

# Energy-reflection symmetry of Lie-algebraic problems: Where the quasiclassical and weak-coupling expansions meet

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We construct a class of one-dimensional Lie-algebraic problems based on  $sl(2)$ , where the spectrum in the algebraic sector has a dynamical symmetry  $E \leftrightarrow -E$ . All  $2j+1$  eigenfunctions in the algebraic sector are paired and inside each pair are related to each other by a simple analytic continuation  $x \rightarrow ix$ , except the zero mode appearing if  $j$  is integer. At  $j \rightarrow \infty$  the energy of the highest level in the algebraic sector can be calculated by virtue of the quasiclassical expansion, while the energy of the ground state can be calculated as a weak-coupling expansion. Both series coincide identically. [S1050-2947(99)00703-9]

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## I. INTRODUCTION

A hidden algebraic structure of the quasiexactly solvable (QES) Hamiltonians [1–4] leads to nontrivial dynamical properties of the QES systems. One such property was observed in [5]: It was noted that all levels in the algebraic sector of the simplest QES problem [see Eq. (1) below] are symmetric under  $E \leftrightarrow -E$ . This property will be referred to as the *energy-reflection* (ER) symmetry. In the present paper we derive a class of one-dimensional QES Hamiltonians with this property and explore the consequences of the ER symmetry. A relation between the weak-coupling and quasiclassical expansions will be established.

One-dimensional QES problems are based on a (hidden)  $sl(2)$  algebra; they are characterized by one quantized (cohomology) parameter  $j$ , where  $j$  is a half integer [6,7]. The number of levels in the algebraic sector is  $2j+1$ . In the systems to be constructed below each state in the algebraic sector with the energy eigenvalue  $-E$  ( $E > 0$ ) is accompanied by a counterpart with energy  $E$  if  $2j+1$  is even. If  $2j+1$  is odd, a zero mode exists, while the remaining  $2j$  levels come in pairs  $\{\psi_{-E}, \psi_E\}$ . The eigenfunctions of the ER-symmetric levels are related to each other by a straightforward analytic continuation

$$x \rightarrow ix, \quad \psi_E \rightarrow \psi_{-E}.$$

At large  $j$  the number of states in the algebraic sector is large. The highest levels still belonging to the algebraic sector can be regarded as highly excited states and as such are amenable to the quasiclassical treatment [8]. The parameter of the quasiclassical expansion is  $1/j$ . At the same time, the lowest levels from the algebraic sector are close to those of the harmonic oscillator. The anharmonicity is small and is determined by a small parameter related to  $1/j$ . Under the

circumstances, one can develop a standard weak-coupling perturbation theory and calculate  $E$  as a series in the weak coupling. Since the energy eigenvalues of the highly excited and low-lying ER partners coincide, up to sign, the quasiclassical expansion and the weak-coupling expansion in the QES problems with the ER symmetry *must be identical*. We discuss how this identity is implemented, taking as a representative example the ground state and its counterpart.

The simplest QES problem with the ER symmetry known for a long time [5] is the sextic anharmonic oscillator, with a quantized coefficient in front of  $x^2$ ,

$$H = \frac{1}{2}[p^2 + (x^6 - (8j+3)x^2)], \quad j = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots \quad (1)$$

In this case the algebraic sector consists of  $2j+1$  levels of positive parity. The general QES potential possessing the ER symmetry involves certain elliptic functions and is related to some problems of practical importance.

## II. GENERALITIES

The strategy we follow is described in [1] (see also [7]) while the notation is borrowed from [5]. The generators of the  $sl(2)$  algebra are defined as

$$T^+ = 2j\xi - \xi^2 \frac{d}{d\xi}, \quad T^0 = -j + \xi \frac{d}{d\xi}, \quad T^- = \frac{d}{d\xi}. \quad (2)$$

If  $j$  is a non-negative half-integer number, a finite-dimensional irreducible representation exists,

$$R_{2j+1} = \{\xi^0, \xi^1, \dots, \xi^{2j}\}, \quad (3)$$

where the subscript indicates the dimension of the representation. In general, the generators  $T^\pm$  have the meaning of the raising (lowering) operators

$$T^\pm R_n \rightarrow R_{n\pm 1}, \quad \text{while} \quad T^0 R_n \rightarrow R_n.$$

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The generic QES  $sl(2)$ -based Hamiltonian is representable as a quadratic combination of the generators  $T^\pm$  and  $T^0$ ,

$$\hat{H} = \sum_{\pm, 0} (C_{ab} T^a T^b + C_a T^a) + C, \tag{4}$$

where  $C_{ab}$ ,  $C_a$ , and  $C$  are parameters. One can always eliminate  $C_{0+}$  and  $C_{-0}$  in favor of  $C_{+0}$  and  $C_{0-}$ , respectively, due to the  $sl(2)$  commutation relations. Moreover,  $C_{+-}$  and  $C_{-+}$  can be eliminated in favor of  $C_{00}$ , as a consequence of the irreducibility of the representation  $R_{2j+1}$  [i.e.,  $T^+ T^- + T^- T^+ + 2T^0 T^0 = 2j(j+1)$ ]. The reference point for the energy is fixed by setting  $C = 0$ .

