# Radiation modes of a cavity with a resonantly oscillating boundary

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We present exact analytic solutions to describe the quantum dynamics of the radiation modes in a one-dimensional cavity vibrating at any one of its resonance frequencies. For a cavity vibrating at its $m$th ($m \geqslant 2$) eigenfrequency, $m$ traveling wave packets emerge in the finite part of the field energy density and their amplitudes grow while their widths shrink in time, representing a large concentration of energy. There also exists a ''sub-Casimir'' region where the energy density equals $m^2$ times the static Casimir energy density. [S1050-2947(99)09902-3]

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Since Moore's pioneering work in 1970 [1], there have been intensive studies on the quantum theory of the electromagnetic field in a cavity with moving boundaries [2–11]. The topic is of fundamental theoretical interest in that it reveals a number of delicate features of quantum physics such as the dynamical modification of the Casimir force [3] and the vacuum emission of photons with nonclassical photon statistics [4–6,10]. On the other hand, the subject is also of practical importance since it may be closely related to sonoluminescence [11,12], high-precision optical interferometry [13], the generation of squeezed light [14], and quantum nondemolition measurements [15], etc.

Among these investigations, much effort has been focused on the radiation modes of a one-dimensional cavity with one fixed and one oscillating mirror, particularly when it oscillates periodically at the eigenfrequencies of the unperturbed cavity modes. Such resonant couplings could greatly distort the vacuum field and lead to pronounced modifications of a number of quantum effects of the vacuum. Theoretical studies of such systems have so far been mainly limited to the small-oscillation-amplitude regime by using various small-parameter expansions (see, e.g., [5]). However, these schemes break down when the oscillation amplitudes are large, just when the physics become interesting. It is thus desirable to obtain exact solutions to the resonant coupling problem. Law made progress to this end by obtaining an exact solution for the particular case of a cavity vibrating at the second resonance and finding a coherent structure emerging as narrow wave packets in the energy density [6]. To our knowledge, no one has succeeded in obtaining any other exact solution to the resonant coupling problem until now.

On the other hand, Cole and Schieve [7] have recently developed a simple and elegant method to calculate numerically the dynamics of the quantum vacuum field in a one-dimensional cavity with an arbitrarily moving boundary. Applying this numerical method to the resonant coupling problem, they [7] found that the long-time solution of a system with undamped resonant mirror motion is determined by just two factors: the points in time at which the wall returns to its unperturbed position and the direction the wall was displaced in between these points in time.

In this paper, we present a family of exact analytical solutions to the resonant coupling problem, with each solution corresponding to a particular resonant frequency at which the mirror oscillates. In the particular case of the second resonant mode, our solution reduces to the one obtained by Law [6]. We show that the concentration of energy into narrow wave packets is a general phenomenon for *all* resonance vibrations of the cavity. Furthermore, the amplitudes of these energy wave packets grow rapidly in time, producing sharp and intense pulses of photons.

We consider the dynamics of the quantum vacuum in a one-dimensional cavity formed by two ideal mirrors: one fixed at $x=0$ and the other moving in a prescribed trajectory $x=q(t)$, where $q(t\leqslant 0)$ is assumed to be a constant $L_0$. The field quantization of this system is done by expressing the vector potential as [1]

$$A(x,t)=\sum_{k=1}^{\infty}[a_k\phi_k(x;t)+a_k^\dagger\phi_k^*(x;t)], \quad (1)$$

where $a_k$ and $a_k^\dagger$ are the time-independent annihilation and creation operators defined for the mode structure existing for $t\leqslant 0$. The mode functions $\phi_k(x;t)$ are given in terms of an auxiliary function $R(z)$ as

$$\phi_k(x;t)=\frac{i}{\sqrt{4\pi k}}(e^{-ik\pi R(z_+)}-e^{-ik\pi R(z_-)}), \quad (2)$$

