PHYSICAL REVIEW A VOLUME 59, NUMBER 2 FEBRUARY 1999

Coherent disintegration and stability of vortices in trapped Bose condensates

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(Received 10 August 1998)

We consider the intrinsic stability of the vortex states of a pure Bose-Einstein condensate confined in a harmonic potential under the effects of a coherent atom-atom interaction. We find that stable vortices can be supported and vortex stability can be controlled by changing the interparticle interaction strength. At unstable regimes, a vortex will spontaneously disintegrate into states with different angular momenta even without external perturbations, with the lifetime determined by its imaginary excitation frequencies.
[S1050-2947(99)07602-7]

PACS number(s): 03.75.Fi, 05.30.Jp

I. INTRODUCTION

Vortices and their motions have long been an important branch of fluid mechanics. With the discovery of superfluid helium II, a different idea developed: that the circulation in a superfluid vortex must be quantized [1]. The consequences of these quantized vortices are profound and the understanding of vortex dynamics plays a key role in the current understanding of superfluidity. Moreover, the detection of individual singly quantized vortices has vividly established the true macroscopic quantum nature of these remarkable degenerate fluids. Intimately related to the observation of the superfluid state of ^{4}He is the evidence for the concurrent formation of a Bose-Einstein condensate (BEC) [2]. An important challenge is to clarify the link between these fundamental and important phenomena: Bose-Einstein condensation, superfluidity, and the formation of a macroscopic quantum state.

The recent observation of Bose-Einstein condensation in dilute alkali-metal-atom vapors [3–5] has presented a striking system for investigation, that of the dilute degenerate Bose *gas*. The alkali-metal BECs differ fundamentally from the helium BEC in several crucial ways. BECs in both bulk liquid helium and the dilute helium ''gas'' are free systems (the gas BEC is created by introducing helium into a porous glass known as Vycor [6]). By contrast, the alkali-metal-atom vapor BECs, although free of container walls (and/or the Vycor host), are created within the confines of a trapping potential. There is another major difference: In the trapped alkali-metal condensates, samples can be prepared in which essentially *all* of the atoms are Bose condensed. By contrast, in bulk superfluid ^{4}He, although the superfluid fraction can be near unity, momentum distribution measurements have shown that the bulk condensate fraction is closer to 0.1 with the remainder of the particles in finite momentum states. As researchers improve their ability to create and manipulate these trapped gaseous condensates, a series of important questions naturally arise. Does the gaseous BEC support superflow? Is it indeed a superfluid? Are there stable vortices? This last question is the subject of this paper.

The problem of vortex state excitations has been recently treated by others. Sinha [7] investigated the low-lying modes under Thomas-Fermi limit and Dodd *et al.* [8] obtained the normal mode spectrum of a single quantized vortex state as a function of the number of condensate atoms for a BEC confined in a time-averaged orbiting potential trap [3]. However, the important question of vortex stability was not addressed by these authors. More recently, Rokhsar [9] studied the stability properties of the trapped vortices and argued that vortices are unstable due to the existence of a bound state inside the vortex core. However, throughout his analysis, the transition from a vortex state to a core state requires the presence of thermal atoms that serve as a reservoir to conserve the energy and angular momentum in the process. Hence, the more fundamental question concerning the intrinsic stability of an isolated vortex (i.e., without the external influence of thermal atoms) remains unanswered.

In this paper we approach the problem by assuming that all atoms are in the condensate such that scattering with thermal background atoms can be neglected. This allows us to focus on the *intrinsic coupling* within and between different vortex states and on the effect of this coupling on vortex stability. (Here we use the word ''intrinsic'' to emphasize coherent coupling between the condensate atoms.) We find that stable vortex states can in fact be supported and show that whether a vortex state is stable or not is determined by its angular momentum and the nonlinear interparticle interaction strength. Furthermore, we point out that the lifetime of an unstable vortex can be directly determined from the frequencies of the collective excitations.

The paper is organized as follows. In Sec. II, we describe our physical model and define the stability criterion. Our main results are presented in Sec. III, where the stable and unstable regions of trapped vortices are identified. We also present a physical interpretation of the meaning of the instability. Finally, we give a summary and compare our work with others in Sec. IV.