After a change of variable and a (quasi)gauge transformation the operator  $\hat{H}$  can be always reduced to the Schrödinger form

$$\hat{H} \rightarrow H \equiv e^{-a(\xi)} \hat{H} e^{a(\xi)} \Big|_{\xi=\xi(x)} = -\frac{1}{2} \frac{d^2}{dx^2} + V(x). \tag{5}$$

The key element in constructing the QES Hamiltonians with the ER symmetry is the following observation [9]: Any tridiagonal matrix of the form

$$\begin{pmatrix} 0 & u_1 & 0 & 0 & \dots & 0 \\ \ell_1 & 0 & u_2 & 0 & \dots & 0 \\ 0 & \ell_2 & 0 & u_3 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & u_n \\ 0 & 0 & 0 & \dots & \ell_n & 0 \end{pmatrix} \tag{6}$$

leads to the characteristic equation

$$\begin{aligned} EP_{n/2}(E^2) &= 0 \quad (n \text{ even}), \\ \tilde{P}_{(n+1)/2}(E^2) &= 0 \quad (n \text{ odd}), \end{aligned} \tag{7}$$

where  $P_{n/2}(z)$  and  $\tilde{P}_{(n+1)/2}(z)$  are polynomials of  $z$  of degree  $n/2$  and  $(n+1)/2$ , respectively, and  $n$  is defined in Eq. (6). Thus any matrix of the form (6) guarantees the ER symmetry of the spectrum. It is evident that the Lie-algebraic Hamiltonian (4) has the matrix representation (6) provided the sum in Eq. (4) does *not* include the terms  $T^+ T^+$ ,  $T^- T^-$ ,  $T^0 T^0$ , and  $T^0$ . Thus the most general form of the Lie-algebraic Hamiltonian compatible with Eq. (6), which, as was explained, ensures the ER symmetry in the algebraic sector, is<sup>1</sup>

$$\begin{aligned} \hat{H} &= \alpha T^+ T^0 + \beta T^0 T^- + \gamma T^+ + \delta T^- \\ &= A(\xi) \frac{d^2}{d\xi^2} + B(\xi) \frac{d}{d\xi} + C(\xi), \end{aligned} \tag{8}$$

where  $\alpha$ ,  $\beta$ ,  $\gamma$ , and  $\delta$  are numerical constants and  $A$ ,  $B$ , and  $C$  are polynomials in  $\xi$  of the third, second, and first degree, respectively,

$$\begin{aligned} A(\xi) &= -\alpha \xi^3 + \beta \xi, & B(\xi) &= [\alpha(3j-1) - \gamma] \xi^2 + (\delta - \beta j), \\ C(\xi) &= 2(\gamma j - \alpha j^2) \xi. \end{aligned} \tag{9}$$

Not all of the four constants above represent physically interesting parameters. In general, two constants can be fixed by a combination of rescalings of the variable  $\xi$  and the energy. Using this freedom and starting from nonvanishing  $\alpha$  and  $\beta$ , one can always reduce them to the ‘‘standard’’  $\alpha = \beta = -2$  (see below). The parameters  $\gamma$  and  $\delta$  remain free.

Requiring the matrix (6) to have nonvanishing eigenvalues leads to a constraint that neither the parameters  $(\alpha, \gamma)$  nor  $(\beta, \delta)$  can be set to zero. One of the parameters in each pair can vanish however. For instance, if  $\alpha = 0$  the general elliptic potential degenerates into a polynomial potential. Thus the example presented in Eq. (1) is nothing but a degenerate case of Eq. (8): It corresponds to  $\alpha = 0$ ,  $\beta = \delta = -2$ , and  $\gamma = -(2j+1)$ .

Needless to say,  $j$  is an additional free parameter taking a discrete set of values. Thus we deal with the three-parameter family of potentials: two continuous and one discrete.

### III. ELLIPTIC POTENTIALS: SPECIAL CASE

Prior to considering the general QES potentials with the ER symmetry we find it illuminating to discuss a few representative examples. We start from

$$\alpha = \beta = -2, \quad \gamma = -(8\nu + 6j + 1), \quad \delta = -(2j + 1), \tag{10}$$

where  $\nu$  is a constant. Since  $\nu$  is free, so is  $\gamma$ ; the parametrization of  $\gamma$  above, in terms of  $\nu$  and  $j$ , will be considered standard. Physical arguments (e.g. the stability of the potential) require  $\nu$  to be non-negative. The parameter  $\delta$  is fixed for the time being. Later on we will let  $\delta$  vary too.

The physical variable  $x$  in Eq. (5) is determined by the inversion of the equation

$$\left(\frac{d\xi}{dx}\right)^2 = 4\xi - 4\xi^3 \tag{11}$$

(see [1,3]). Equation (11) has the solutions  $-\mathcal{P}(x)$  and  $1/\mathcal{P}(x)$ , where  $\mathcal{P}$  is the Weierstrass function. One could use either of them; the second solution is more convenient for our purposes. Thus

$$\xi(x) = \frac{1}{\mathcal{P}(x)}, \quad g_2 = 4, \quad g_3 = 0, \tag{12}$$

where  $g_{2,3}$  are the invariants of the Weierstrass function. For the time being it is assumed that  $\xi \in [0, 1]$  and  $x \in [0, x_*]$ , where

$$x_* = \int_0^1 \frac{d\xi}{2\sqrt{\xi - \xi^3}} = \frac{\sqrt{\pi}\Gamma(5/4)}{\Gamma(3/4)} \approx 1.311. \tag{13}$$

<sup>1</sup>The representation (6) is a sufficient but not necessary condition. At some specific values of  $n$  there may exist QES systems with the ER symmetry that do not fall in the class of systems we built.