where $z_\pm\equiv t\pm x$ and we have taken $c=\hbar=1$ for convenience. Here $R(z)$ is a real function that satisfies

$$R(t+q(t))=R(t-q(t))+2 \quad (3)$$

and $R(z\leqslant L_0)=z/L_0$. A unique solution for $R(z>L_0)$ is then defined by the recursion relation (3) once a mirror trajectory $q(t\geqslant 0)$ is given. Here we consider a family of mirror trajectories described by

$$q_m(t\geqslant 0)=L_0+\frac{L_0}{m\pi}\left\{\sin^{-1}\left[\sin\theta_m\cos\frac{m\pi t}{L_0}\right]-\theta_m\right\}, \quad (4)$$

 

where $m \geqslant 1$ can be any positive integer and the principal value of the function $\sin^{-1}x$ is assumed. The time-independent parameter $\theta_m(|\theta_m|<\pi/2)$ can be either positive or negative and is defined by $\tan\theta_m \equiv m\epsilon\pi/2L_0$, where $\epsilon$ can be of both signs and its magnitude characterizes the oscillation amplitude of the moving mirror. Obviously, $q_m$ describes the trajectory of a mirror oscillating at a frequency $\Omega_m = m\pi/L_0$, which is the $m$th eigenfrequency of the unperturbed cavity. The trajectory considered by Law [6] is a member of this family ($m=2$ and $\theta_2>0$). In general, if $\epsilon$ is small (i.e., $\epsilon\Omega_m \ll 1$),

$$q_m(t \geqslant 0) = L_0 - \epsilon \sin^2\frac{\Omega_m t}{2} + O(\theta_m^3). \tag{5}$$

Thus $\theta_m<0(>0)$ corresponds to the case of a vibrating mirror that always displaces outward (inward) from its original position during each oscillation. In addition, $q_m(t\geqslant 0)$ is well described by a simple harmonic motion with the frequency $\Omega_m$ as long as the oscillation amplitude $m\epsilon$ is small compared to the original cavity length $L_0$.

Using $q_m(t\leqslant 0)=L_0$ and $q_m(t\geqslant 0)$ from Eq. (4), the corresponding unique exact solution for the recursion equation (3) is $R(z\leqslant L_0)=z/L_0$ and

$$R_m[(2n-1)L_0+\xi] = 2n-1+\frac{2}{m\pi}\cot^{-1}\left[\cot\left(\frac{\Omega_m\xi}{2}\right)-2n\tan\theta_m\right], \tag{6}$$

where $R_m$ is the $R(z)$ corresponding to the trajectory $q_m(t)$, $n\geqslant 1$ is any positive integer, and $\xi\in(0,2L_0]$ is a variable. Here the branch of the multivalued function $\cot^{-1}(x)$ should be properly chosen to avoid any discontinuity in $R(z)$. The particular solution $R_2$ for positive $\theta_2$ can be easily shown to be identical to the solution found by Law [6] except for the different notation. We have checked our analytic solutions for some of the trajectories in Eq. (3) with numerical solutions using the method in Ref. [7] and found excellent agreement.

We plot $R_m(z)$ for $m=1,2,3,4$, $\epsilon=0.01L_0$, and positive $\theta_m$ in Fig. 1 as an illustration. It can be seen from Fig. 1 and Eq. (6) that $R_m(z)$ has the staircase structure that develops as $z$ increases. As $z\to\infty$, $R_m(z)$ approaches a ''perfect'' staircase with arbitrarily flat steps of widths $2/m$ and arbitrarily small transition regions between the steps of heights $2/m$. These steps of the phase factors in the mode functions $\phi_k(x;t)$ impart to the vacuum a coherent structure that is exhibited as narrow wave packets in the energy density. As mentioned above, the long-time solution of $R(z)$ for a system with an undamped resonant wall motion is shown [7] to be determined by only two factors, namely, the points in time at which the wall returns to its unperturbed position, and the direction the wall is displaced in between these points in time. Hence each one of our solutions, say, for a fixed $m$ and $\theta_m>0$ $(<0)$, is representative of a class of motions where the wall is displaced inward (outward) at each oscillation and returns to its unperturbed position with period $2L_0/m$. Therefore, our exact solutions represent in fact all possible kinds of resonant perturbations as far as the long-time asymptotics are concerned.