II. PHYSICAL MODEL

Many nonlinear physical systems exhibit an instability that leads to a self-induced modulation of the steady state. This phenomenon has been studied in such diverse fields as fluid dynamics, nonlinear optics, and plasma physics [10].

The accepted method to study the stability of such systems is first to linearize the appropriate governing nonlinear equations for the system around the stationary state and then to calculate the excitation frequencies corresponding to small oscillations around the steady state. A universally accepted signature of instability is the existence of frequencies with a nonzero imaginary part, which means that excitations in those modes will experience exponential growth.

Here we also adopt $\mathrm{Im}(\omega)=0$ as our stability criterion (this criterion is also used to study the vortex stability in superfluid helium [12]). To simplify our calculations, we consider a condensate confined in a two-dimensional (2D) isotropic harmonic potential with trap frequency ω_0 at zero temperature. In current experiments, condensates are achieved in 3D traps with cylindrical symmetry. A quasi-2D situation can be realized when $\omega_\perp \ll \omega_z$, where $\omega_\perp$ and ω_z are transverse and longitudinal trap frequencies, respectively [11]. In this limit, one can produce a ''pancake''-shaped condensate with all the atoms lying in the lowest harmonic oscillator state in the z direction and hence the degree of freedom in the z coordinate is frozen. Our treatment lies within the Hartree-Fock-Bogoliubov approximation. First, we calculate the macroscopic wave functions of the condensate in a vortex state. We then find the collective excitation frequency ω of that state.

In the Gross-Pitaevskii treatment [13], the energy for N condensed bosons of mass m is given by the functional

$$\frac{E(\Psi_\kappa)}{N}=\int d\mathbf{r}(\Psi_\kappa^*\hat{T}\Psi_\kappa+\hat{V}|\Psi_\kappa|^2+\tfrac{1}{2}NU|\Psi_\kappa|^4). \quad (1)$$

Here

$$\Psi_\kappa(\mathbf{r})=\Phi_\kappa(r)e^{i\kappa\theta}, \quad \kappa=0,\pm1,\pm2,\dots, \quad (2)$$

represents the wave function of the macroscopic vortex state with azimuthal angular momentum $\kappa\hbar$. $\hat{T}=-\hbar^2\nabla^2/2m$ and $\hat{V}=m\omega_0^2r^2/2$ are the kinetic- and potential-energy operators, respectively, and the coupling constant U describes the interactions between condensate atoms. In the quasi-2D situation considered here, the coupling constant takes the form $U=4\sqrt{\pi}\hbar\omega_z\xi_z a$ [14], where $\xi_z=\sqrt{\hbar/2m\omega_z}$ is the harmonic oscillator length in z dimension. In our analysis, the solution that minimizes Eq. (1) is found iteratively using a finite element method (FEM) [15,16]. In our calculations, we normally used 20 elements, with two nodes and three degrees of freedom for each element. This numerical method is very efficient and it typically took no more than a few minutes to find the wave function Ψ_κ on a Cray-YMP2E/232 supercomputer.

With the solution of Ψ_κ at our disposal, we can now calculate collective excitation frequencies by solving Bogoliubov equations [8]

$$(\mathcal{L}-\hbar\omega_\lambda-\mu_\kappa)u_\lambda(\mathbf{r})+NU[\Psi_\kappa(\mathbf{r})]^2v_\lambda^*(\mathbf{r})=0, \quad (3a)$$

$$NU[\Psi_\kappa^*(\mathbf{r})]^2u_\lambda(\mathbf{r})+(\mathcal{L}+\hbar\omega_\lambda-\mu_\kappa)v_\lambda^*(\mathbf{r})=0, \quad (3b)$$

where μ_κ is the chemical potential for the state $\Psi_\kappa(\mathbf{r})$, $u_\lambda(\mathbf{r})$, and $v_\lambda(\mathbf{r})$ are normal mode functions with mode frequency ω_λ, and $\mathcal{L}=\hat{T}+\hat{V}+2NU|\Psi_\kappa(\mathbf{r})|^2$. It is straightforward to show that if $\Psi_\kappa(\mathbf{r})$ is given by Eq. (2), then $u_\lambda(\mathbf{r})$ and $v_\lambda(\mathbf{r})$ must have definite angular momentum compositions $\kappa_u\hbar$ and $\kappa_v\hbar$, respectively, such that $u_\lambda(\mathbf{r})=\tilde{u}_\lambda(r)e^{i\kappa_u\theta}$, $v_\lambda(\mathbf{r})=\tilde{v}_\lambda(r)e^{i\kappa_v\theta}$, and $\kappa_u+\kappa_v=2\kappa$ [8].