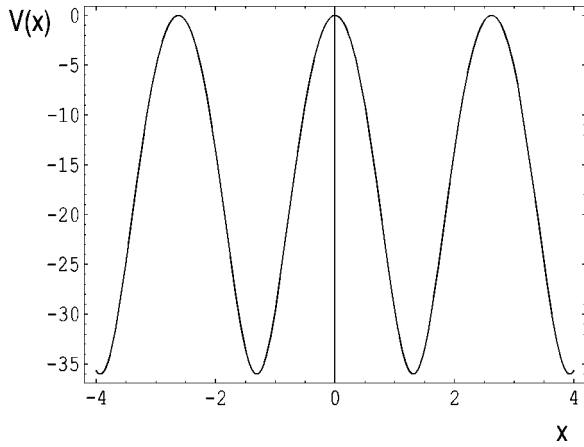


FIG. 1. Periodic QES potential of Eq. (17) at  $\nu=0$  and  $j=2$ .

Later on this constraint will be relaxed. Equation (12) maps the interval  $[0, x_*]$  onto  $[0, 1]$ . The function  $\xi(x)$  is double periodic in the complex plane, with the periods  $2x_*$  and  $2ix_*$ . Thus, under our choice of parameters, the parallelogram of periods of the Weierstrass function becomes square. The symmetry of the square immediately translates in the ER symmetry of the quantal problem at hand. We will use the fact that

$$\xi(-x) = \xi(x), \quad \xi(ix) = -\xi(x), \quad (14)$$

stemming from the properties of the Weierstrass function with the above periods. The expansion of  $\xi(x)$  at  $x=0$  runs in powers of  $x^2$ .

The phase  $a(x)$  of the gauge transformation and the ‘‘gauge potential’’  $\mathcal{A}(x)$  are

$$a(x) = \frac{1}{4} \int \left. \frac{(dA/d\xi) - 2B}{A} d\xi \right|_{\xi=\xi(x)} = -\nu \ln(1 - \xi^2) \Big|_{\xi=\xi(x)} \quad (15)$$

and

$$\mathcal{A} = -\frac{1}{2} \left. \frac{(dA/d\xi) - 2B}{\sqrt{-2A}} = \frac{4\nu\xi^{3/2}}{\sqrt{1-\xi^2}} \right|_{\xi=\xi(x)}. \quad (16)$$

As a result, we get the following potential in the Schrödinger operator (5):

$$V(x) = \left\{ \frac{4\nu(2\nu-1)\xi^3}{1-\xi^2} - (8j^2 + 2j + 16\nu j + 6\nu)\xi \right\}_{\xi=\xi(x)}. \quad (17)$$

[The general formula for calculating the corresponding QES potential in the case at hand reduces to

$$V(x) = \left\{ C(\xi) + \frac{1}{2} [\mathcal{A}(\xi)]^2 - \frac{d\mathcal{A}(\xi)}{d\xi} \sqrt{\xi - \xi^3} \right\}_{\xi=\xi(x)},$$

where  $C(\xi)$  is defined in Eq. (9).]

This Schrödinger problem is quasiexactly solvable and can be considered beyond the original interval  $[0, x_*]$ . For  $\nu=0$  we deal with the periodic potential defined on the entire  $x$  axis (Fig. 1), which is akin to the Lamé problem [10].

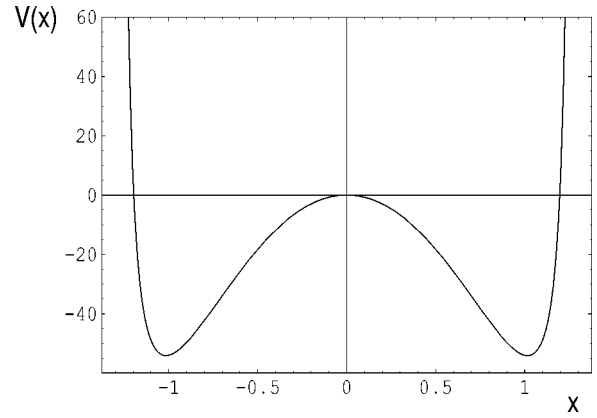


FIG. 2. QES potential of Eq. (17) at  $\nu=1$  and  $j=2$  defined on the interval  $-x_* < x < x_*$ . The potential has a double-well shape.

If  $\nu > 1/2$  the potential is singular at  $x = \pm x_*$  (Fig. 2) and the problem is defined at  $x \in (-x_*, x_*)$ . The condition  $\nu > 1/2$  is necessary for stability. The Hamiltonian changes sign under the transformation  $x \rightarrow ix$ ,

$$H \rightarrow -H, \quad x \rightarrow ix,$$

as follows from Eq. (14). The eigenfunctions  $\{\psi_E, \psi_{-E}\}$  in each pair interchange. The ER symmetry is explicit. Let us consider separately two cases.