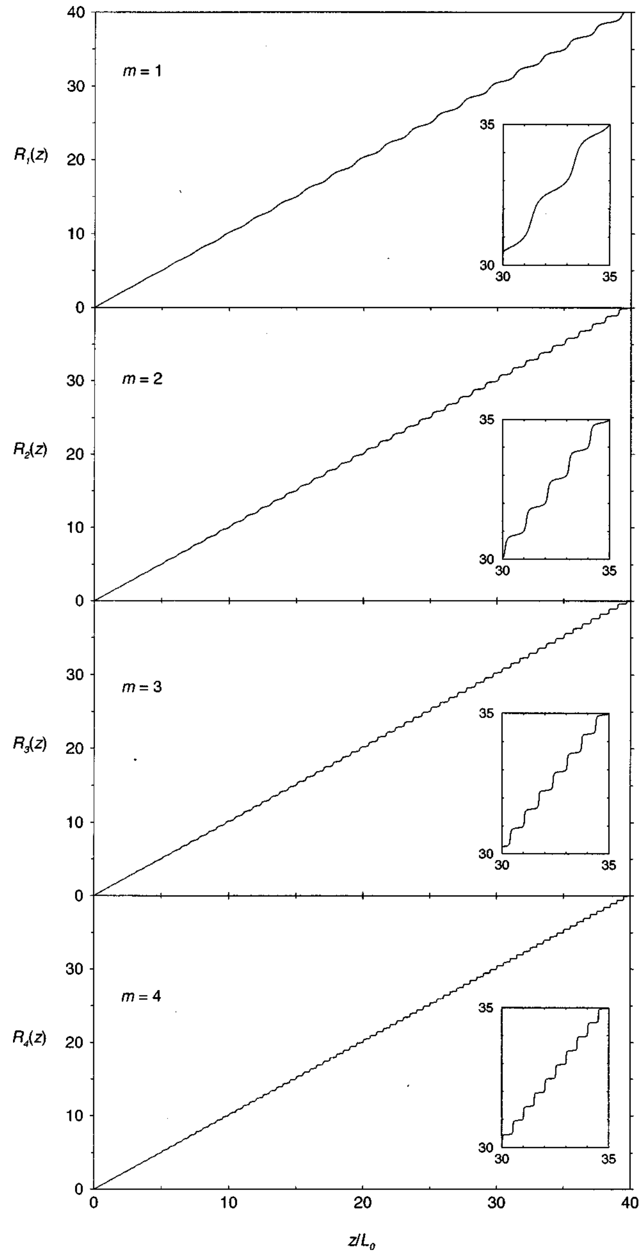


FIG. 1. Phase function $R(z)$ [see Eq. (6)] for a one-dimensional cavity vibrating with an amplitude $\epsilon=0.01L_0$ and positive $\theta_m$ at the first four resonance frequencies $m=1,2,3,4$.

Once the solution $R(z)$ is known, the vacuum energy density can be found immediately by using the general formula derived by Fulling and Davies [8]. The cutoff-independent expression of the vacuum energy is [6,7]

$$\langle T_{00}(x,t)\rangle = -f(t+x)-f(t-x), \tag{7}$$

with

$$24\pi f = \frac{R'''}{R'} - \frac{3}{2}\left(\frac{R''}{R'}\right)^2 + \frac{\pi^2}{2}(R')^2, \tag{8}$$