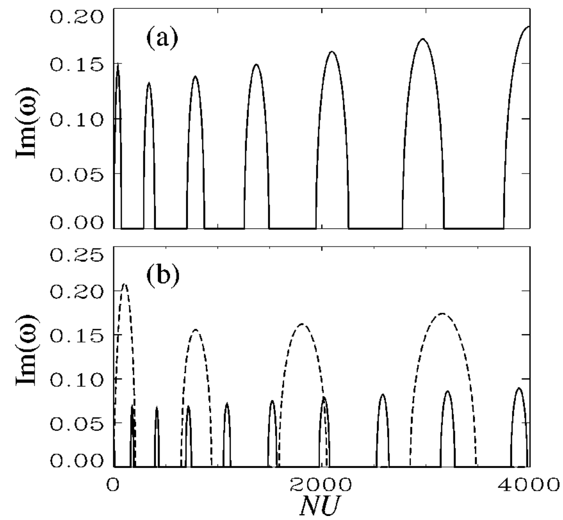

FIG. 1. (a) Imaginary part of the complex frequency of a double quantized vortex state Ψ_2 as a function of interaction strength NU for $\kappa_u=0$ and $\kappa_v=4$. (b) Same as (a) for a triple quantized vortex state Ψ_3. Solid line, $\kappa_u=0$ and $\kappa_v=6$; dashed line, $\kappa_u=1$ and $\kappa_v=5$. The frequency is in units of the trap frequency ω_0 and U in units of $(\hbar\omega_0\xi^2)$, where $\xi=(\hbar/2m\omega_0)^{1/2}$ is the harmonic oscillator length.

III. RESULTS AND INTERPRETATION

Equations (3) were transformed to an eigenvalue problem for a finite-size matrix and solved using the FEM. Our goal here is to find mode frequencies with a nonzero imaginary part in order to determine the vortex stability. As in the case of the ground state, the vortex stability properties for a condensate with a repulsive interparticle interaction are drastically different from that for a condensate with attractive interaction. We will discuss these two cases separately.

Repulsive interaction, i.e., $U>0$. When we calculate the collective excitation frequencies of a single quantized ($\kappa=1$) vortex state Ψ_1, we find that all the excitation frequencies are real, which means that Ψ_1 is *always stable*. Next we consider a double quantized ($\kappa=2$) vortex state Ψ_2. Here we find that complex frequencies exist only for $\kappa_u=0$ and $\kappa_v=4$. (Without loss of generality, we assume that $\kappa>0$ and $\kappa_v>\kappa>\kappa_u$.) We find that for any other pairs of (κ_u,κ_v), the excitation frequencies are all real. Furthermore, for values of NU for which the vortex is unstable, we find that there exists at most one complex frequency. Figure 1(a) shows the imaginary part of the complex frequency as a function of the interaction strength NU. As we can see, in this particular channel [i.e., choice of (κ_u,κ_v)], the parameter space of NU is divided into alternating stable and unstable regions. In the unstable regions, the inverse of $\mathrm{Im}(\omega)$ determines the lifetime of the unstable vortex. For the parameter range described in Fig. 1, the most unstable vortex state will decay after several periods of the harmonic trapping potential. We stress that the details of how the condensate will evolve under these instabilities are beyond the capability of the mean-field treatment and requires further investigation.

For a general state Ψ_κ, our numerical calculations show that there are $\kappa-1$ unstable channels that possess complex excitation frequencies, those with $\kappa_u=0,1,\ldots,\kappa-2$ and $\kappa_v=2\kappa-\kappa_u$. Figure 1(b) shows the imaginary part of the complex frequency for a triple quantized vortex state Ψ_3. We can see a similar pattern to Fig. 1(a), but here there are two unstable channels. Each channel shows its own quasiperiodic behavior as a function of NU. The two channels have quite different ''periods'' and the characteristic width of unstable regions. At first look, this may appear rather unexpected. To interpret this behavior we will show that each unstable region in NU space represents a decay channel in which two atoms from the given vortex state scatter into two new states, with angular momenta $\kappa_u\hbar$ and $\kappa_v\hbar$, respectively, thus inducing instability in that initial vortex state Ψ_κ.