(i)  $\nu=0$ . The potential and the eigenfunctions take the form

$$V(x) = -\frac{2j(4j+1)}{\mathcal{P}(x)}, \quad \psi(x) = P_{2j}(\xi), \quad (18)$$

where  $P_{2j}$  is a polynomial of degree  $2j$ . At positive  $E$  the spectrum is continuous, while the counterpart wave functions  $\psi_{-E}$  with negative-energy eigenvalues correspond to the boundaries of the Bloch zones; all these eigenfunctions are periodic. A very similar system, with a different coefficient in front of  $1/\mathcal{P}(x)$ , emerges at  $\nu=1/2$ .

(ii)  $\nu > 1/2$ . The potential has a double-well form (Fig. 2). The singularity at  $x \rightarrow \pm x_*$  is of the form  $(x \pm x_*)^{-2}$ . The wave functions must vanish at  $x = \pm x_*$  and the spectrum is discrete. The algebraic sector includes the ground state and  $2j$  excited states symmetric under  $x \rightarrow -x$ . If  $j$  is an integer, one level lies exactly at zero;  $j$  levels are below and above zero, respectively. If  $j$  is a half-integer,  $(2j+1)/2$  levels lie below zero and the same number above.

#### IV. GENERIC ELLIPTIC POTENTIALS WITH THE ER SYMMETRY

To proceed to the general case we invoke the only remaining freedom and let  $\delta$  in Eq. (9) float. The following parametrization will be used:

$$\delta = -(2j + 1) - \mu, \tag{19}$$

while  $\alpha$ ,  $\beta$ , and  $\gamma$  are the same as in Eq. (10). If  $\mu \neq 0$  Eqs. (15)–(17) are modified as

$$a(x) = \left\{ -\nu \ln(1 - \xi^2) - \frac{\mu}{8} \ln \frac{\xi^2}{1 - \xi^2} \right\}_{\xi = \xi(x)}, \tag{20}$$

$$A = \left\{ \frac{4\nu\xi^{3/2}}{\sqrt{1 - \xi^2}} - \frac{\mu\xi^{-1/2}}{2\sqrt{1 - \xi^2}} \right\}_{\xi = \xi(x)}, \tag{21}$$

and

$$V(x) = \left\{ \frac{4\nu(2\nu - 1)\xi^3}{1 - \xi^2} - (8j^2 + 2j + 16\nu j + 6\nu)\xi + \frac{\mu}{4\xi} \frac{1}{1 - \xi^2} \left[ \frac{\mu}{2} - (1 - 3\xi^2) \right] - \frac{2\nu\mu\xi}{1 - \xi^2} \right\}_{\xi = \xi(x)}, \tag{22}$$

where  $\xi(x)$  is the same as before; see Eq. (12). By inspecting this potential one concludes on physical grounds that

$$8\nu - 4 > \mu \geq 2. \tag{23}$$

Since the potential (22) is singular at  $x \rightarrow 0$  (it explodes as  $1/x^2$ ) the problem is defined for  $\mu > 2$  on the interval  $[0, x_*]$ . If  $\mu = 2$  there is no singularity at  $x = 0$  and the problem is defined on the interval  $[-x_*, x_*]$ . The potential is depicted in Fig. 3.

**V.  $x \rightarrow ix$  AND  $H \rightarrow -H$**

Since the Hamiltonian we built possesses this property one might ask why the ER symmetry is realized only in the algebraic sector rather than for the whole spectrum. From Fig. 2 it is quite obvious that the whole spectrum cannot have this symmetry: The states at positive  $E$  stretch indefinitely, while for negative  $E$  the lowest level is the ground state. Although the answer to this question is rather obvious, an explanation is in order.

At arbitrary  $E$  the second-order differential equation  $H\psi = E\psi$  has two linearly independent solutions; let us call them  $\psi_{1,2}$ . Generically, both are non-normalizable. For  $E = E_n$  one of the solutions (say,  $\psi_1$ ) is normalizable; the other is not.

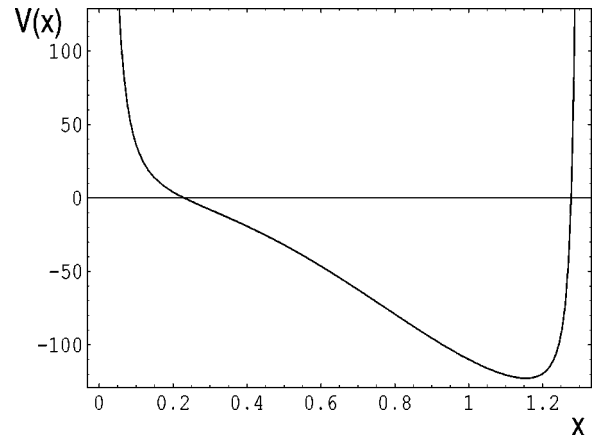


FIG. 3. QES potential of Eq. (22) at  $\nu = 1$ ,  $\mu = 3$ , and  $j = 3$  defined on the interval  $0 < x < x_*$ .