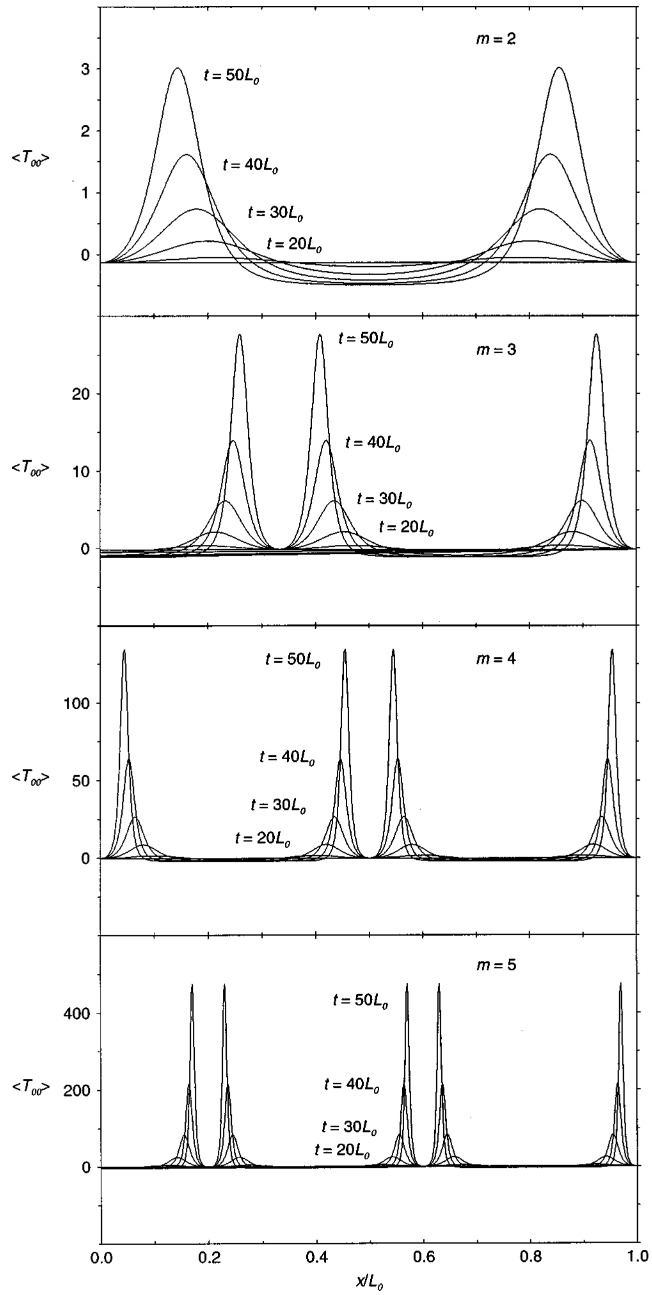


FIG. 2. Snapshots of the energy density in a one-dimensional cavity vibrating with an amplitude $\epsilon=0.01L_0$ and positive $\theta_m$ at four resonance frequencies $m=2,3,4,5$. The curves correspond to four instances $t=20L_0,30L_0,40L_0,50L_0$.