First, let us define a boson field operator as $\hat{\Psi}(\mathbf{r}) \equiv \sqrt{N}\Psi_\kappa(\mathbf{r})+\hat{\psi}(\mathbf{r})$, where the c number $\Psi_\kappa(\mathbf{r})$ denotes the one-body wave function for the condensate and $\hat{\psi}(\mathbf{r})$ is the field operator for the fluctuation part [17]. The second quantized Bogoliubov Hamiltonian reads

$$\hat{K}_B=\int d\mathbf{r}\,\hat{\psi}^\dagger(\mathbf{r})[\mathcal{L}-\mu_\kappa]\hat{\psi}(\mathbf{r}) +\left[\frac{1}{2}NU\int d\mathbf{r}\,\hat{\psi}^\dagger(\mathbf{r})\hat{\psi}^\dagger(\mathbf{r})\Psi_\kappa\Psi_\kappa+\text{H.c.}\right],$$

where the c-number part independent of $\hat{\psi}(\mathbf{r})$ has been neglected. We can further decompose $\hat{\psi}(\mathbf{r})$ as $\hat{\psi}(\mathbf{r}) =\Sigma_{n,\alpha}a_{n,\alpha}\phi_{n,\alpha}(\mathbf{r})$, where $a_{n,\alpha}$ is an annihilation operator associated with a single-particle state $\phi_{n,\alpha}$. The set of states $\{\phi_{n,\alpha}\}$ is defined as the eigenvectors of $\mathcal{L}$ with eigenvalues $\epsilon_{n,\alpha}$, i.e., $\mathcal{L}\phi_{n,\alpha}=\epsilon_{n,\alpha}\phi_{n,\alpha}$, with subscripts (n,α) labeling the radial and angular quantum numbers, respectively. The Hamiltonian $\hat{K}_B$ may then be rewritten as $\hat{K}_B=\hat{H}_0+\hat{H}_I$, where

$$\hat{H}_0=\sum_{n,\alpha}(\epsilon_{n,\alpha}-\mu_\kappa)a^\dagger_{n,\alpha}a_{n,\alpha},$$

$$\hat{H}_I=\sum_{n_u,\kappa_u}\sum_{n_v,\kappa_v}\Lambda(n_u,\kappa_u;n_v,\kappa_v)a^\dagger_{n_u,\kappa_u}a^\dagger_{n_v,\kappa_v}+\text{H.c.},$$

and

$$\Lambda(n_u,\kappa_u;n_v,\kappa_v)=\frac{1}{2}NU\int d\mathbf{r} \times\phi^*_{n_u,\kappa_u}(\mathbf{r})\phi^*_{n_v,\kappa_v}(\mathbf{r})\Psi_\kappa(\mathbf{r})\Psi_\kappa(\mathbf{r}). \tag{4}$$

In the interaction picture, $a^\dagger_{n,\alpha}(t)=a^\dagger_{n,\alpha}e^{i(\epsilon_{n,\alpha}-\mu_\kappa)t}$ and the Hamiltonian is given by

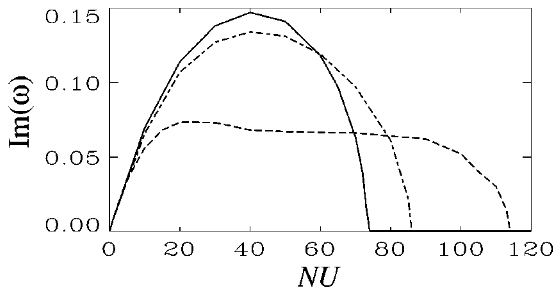

FIG. 2. Solid line, imaginary part of the complex frequency of a double quantized vortex state Ψ_2 as a function of interaction strength NU for $\kappa_u=0$ and $\kappa_v=4$; dashed line, decay rate of the vortex state Ψ_2 as a function of NU calculated using the Hamiltonian (4) [only contributions from the resonant states $(\phi_{0,0},\phi_{0,4})$ are included]; dot-dashed line, same as dashed line, but including contributions from two more states $(\phi_{1,0},\phi_{1,4})$.