Generically, for arbitrary  $n$ , the transformation  $x \rightarrow ix$  connects a normalizable solution at positive  $E$  to a non-normalizable one at negative  $E$ . The latter does not lie in the physical Hilbert space and there is no physical symmetry  $E \leftrightarrow -E$ . However, if  $n$  belongs to the algebraic sector, the transformation  $x \rightarrow ix$  does not generate a non-normalizable solution since the phase factor  $a$  is invariant under  $x \rightarrow ix$ . Rather, in this case  $x \rightarrow ix$  connects a normalizable solution at positive  $E$  to another normalizable solution at negative  $E$ . Both belong to the physical Hilbert space and  $E \leftrightarrow -E$  is a valid symmetry.

The argument above is somewhat simplified; we cut corners. Although the conclusion is perfectly valid, the careful treatment would require the explicit introduction of the Stokes lines and consideration of sectors in the complex plane. It is important that in the problems with the ER symmetry under consideration, one does *not* jump from one branch onto another in the process of the analytic continuation from the purely real to purely imaginary values of  $x$ . In this respect the situation is different from that discussed in [11], where the emphasis was on problems with the branch intertwining and the leaps from one branch onto another.

**VI. THE ZERO MODE**

If  $j$  is integer, there always exists a solution of the equation  $\hat{H}\tilde{\psi} = E\tilde{\psi}$  with the vanishing energy eigenvalue  $E = 0$ . The corresponding wave function  $\tilde{\psi}_0(\xi)$  contains only even powers of  $\xi$ , so that it is invariant under  $x \rightarrow ix$ ; it can be given in the closed form

$$\tilde{\psi}_0(\xi) = \sum_{k=0}^j \frac{j!}{k!(j-k)!} \frac{[\gamma - \alpha j][\gamma - \alpha(j-2)] \cdots [\gamma - \alpha(j-2k+2)]}{[(j-1)\beta - \delta][\beta - \delta] \cdots [(j-2k+1)\beta - \delta]} \xi^{2k}. \tag{24}$$

**VII. QUASICLASSICAL VS WEAK-COUPLING EXPANSIONS**

Consider the lowest and the highest levels in the algebraic sector in the limit  $j \gg 1$ . We will denote them by  $\psi_{-E_0}$  and  $\psi_{E_0}$ , respectively. The highest level is highly excited and as such is amenable to the quasiclassical treatment. The quasiclassical calculation of the energy of  $\psi_{E_0}$  was carried out<sup>2</sup> in Ref. [8]. On the other hand,  $\psi_{-E_0}$  corresponds to a system at the bottom of the well. This system is close to the harmonic oscillator, with a weak anharmonicity. One can develop the standard perturbation theory. The quasiclassical expansion and the weak-coupling expansion have one and the same parameter and coincide term by term, up to the overall sign [12].

Although the assertion above is quite general and refers to all QES problems with the ER symmetry, we will elucidate it using the simplest example. This will allow us to avoid bulky formulas. The generalization to the general case is transparent.

As was noted in [5], the simplest QES problem with the ER symmetry is that of Eq. (1). It is convenient to introduce a slightly different notation

$$\hat{H} = -2T^0T^- - (2j+1)T^- - 2T^+,$$

$$H\psi \equiv \left\{ -\frac{1}{2} \frac{d^2}{dx^2} + \left[ \frac{x^6}{2} - \frac{\kappa}{2}x^2 \right] \right\} \psi(x) = E\psi(x), \quad (25)$$

where  $\kappa = 8j + 3 = 3, 7, 11, 15, \dots$ . We are interested in the limit  $\kappa \rightarrow \infty$ .

At large  $\kappa$  the depth of the double-well potential becomes large and the well width small. The minima of  $V(x)$  lie at

$$x = \pm x_0, \quad x_0 \equiv (\kappa/3)^{1/4}. \quad (26)$$

Near, say, the right minimum

$$V(x) = -\frac{(\kappa)^{3/2}}{3\sqrt{3}} + 2\kappa(x-x_0)^2 + 10\left(\frac{\kappa}{3}\right)^{3/4}(x-x_0)^3 + \dots, \quad (27)$$

where the ellipsis denotes quartic- and higher-order terms; a similar expansion is valid near the left minimum. From Eq. (27) it is easy to find the ground-state energy in the form of an expansion in  $1/\kappa$ . Indeed, if we neglect exponential terms of the type  $\exp(-C\kappa)$ , where  $C$  is a positive constant, arising due to tunneling from one well into another,<sup>3</sup> the ground-state level can be considered as that of the harmonic oscillator slightly perturbed by cubic, quartic, etc., terms. The lead-

<sup>2</sup>Note that the unnumbered equation after Eq. (12) in [8] contains errors in signs. These errors miraculously combine with another error [the parameter of the quasiclassical quantization  $n$  was taken to be  $(\kappa+1)/2$  in Ref. [8], while actually it is  $(\kappa-3)/2$ ] to annihilate each other. The final expansion for the energy  $E_0$  presented in Eq. (13) of Ref. [8] is perfectly valid. Note that our  $\kappa$  corresponds to  $4J-1$  in [8].