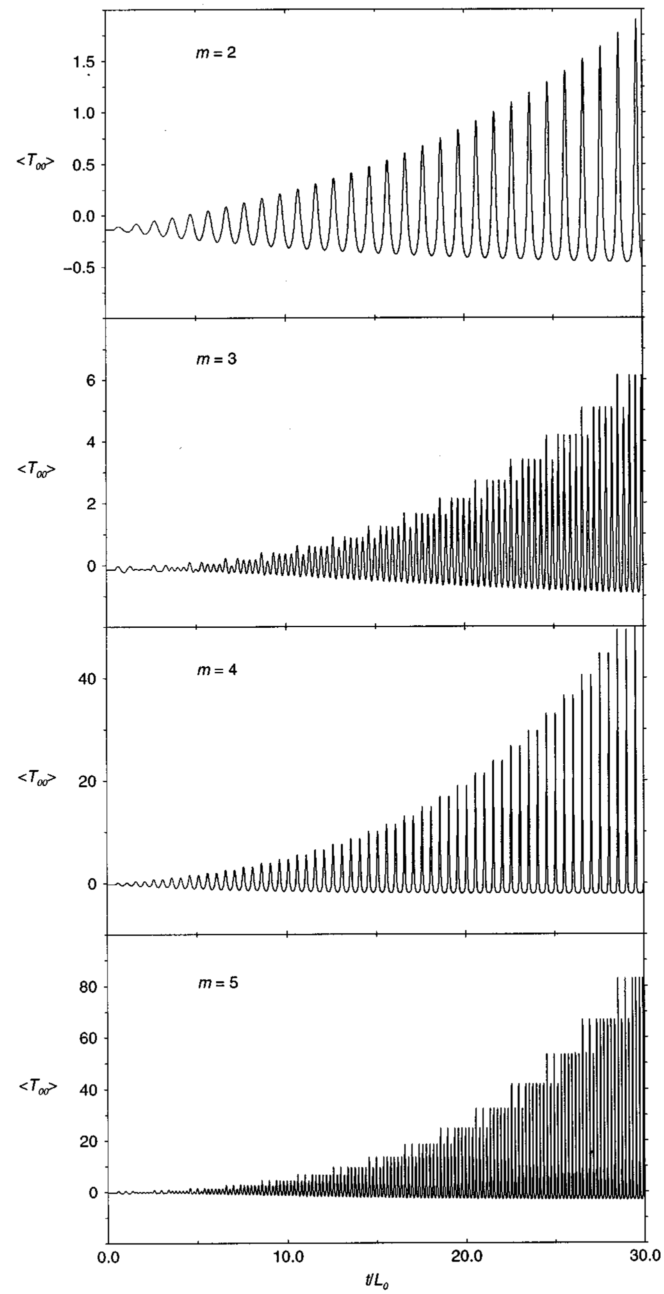


FIG. 3. Energy density at $x=0.5L_0$ vs time in a one-dimensional cavity vibrating with an amplitude $\epsilon=0.01L_0$ and positive $\theta_m$ at four resonance frequencies $m=2,3,4,5$.

where $R'$, $R''$, etc., are the derivatives of $R$. Substituting the form $R_m$ into this equation, we obtain the corresponding $f$ (denoted $f_m$) as

$$f_m[(2n-1)L_0+\xi]=-\frac{m^2}{2}E_{\rm sct}-\frac{(m^2-1)\pi}{48L_0^2D_{n,m}^2(\xi)}+\frac{m^2\pi\epsilon}{24L_0^2}\delta(\xi-2L_0), \tag{9}$$

where $E_{\rm sct}=-\pi/24L_0^2$ is the static Casimir term that is present even when the mirrors are at rest and

$$D_{n,m}(\xi)=1+2\tau_{n,m}^2(1-\cos\Omega_m\xi)-2\tau_{n,m}\sin\Omega_m\xi, \tag{10}$$

with $\tau_{n,m}\equiv n\tan\theta_m=n\epsilon\Omega_m/2$. Again, the expression $f_2$ for positive $\theta_2$ is easily shown to be identical to that found by Law [6].

The first term on the right-hand side of Eq. (9) is independent of the space-time variable and determines the long-time energy density in the so-called sub-Casimir region, i.e., the region far from those of the wave packets [6]. It contributes $m^2E_{\rm sct}$ to the vacuum energy density. The $\delta$-function term is caused by an abrupt initial acceleration of the moving mirror at $t=0$ [6,7]. The second term on the right-hand side of Eq. (9) describes a coherent structure exhibited as narrow wave packets in the energy density. A straightforward analysis of the function $D_{n,m}$ shows that for a given value of $n$ and $m \geqslant 2$, $f_m$ has $m$ identical peaks located within the interval $\xi \in (0,2L_0)$, representing $m$ wave packets traveling back and forth inside the cavity with the speed of light, as shown in Fig. 2. Their heights and widths are proportional to $\tau_{n,m}^4$ and $\tau_{n,m}^{-2}$, respectively, as $|\tau_{n,m}| \gg 1$, which means that the amplitudes of the traveling wave packets grow while their widths shrink in time. This is shown clearly in Fig. 3, where the energy density at $x=0.5L_0$ is plotted as a function of time.

