_{\substack{1\\(2)}}\equiv\Phi^{\mp}=\frac{1}{\sqrt{2}}(|\uparrow\uparrow\rangle\mp|\downarrow\downarrow\rangle),$$

$$\psi_{\substack{3\\(0)}}\equiv\Psi^{\pm}=\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle\pm|\downarrow\uparrow\rangle). \tag{7}$$

Let us now apply the Jaynes inference scheme to these data. Then the Jaynes state, calculated directly by use of formula (3) is given by

$$\varrho_J=\tfrac{1}{4}\Bigg[\left(1-\frac{b}{\sqrt{2}}+\frac{b^2}{8}\right)|\Phi^+\rangle\langle\Phi^+| +\left(1+\frac{b}{\sqrt{2}}+\frac{b^2}{8}\right)|\Psi^-\rangle\langle\Psi^-| +\left(1-\frac{b^2}{8}\right)(|\Psi^+\rangle\langle\Psi^+|+|\Phi^-\rangle\langle\Phi^-|)\Bigg]. \tag{8}$$

We see that the state is diagonal in the Bell basis. Let us now check the separability of the state. There is a simple separability criterion for the Bell diagonal states [15], namely, such a state is separable if, and only if, all its eigenvalues do not exceed $\frac{1}{2}$. Then, we obtain that for $b>4-2\sqrt{2}$, the state is *inseparable*. Thus for that value of b using the Jaynes principle we infer the presence of entanglement. Now we will see that for $b\leqslant\sqrt{2}$ there exists a *separable* state, which satisfies the considered data. This is the following state:

$$\varrho=\frac{b}{2\sqrt{2}}|\Phi^+\rangle\langle\Phi^+|+\left(\frac{1}{2}-\frac{b}{4\sqrt{2}}\right)(|\Psi^+\rangle\langle\Psi^+|+|\Phi^-\rangle\langle\Phi^-|) \tag{9}$$

for $b\leqslant\sqrt{2}$. So within the region $4-2\sqrt{2}<b\leqslant\sqrt{2}$, the Jaynes principle produces fake entanglement: the Jaynes state is *inseparable*, while there is a *separable* state (9) satisfying the data.

One could think that this difference between the two types of inference is due to the fact that the used observable is nonlocal, i.e., it cannot be measured itself without interchange of quantum information between the observers. If the measurments are performed locally, then the mean value of the Bell-CHSH observable is not the only measured quantity as we simultaneously obtain the mean values of the product observables, which add up to the observable. Moreover, by measuring the product observable, we gain additional information. Indeed, if the correlations are measured, the marginal distributions are also obtained. To show that nonlocality of observables is not necessary to get fake entanglement via the Jaynes principle, let us consider the following data, which could be obtained by distant observers (who can communicate only by means of classical bits),

$$\langle\sqrt{2}\sigma_x\otimes\sigma_x\rangle=\langle\sqrt{2}\sigma_z\otimes\sigma_z\rangle=\frac{b}{2},$$

$$\langle\sigma_x\otimes I\rangle=\langle\sigma_z\otimes I\rangle=\langle I\otimes\sigma_x\rangle=\langle I\otimes\sigma_z\rangle=0. \tag{10}$$

One can easily see that the Jaynes state is now the same as in the previous case, and the state (9) still satisfies these constraints. So we have obtained that even if the data come from measurements, which do not involve quantum-correlated observables, the Jaynes principle still produces wrong inference about entanglement.

Then, it follows that the above result has no classical counterpart. Indeed, in a classical case, the Jaynes principle can also fake correlations, provided that the used observables are *correlated*. Here, we applied observables that are correlated, but are not *quantum* correlated. Still, however, the Jaynes principle fakes the quantum correlations.

III. POSSIBLE INFERENCE SCHEME FOR QUANTUM ENTANGLEMENT PROCESSING: MINIMIZATION OF ENTANGLEMENT

A. Inference scheme

In the previous section we showed that the standard method of state estimation, which is the Jaynes principle, is not universal. Since one cannot imagine a fully universal inference scheme, there is no way to obtain full knowledge from partial knowledge; the used inference scheme must depend on the particular context, i.e., on the purposes for which the estimated state is needed. Here we deal with entanglement processing (that is, we would like to use entanglement for some practical purposes). Let us now propose a natural inference scheme for this situation. As mentioned before, the scheme must produce a separable state if only there exists a separable state compatible with the data. A natural procedure, which allows us to avoid producing fake entanglement, is *minimization of entanglement*. Clearly the latter must be somehow quantified. To this end one uses the so-called measures of entanglement, which vanish for separable states (the latter represent no entanglement) [16]. Hence, a reasonable inference scheme should involve minimization of a chosen measure of entanglement. Then one can be sure that if the data could be produced by a separable state, the estimated state would also be separable.