$$\hat{H}_I(t)=\sum_{n_u,\kappa_u}\sum_{n_v,\kappa_v}\Lambda(n_u,\kappa_u;n_v,\kappa_v) \times e^{i(\epsilon_{n_u,\kappa_u}+\epsilon_{n_v,\kappa_v}-2\mu_\kappa)t}a^\dagger_{n_u,\kappa_u}a^\dagger_{n_v,\kappa_v}+\text{H.c.} \tag{5}$$

$\Lambda(n_u,\kappa_u;n_v,\kappa_v)$ is nonzero only when $\kappa_u+\kappa_v=2\kappa$, which is a direct consequence of the conservation of angular momentum. The interaction described by the Hamiltonian (5) is analogous to parametric processes in quantum optics where instability can occur under certain conditions. For example, one can build up large numbers of photon pairs (signal and idle) from the vacuum via parametric down-conversion if the field frequencies satisfy a parametric resonance condition. In our case, the fluctuation in mode pair $(\phi_{n_u,\kappa_u},\phi_{n_v,\kappa_v})$ grows exponentially when

$$|\epsilon_{n_u,\kappa_u}+\epsilon_{n_v,\kappa_v}-2\mu_\kappa|<\Lambda(n_u,\kappa_u;n_v,\kappa_v) \tag{6}$$

and hence the vortex is unstable under such a resonance condition. We emphasize that the instability implied in this picture is purely quantum mechanical. The atoms in the vortex can *spontaneously disintegrate* into ϕ_{n_u,κ_u} and ϕ_{n_v,κ_v} states without the need of external (classical) perturbations, such as the interaction with the thermal background gases or perturbation of the trap.

For a $\kappa=1$ vortex, our numerical calculations show that there exists *no* particle states that satisfy the resonance condition (6), in support of our prediction that a single quantized vortex state is always stable for $U>0$. For a $\kappa=2$ vortex, we find that a pair of particle states $(\phi_{0,0},\phi_{0,4})$ indeed satisfy the inequality (6). In the weak coupling limit, we can calculate the decay rate of the double quantized vortex state Ψ_2 using the Hamiltonian (5) by neglecting all the nonresonant terms (i.e., keeping only terms with $n_u=n_v=0$, $\kappa_u=0$, and $\kappa_v=4$). The results are shown in Fig. 2 along with the imaginary part of the complex excitation frequencies of the vortex state Ψ_2. We can see a clear qualitative agreement between the two results. The agreement can be significantly improved if the contribution from states $(\phi_{1,0},\phi_{1,4})$ is also included in calculating the decay rate (see Fig. 2). We remark that although useful for interpreting our vortex stability predictions, the parametric resonance picture is valid only for the weak-interaction regime. This is because a strong inter-

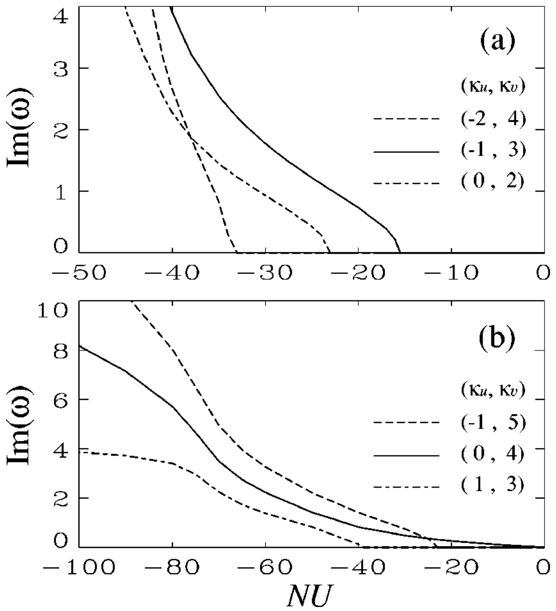

FIG. 3. Imaginary part of the complex frequency of (a) a single and (b) a double quantized vortex state as a function of interaction strength NU, with $NU<0$.

action can drastically change the frequencies of the oscillators and introduce mixing among different particle states. Further work would be necessary in order to understand all aspects of the complex structure shown in Fig. 1, particularly for large NU.