For the special case of $m=1$, the second term on the right-hand side of Eq. (9) vanishes because it is proportional to $(m^2-1)$ and hence there exists no traveling wave packet even though $D_{n,m=1}^{-2}$ does have a maximum value within the interval $\xi \in (0,2L_0)$ proportional to $\tau_{n,m=1}^4$ as $|\tau_{n,m=1}| \gg 1$. A staircase structure does develop in $R_1(z)$ and as $z \to \infty$, it approaches a perfect staircase with a height of 2, the highest jump height of all $m$ (note that the jump height is $2/m$). However, the combination of the derivatives $R_1'$, $R_1''$, and $R_1'''$ in the expression of the vacuum energy density happens to exactly cancel out the effect of the jump in $R_1$, resulting in no traveling wave packet at all.

We emphasize that the emergence of the $m$ $(\geqslant 2)$ traveling wave packets in the $m$th resonant perturbation case is a generic and robust result. The existence of these wave packets is insensitive to the exact form of $q(t)$. However, their absence in the fundamental resonant case $(m=1)$ is not a generic behavior. In other words, a wave packet may emerge if the mirror trajectory deviates only a little from our chosen $q_1$. This can be understood by noting that the cancellation of the jump effect ceases to be exact if some or all of the derivatives $R_1'$, $R_1''$, and $R_1'''$ are slightly modified. However, even a partial cancellation should greatly reduce the amplitude of the wave packet for $m=1$ compared to other resonances [7]. This point, together with the fact that the energy density in the sub-Casimir region $m^2 E_{\rm sct}$ is identical to the static Casimir energy $E_{\rm sct}$ for $m=1$, reflects the reluctance of the vacuum to respond to the fundamental resonant perturbation. In other words, the fundamental resonant perturbation is not effective in creating photons and in enhancing the Casimir force. This conclusion has previously been drawn for small-amplitude oscillations only [3], while here we show that it is true in general. As discussed above, our exact analytical expression for the vacuum energy density for the trajectory family $q_m(t)$ captures all the robust and generic features of the vacuum energy density for all possible kinds of resonant perturbations $q(t)$.

The emergence of narrow wave packets in the energy density implies a large concentration of energy within the cavity. By the action of driving a cavity to vibrate at one of its resonance frequencies, one can *squeeze* an initially uniform field of energy into extremely sharp pulses, even though the governing equation for the cavity field is a simple linear wave equation. The resulting photon production rate can be easily calculated using our exact results for $R$ and a formula developed by Dodonov and Klimov [5]; we plan to present and discuss these elsewhere. It will be extremely interesting to observe this effect experimentally. We are also exploring the possibility that this mechanism may explain the phenomenon of single-bubble sonoluminescence, in which an air bubble suspended in water driven to oscillate by ultrasound emits an intense light pulse in every cycle, representing an enormous energy concentration [12].

In summary, we have presented a family of exact analytical solutions for the quantum vacuum field in a one-dimensional cavity vibrating in any one of its resonance frequencies. We show that for *all* harmonic resonances, staircase structures develop in the phase function $R$ and wave packets emerge in the energy density for $m \geqslant 2$, whose amplitudes grow while their widths shrink in time, greatly concentrating the energy. Outside these wave packets is the sub-Casimir region where the energy density equals $m^2$ times the static Casimir energy density. In view of the conclusion from numerical solutions that the long-time behavior of the system is insensitive to the exact trajectories of the moving mirror [7], our exact analytical solutions presented here contain all the main generic features for all possible kinds of resonant vibrations of a one-dimensional cavity.

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