The problem is that the measures are usually not strictly convex. As a consequence, under a given set of constraints of type (1), the state of minimum entanglement does not need to be unique. To overcome the difficulty, we propose to maximize the von Neumann entropy *after* minimization of entanglement. Such a procedure produces a *unique representative state* [19]. We shall denote it by ϱ_E where E is the used measure of entanglement.

B. Bell constraints

Before we provide an example of the use of our scheme, let us introduce the notion of the so-called Bell constraints. They are characterized by the following condition: any state that fulfills the constraints would also satisfy them if subjected to dephasing (removing off-diagonal elements) in the Bell basis. It turns out that for the Bell constraints, the number of the state parameters, which are to be varied within the minimization procedure, can be considerably reduced. This follows from the following lemma.

Lemma. For the Bell constraints the representative state ϱ_E is diagonal in the Bell basis (independently on the used entanglement measure).

Proof. To prove the lemma we note two important properties of the operation of dephasing in the Bell basis (7). Namely, such an operation (i) does not increase entanglement (for any possible entanglement measure) and (ii) does not decrease entropy. In other words, for any state ϱ, we have

$$E(\varrho_B)\leqslant E(\varrho),\quad S(\varrho_B)\geqslant S(\varrho), \tag{11}$$

where ϱ_B is the state resulting from ϱ after performing dephasing in the Bell basis

$$\varrho\rightarrow\varrho_B=\sum_{i=0}^{3} P_i^B\varrho P_i^B \tag{12}$$

with $P_i^B=|\psi_i\rangle\langle\psi_i|$. To see that (i) holds, it suffices to note that the above operation can be alternatively represented as a random application of one of four local unitary transformations (cf. Appendix A in Ref. [20]),

$$\varrho_B=\frac{1}{4}\sum_{i=x,y,z,0}\sigma_i\otimes\sigma_i\varrho\,\sigma_i\otimes\sigma_i, \tag{13}$$

where σ_i, $i=x,y,z$ are Pauli matrices and $\sigma_0=I$ is identity. As such an operation does not require exchange of quantum information between Alice and Bob, it cannot increase entanglement. This is independent on possible used measures. The property (ii) follows from the fact that removing off-diagonal elements in any basis does not decrease the entropy [6].

Let us take the state ϱ_E, which is representative under some Bell constraints. Consider a new state ϱ_E^B given by

$$\varrho_E^B=\sum_i P_i^B\varrho_E P_i^B. \tag{14}$$

By definition of the Bell constraints, ϱ_E^B also satisfies them. According to the properties (i) and (ii), we have $E(\varrho_E^B)\leqslant E(\varrho_E)$ and $S(\varrho_E^B)\geqslant S(\varrho_E)$. But as ϱ_E is the representative state, no other state satisfying the constraints can be less entangled; hence, $E(\varrho_E^B)=E(\varrho_E)$. As the state ϱ_E is unique among the states of minimum entanglement, no other state can have entropy greater than or equal to ϱ_E. As a result we have $\varrho_E^B=\varrho_E$. But this means that the state ϱ_E does not change under the measurement in the Bell basis. This is possible if, and only if, ϱ_E is *diagonal* in this basis. This ends the proof of the lemma.

From the lemma it follows that for the Bell constraints, one can perform the procedure of minimization of entanglement (and subsequent maximization of entropy) *only* over the Bell diagonal states and in this way would obtain the same result as if the procedure were performed over the *whole* set of states satisfying the constraints. Let us stress here that the lemma applies to any entanglement measure, as it uses general features of entanglement, which must be satisfied by any measure.

C. Examples

Here we will apply our scheme to the constraints (9). Of course, as B is diagonal in the Bell basis, it forms Bell constraints. Indeed, for any state ϱ, we have

$$b=\mathrm{Tr}\,\varrho B=\mathrm{Tr}\left(\varrho\sum_i P_i^B B P_i^B\right)=\mathrm{Tr}\left(\sum_i P_i^B\varrho P_i^B B\right). \tag{15}$$

Hence the state after measurement still satisfies the constraints. So we can now deal only with Bell diagonal states, which satisfy the constraints.

In our analysis we will use two measures: entanglement of formation E_f [20] and relative entropy entanglement E_r [21]. Both of them are calculated for the two spin-$\frac{1}{2}$ states diagonal in the Bell basis (7) [22]. In this case both the measures depend only on the largest eigenvalue F of a given state and are increasing functions of F [20,21],

$$E_f=H[\tfrac{1}{2}+\sqrt{F(1-F)}],\quad E_r=\ln 2-H(F), \tag{16}$$

for $F>\frac{1}{2}$ and $E_r=E_f=0$; otherwise, here $H(x)=-x\ln x-(1-x)\ln(1-x)$. Therefore if the state of minimum entanglement is diagonal in the Bell basis, it is of the *same* form for both measures (i.e., $\varrho_{E_f}=\varrho_{E_r}$). So, in our case, the two measures will produce the same representative state, call it ϱ_E. To obtain it we need to minimize the largest eigenvalue of the state of the form