<sup>3</sup>The impact of tunneling and the issue of the analytic structure in  $\kappa$  are discussed in more detail in Sec. VIII.

ing term in the ground-state energy  $E_0$  is just the classical energy of the particle at rest in the minimum, i.e., at  $x_0$ ,

$$-\frac{(\kappa)^{3/2}}{3\sqrt{3}}.$$

The next-to-leading term is the zero-point oscillation energy of the harmonic oscillator

$$\frac{\omega}{2} = \kappa^{1/2}.$$

Then come the corrections due to the anharmonic terms in the potential (27). The first-order correction due to the cubic term obviously vanishes. Therefore, the next term in the  $1/\kappa$  expansion of  $E_0$  comes from the quartic term in Eq. (27) (treated as a first-order perturbation) plus the second-order perturbation generated by  $10(\kappa/3)^{3/4}(x-x_0)^3$ :

$$\frac{135}{32} \frac{1}{3\sqrt{3}\kappa^{1/2}} - \frac{275}{32} \frac{1}{3\sqrt{3}\kappa^{1/2}},$$

respectively. Assembling all these terms together we arrive at the following expansion for the ground state energy  $-E_0$ :

$$-E_0 = -\left(\frac{\kappa}{3}\right)^{3/2} \left[ 1 - \frac{3\sqrt{3}}{\kappa} + \frac{35}{8\kappa^2} + O(\kappa^{-3}) \right]. \quad (28)$$

So far only the lowest level was discussed. What can be said about the highest level in the algebraic sector? The ER symmetry implies that the last level belonging to the algebraic sector has the energy

$$E_0 = \left(\frac{\kappa}{3}\right)^{3/2} \left[ 1 - \frac{3\sqrt{3}}{\kappa} + \frac{35}{8\kappa^2} + O(\kappa^{-3}) \right]. \quad (29)$$

Being considered as an excited state from the full set of states of the Hamiltonian (25), this level should have been labeled by  $(\kappa-3)/2$ . Indeed,  $\psi_{-E_0}$  is the ground state, then comes the first  $P$ -odd state, the first  $P$ -even excitation, etc. The ground state and  $2j$   $P$ -even excitations belong to the algebraic sector. The last ( $P$ -even) state from the algebraic sector has

$$n = \frac{\kappa-3}{2}, \quad (30)$$

where  $n$  is the number of zeros in the corresponding wave function.

Let us discuss now how the very same expansion for  $E_0$  emerges in the WKB approximation [8]. It is instructive to start from the leading WKB approximation. Bohr and Sommerfeld's quantization rule at large  $n$  implies that

$$\int_b^a p dx = n\pi = \frac{\kappa\pi}{2}, \quad \kappa \rightarrow \infty, \quad (31)$$

where

$$p = \sqrt{2E - x^6 + \kappa x^2} \quad (32)$$

and  $a$  and  $b$  are the turning points. We check that Eq. (31) is satisfied at  $E = E_0 = (\kappa/3)^{3/2}$ . It is convenient to rescale the coordinate  $x$ ,

$$x = 2^{1/2} 3^{-1/4} \kappa^{1/4} y. \tag{33}$$

Then

$$\begin{aligned} p(E_0) &= \sqrt{2} \left(\frac{\kappa}{3}\right)^{3/4} \sqrt{1 - 4y^6 + 3y^2} \\ &= \sqrt{2} \left(\frac{\kappa}{3}\right)^{3/4} \sqrt{(1 - y^2)(1 + 2y^2)}. \end{aligned} \tag{34}$$

At  $E = E_0$  the expression for  $E - V$  factorizes and the integral  $\int p dx$ , which in general is representable through elliptic functions, in fact reduces to elementary functions [8]. Owing to factorization, we immediately see that the turning points are at  $y = \pm 1$ ,

$$\int_b^a p dx = \frac{4}{3} \kappa \int_0^1 dy (1 + 2y^2) \sqrt{1 - y^2} = \frac{\kappa \pi}{2}. \tag{35}$$

Q.E.D.

The first correction in the quasiclassical expansion can be calculated as easily as the leading term. Indeed, at this level the only change to be done is the substitution

$$n \rightarrow n + \frac{1}{2} = \frac{\kappa}{2} - 1 \tag{36}$$

in the WKB quantization condition (31) and

$$E \rightarrow E_0 = \left(\frac{\kappa}{3}\right)^{3/2} \left(1 - \frac{C_1}{\kappa}\right), \tag{37}$$

where  $C_1$  is a numerical coefficient, to be determined from the quantization condition

$$\int_b^a p(E_0) dx = \left(\frac{\kappa}{2} - 1\right) \pi. \tag{38}$$

Now  $p(E_0)$  takes the form

$$\begin{aligned} p(E_0) &= \sqrt{2} \left(\frac{\kappa}{3}\right)^{3/4} \sqrt{1 - \frac{C_1}{\kappa} - 4y^6 + 3y^2} \\ &= \sqrt{2} \left(\frac{\kappa}{3}\right)^{3/4} \left[ \sqrt{(1 - y^2)(1 + 2y^2)^2} \right. \\ &\quad \left. - \frac{C_1}{2\kappa} \frac{1}{\sqrt{(1 - y^2)(1 + 2y^2)^2}} + \dots \right]. \end{aligned} \tag{39}$$

We have already checked that the  $O(\kappa)$  term in Eq. (38) (which corresponds to keeping the first term in the square brackets) implies  $E_0 = (\kappa/3)^{3/2}$ . Matching of the  $O(\kappa^0)$  term in Eq. (38) (which corresponds to the second term in the square brackets) yields

$$C_1 = 3\sqrt{3}, \tag{40}$$

in full accord with Eq. (29).