$$\varrho=p_1|\Phi^+\rangle\langle\Phi^+|+p_2|\Psi^-\rangle\langle\Psi^-|+p_3|\Psi^+\rangle\langle\Psi^+|+p_4|\Phi^-\rangle\times\langle\Phi^-|, \tag{17}$$

where $\Sigma_i p_i=1$, $p_i\geqslant 0$, and $p_1-p_2=b/2\sqrt{2}$. Note that if $b\leqslant\sqrt{2}$, then for $p_2=\frac{1}{2}-b/2\sqrt{2}$ the state is separable as then the largest eigenvalue is $p_1=\frac{1}{2}$. This is compatible with the results of Sec. II. We will not calculate the state ϱ_E in this case: if the estimated state is separable (hence useless for quantum communication), one is not especially interested in its particular form. For $b>\sqrt{2}$, the state (17) is always inseparable as $p_1 > \frac{1}{2}$. The latter is minimal if $p_2 = 0$. Then we obtain the family of states with minimal entanglement of the form

$$\varrho = \frac{b}{2\sqrt{2}} |\Phi^+\rangle\langle\Phi^+| + p_3 |\Psi^+\rangle\langle\Psi^+| + p_4 |\Phi^-\rangle\langle\Phi^-|. \tag{18}$$

Subsequently, maximizing the von Neumann entropy, we obtain the representative state ϱ_E of the form

$$\varrho_E = \frac{b}{2\sqrt{2}} |\Phi^+\rangle\langle\Phi^+| + \left(\frac{1}{2} - \frac{b}{2\sqrt{2}}\right)(|\Psi^+\rangle\langle\Psi^+| + |\Phi^-\rangle\langle\Phi^-|), \tag{19}$$

for $b > \sqrt{2}$. As seen, the scheme of minimization of entanglement allows us to check whether there exists a separable state for given constraints. Here we obtained that only for $b \leq \sqrt{2}$ it is the case (we used it in the previous section). For $b > \sqrt{2}$ the data do not admit separable state; still, however, we have a quantitative difference: the Jaynes state exhibits a too ''optimistic'' value of entanglement. Indeed, the largest eigenvalue of ϱ_J is greater than the one of ϱ_E. Hence, following remarks on entanglement measure (16), the first state has greater entanglement than the second one.

Let us now consider the constraints given by the projector P_- corresponding to the singlet state vector Ψ^-. One can check that here $\varrho_E = \varrho_J$ for any mean value $F = \mathrm{Tr}\, \varrho P_-$ and both the states are equal to a suitable Werner state [11,23] (we again deal with Bell constraints). So, in this case, if the Jaynes state is inseparable, then the data certainly do not come from any separable state. This involves an interesting problem: for which type of constraints does the Jaynes scheme fail? However, it goes beyond the scope of this paper.

Finally, it is worth mentioning how the problem of the entanglement processing with incomplete data appeared implicitly in the context of the protocols of entanglement distillation. Namely, the first proposed distillation scheme [24] is based on information about the state given just by the projector P_-. As a result, in contrast with more general schemes that involve full knowledge about the state [25], it works only for $F > \frac{1}{2}$. In the present context, this appears to be a consequence of the fact that the minimum entanglement state for $F \leq \frac{1}{2}$ is separable. Now, if the real state is in fact inseparable, we must gain some more information (i.e., increase the number of observables) to be able to distill the state.

IV. CONCLUDING REMARKS

In conclusion, we have considered the problem of statistical inference of incomplete data in the context of entanglement processing. We have shown that the Jaynes principle when applied to composite quantum systems can produce fake entanglement. We have suggested a statistical inference scheme based on minimization of entanglement.

One can ask what is the place of the two inference schemes (the entropic one and the entanglement one) in quantum communication theory. It seems that they are, in a way, complementary. As the quantum noisy channels are usually described in terms of entanglement [20,26], the scheme proposed in this paper could be a suitable tool for estimation of parameters of quantum channels. On the other hand, the capacity of a quantum source is described by von Neumann entropy [27]. Thus the Jaynes principle is here the natural scheme. Indeed, it has been recently shown that the Jaynes scheme offers optimal compression of quantum information if we have partial knowledge about the source parameters [28].

We hope that the present results will stimulate further effort to clarify the problem of statistical inference for entanglement processing. As the Jaynes scheme plays a fundamental role in the understanding of the statistical thermodynamics [3], we believe that the development of the theory of entanglement (or more generally, quantum information) processing of incomplete data will be crucial in the construction of a kind of ''thermodynamics of entanglement'' [29,17,30]. In particular, in Refs. [30] the distillable entanglement was proposed to be an analogue of free energy. Then, if we used the distillable entanglement as a suitable entanglement measure in our scheme, we would obtain an interesting analogue of minimization of free energy (condition of thermodynamical equilibrium).

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