Next-to-leading corrections in the quasiclassical expansion are calculated too [8]; see also footnote 1 above. The third and higher terms in the expansion require certain modifications of the WKB quantization condition that go beyond Eq. (38). From what we already know about the QES systems under consideration, it is clear that the  $1/\kappa$  expansion of  $E_0$  obtained through the WKB quantization condition must match the weak-coupling expansion. Six terms in the quasiclassical expansion of  $E_0$  were found in Ref. [8]. Although it was expected, it was amusing to observe the coincidence with the first six terms in the weak-coupling expansion.

### VIII. HIGH-ORDER BEHAVIOR OF THE EXPANSION

It goes without saying that the weak-coupling expansion (28) is asymptotic. This is due to the possibility of the ‘‘leakage’’ from the right to the left well. The high-order terms are factorially divergent and of the same sign. The behavior of the high-order terms in the  $1/\kappa$  series for the ground-state energy is determined [13] by the action of the instanton, the classical trajectory connecting the left and right minima in the Euclidean time. Let the instanton action be  $S_0 \kappa$ , where  $S_0$  is a number that we will calculate shortly. The Borel-resummed expression for the ground-state energy has the form

$$E_0 \sim \int dg \frac{e^{-1/g}}{g - (2S_0 \kappa)^{-1}}, \tag{41}$$

where the principle-value prescription applies. The imaginary part of the integral is canceled by the imaginary part coming from the instanton–anti-instanton transition, which in turn is proportional [13] to  $\exp(-2S_0 \kappa)$ . The condition of cancellation fixes the denominator of the integrand. Expanding Eq. (41) in  $1/\kappa$ , we find the high-order tail of the  $E_0$  expansion

$$-E_0 \sim -\kappa^{3/2} \sum_{n > n_0} n! \frac{1}{(2S_0 \kappa)^n}, \tag{42}$$

where  $n_0$  is an integer large enough for the asymptotics to set in. The instanton action is readily calculable

$$S_0 \kappa = \int_{-x_0}^{x_0} dx \sqrt{\frac{2\kappa^{3/2}}{3\sqrt{3}} + x^6 - \kappa x^2}, \quad x_0 = \left(\frac{\kappa}{3}\right)^{1/4}, \tag{43}$$

from where we obtain

$$S_0 = \ln \frac{1 + \sqrt{3}}{\sqrt{2}} \approx 0.658. \tag{44}$$

The ER symmetry and Eq. (42) imply that the very same factorial divergence is inherent to the quasiclassical expansion for energies of the highly excited states. Certainly, this phenomenon is known in the literature [14]. We find the argument above to be an illuminating way of demonstrating the asymptotic nature of the quasiclassical expansion. In fact, it is likely that the asymptotic regime starts quite early. Indeed, the first five coefficients in the quasiclassical expansion

can be inferred from Ref. [8]. Denote the coefficients in front of  $1/(2S_0\kappa)^n$  by  $C_n$ . Then, from Eq. (13) of this work we get

$$C_3 \approx 6.57, \quad C_4 \approx 20.2, \quad C_5 \approx 117,$$

to be compared with the asymptotic prediction (42)

$$C_3 = 3! = 6, \quad C_4 = 4! = 24, \quad C_5 = 5! = 120.$$

Barring the possibility of a coincidental proximity, we conclude that  $n_0$  can be as low as 3.

The parameter  $\kappa$  is related to the cohomology parameter and is quantized. The nature of the  $1/\kappa$  expansions is closely related to the singularity structure in the complex  $\kappa$  plane. In discussing this structure one should exercise caution since the analytic continuation is performed from a discrete set of points  $\kappa = 3, 7, 11, \dots$ . This is one of the reasons why the singularity structure in the complex  $\kappa$  plane turns out to be totally different from that discussed in earlier works [11], devoted to the analytic continuation in continuous parameters in the QES problems. There are also some other reasons responsible for the distinctions, e.g.,  $\kappa$  appears as a coefficient of a subleading term in the potential, which is important. We do not dwell on this issue here since it deserves a dedicated analysis.

The quasiclassical quantization and the associated expansion imply that  $\kappa$  is an integer (more exactly,  $\kappa = 3, 7, 11, \dots$ ). At the same time, the weak-coupling expansion (28) is the same independently of whether or not  $\kappa \in \{3, 7, 11, \dots\}$ . It holds for any sufficiently large  $\kappa$ . Both expansions coincide order by order, to any finite order; yet if  $\kappa \notin \{3, 7, 11, \dots\}$  the physical ER symmetry is absent and there is no reason for the coincidence of the absolute values of energy. This means that the factorially divergent weak-coupling series and the quasiclassical expansion, presented in the square brackets in Eqs. (28) and (29), respectively, define, generally speaking, two distinct functions, despite the fact that the expansions *per se* are identical, order by order. The difference between these two functions is of the type  $\sin(\pi\kappa)\exp(-C\kappa)$ ; it vanishes at  $\kappa = 3, 7, 11, \dots$ . For these and only these values of  $\kappa$ , making a full  $2\pi$  circle in the complex plane around  $\kappa = \infty$ , starting from a positive  $\kappa$  and returning to the very same point, we smoothly interpolate between the lowest and the highest levels in the algebraic sector; their positions interchange.

### IX. THE ER SYMMETRY IN THE FINITE-DIFFERENCE PROBLEMS

We have to mention that the energy reflection symmetry appears also in quantum-algebraic problems with the Hamiltonians built from finite-difference operators (such problems naturally emerge in solid-state physics). In order to display this property let us consider, for instance, the dilatation-invariant discrete operator  $D_\xi$  defined as

$$D_\xi f(\xi) = \frac{f(\xi) - f(q\xi)}{(1-q)\xi}, \quad (45)$$

also known as the Jackson derivative (see, e.g., [15,16]). Here  $q$  is a complex number. In the limit  $q \rightarrow 1$  the Jackson symbol obviously goes into the conventional derivative.

Now one can easily introduce [17] a finite-difference analog of the algebra of the differential operators (2) based on the operator  $D_\xi$ , instead of the continuous derivative (for a discussion see [7])

$$\tilde{T}^+ = \{n\} \xi - \xi^2 D_\xi, \quad \tilde{T}^0 = -\hat{n} + \xi D_\xi, \quad \tilde{T}^- = D_\xi, \quad (46)$$

where

$$\{n\} = \frac{1 - q^n}{1 - q}$$

is the so-called  $q$  number and

$$\hat{n} \equiv \frac{\{n\}\{n+1\}}{\{2n+2\}}.$$

It is easy to check that the operators (46) obey the commutation relations of the quantum algebra  $sl_{2q}$  for any value of the parameter  $j = n/2$  (see, e.g., [18]). If  $j$  is a non-negative integer, the finite-dimensional representation (3) of the algebra (46) exists; it is irreducible when  $q$  is not a prime root of unity. The same line of reasoning that we followed to demonstrate the ER symmetry of the Hamiltonian (8) can be used in the case of the finite-difference generators (46). In this way we arrive at the conclusion that the discrete Hamiltonian

$$\begin{aligned} \hat{H} &= \alpha \tilde{T}^+ \tilde{T}^0 + \beta \tilde{T}^0 \tilde{T}^- + \gamma \tilde{T}^+ + \delta \tilde{T}^- \\ &= \tilde{A}(\xi) D_\xi^2 + \tilde{B}(\xi) D_\xi + \tilde{C}(\xi) \end{aligned} \quad (47)$$

possesses the ER symmetry. Here  $\alpha$ ,  $\beta$ ,  $\gamma$ , and  $\delta$  are numerical constants and  $\tilde{A}$ ,  $\tilde{B}$ , and  $\tilde{C}$  are polynomials of the third, second, and first degree in  $\xi$ , respectively,

$$\begin{aligned} \tilde{A}(\xi) &= -\alpha q \xi^3 + \beta \xi, \\ \tilde{B}(\xi) &= [\alpha(\{n\} + \hat{n} - 1) - \gamma] \xi^2 + (\delta - \beta\{n\}), \end{aligned} \quad (48)$$

$$\tilde{C}(\xi) = \{n\}(\gamma - \alpha \hat{n}) \xi,$$

where  $n = 2j$ .

It is remarkable that a particular form of this quantum-algebraic Hamiltonian (with a slightly different definition of the discrete derivative) appears in the Azbel-Hofstadter problem of the electron motion on the two-dimensional lattice in the transverse constant magnetic field [19,20]. In this case the parameter  $q$  is a prime root of unity; it is related to the magnetic flux through the lattice plaquette (the flux is given by a rational number with an even denominator).

## X. COMMENT ON THE LITERATURE

In Ref. [21] a certain ‘‘duality’’ transformation was suggested for the QES systems that inverts the signs of all levels belonging to the algebraic sector and simultaneously changes the form of the potential in a concerted way. It was observed that the potential (1) is self-dual. Thus the ER symmetry of the Schrödinger problem (1) was rediscovered. It was noted then that the quasiclassical treatment of the QES problems should be qualitatively different from that of ‘‘conventional’’ problems, where there is no (quasi)exact solvability. The corresponding remark in [21] is rather vague and we feel that an explanation is in order here.

Suppose that the wave functions of a quantal system are treated in the WKB approximation. The WKB asymptotics, being considered in the complex  $x$  plane, contains singularities at the points where the classical momentum vanishes. The Stokes lines are attached to these points; they divide the complex  $x$  plane into several sectors. The appropriate WKB expression for the wave function in the given sector, when analytically continued across a Stokes line, may or may not match the appropriate WKB expression in another sector. In other words, distinct asymptotics may apply in the different sectors in the complex plane. This is a general situation. In the QES problems, for those levels that are determined algebraically, the wave function is analytic everywhere except

infinity.<sup>4</sup> One and the same asymptotics remains valid in all sectors; one can freely do analytic continuations across the Stokes lines. The singularities of separate parts of the WKB expressions for the wave functions are superficial; they cancel when all parts are assembled together. This property is well known in the harmonic oscillator; it extends to all QES systems however.

The observation above belongs to Vainshtein. He pointed out that the requirement of cancellation of these apparent singularities can be used in order to generate QES potentials. This requirement acts as a substitute of the algebraic structure within the Lie-algebraic approach.

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<sup>4</sup>It is implied that the original problem is defined on  $x \in (-\infty, \infty)$ , as in Eq. (1). If the original problem is formulated on a finite interval, the wording must be changed appropriately.

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