Attractive interaction, i.e., $U<0$. A condensate with a strong attractive interparticle interaction is known to be subject to collapse. However, a metastable condensate with a small number of atoms can still exist [4,18]. Figure 3 shows the imaginary part of the complex excitation frequency for a single and a double quantized vortex state as functions of NU. Figure 3(a) shows that Ψ_1 is stable for a sufficiently small attractive interaction, but unstable for a larger interaction strength. For Ψ_2, as we can see from Fig. 3(b), the channel ($\kappa_u=0,\kappa_v=4$) possesses complex frequency for all negative values of NU instead of showing a quasiperiodic pattern as in the case of the repulsive interaction. Furthermore, we find that, similar to Ψ_1, other channels that are stable for $NU>0$ become consistently unstable for sufficiently large $|NU|$ [we only show two such channels in Fig. 3(b)]. Our calculations show that for $U<0$, stable vortices exist only for a single quantized vortex state in the weak interaction regime [see Fig. 3(a)]; a multiple quantized vortex state (i.e., $\kappa>1$) is always unstable. It has been speculated that the existence of vortices may help stabilize a condensate with negative scattering length [19]. However, as we show here, although such vortices may seem to be more stable against the collapse when compared to the ground state, they remain fundamentally unstable and small fluctuations will eventually destroy such vortices.

IV. SUMMARY AND DISCUSSION

In summary, we have calculated the collective excitation frequencies of a Bose-Einstein condensate in a vortex state and have established intrinsic stability regions for these vortices. We have shown that, even without any perturbation, an unstable vortex can still decay spontaneously. For a repulsive interparticle interaction, we found that single quantized vortices are always stable, while imaginary excitation modes divide the interaction energy axis (NU) of multiple quantized vortices ($\kappa>1$) into alternating stable and unstable regions. Hence one can control the vortex stability by varying the value of interaction strength, which in turn can be achieved by changing the scattering length [20,21], particle number, or trap frequency. This provides us with the possibility of studying condensate evolution under the effect of imaginary modes.

In the unstable regime, the mean-field equations may become inapplicable to describe the long-term evolution of the system due to the exponential growth of the fluctuation. Nevertheless, the mean-field theory is able to predict when the system becomes unstable and the magnitude of the imaginary frequency provides an estimation for how fast the vortex will break down. This information is particularly useful if one wants to create a vortex state from the ground state of the condensate [26]. A familiar behavior of the systems with modulation instability is that the steady state will disintegrate into filaments, wave packets, or solitons [10]. Instability has also been studied in the context of a two-species Bose-Einstein condensate [22]. Gordon and Savage showed that imaginary excitation modes may break the spatial symmetry of the ground state of the two-species condensate [23]. Here we show that an unstable vortex state will disintegrate into states with different angular momenta.

For a condensate in the vortex state, there may exist quasiparticle states with negative frequencies. One such negative frequency state was identified by Dodd *et al.* in Ref. [8]. The presence of negative frequencies implies that there exist states with lower energy. However, this does not necessarily mean that the condensate is unstable if no mechanism exists to drive the system to these lower-energy states [24]. In Ref. [9] Rokhsar considered the instability arising from the incoherent interactions between condensate and thermal atoms, which induce the transition to the negative frequency core state. In contrast, in the present paper we study the *intrinsic stability of vortices in a pure condensate*, excluding such incoherent processes while focusing on the coherent interactions within the condensate. In our work the disintegration of an unstable vortex occurs as a *coherent process*. We found that stable vortices *can* be supported in harmonic traps as long as the temperature is low enough such that the effects of thermal atoms are insignificant. At temperatures when thermal atoms cannot be neglected, both coherent and incoherent processes will be present and each will have its effect on vortex stability. It remains to be seen which process will be dominant. Further investigations should also include the possible influence of trap anisotropy and the dynamics of the disintegration processes.

Our analysis concerns the case of a 2D system; however, we believe that the qualitative stability characters of a 3D condensate in a vortex state will not be very different from its 2D counterpart. To support this, we note that our calculation shows that the excitation spectrum of a 2D system is essentially identical to that of a 3D system [11]. However, the presence of the third dimension may change the resonance condition and hence shift the stable/unstable region. Recently, vortex stability in 2D harmonic trap was studied by Caradoc-Davies *et al.* through a direct numerical simulation [25]. In that study, a blue detuned laser beam is applied to perturb the condensate in a vortex state. They found that the single quantized vortex is indeed stable, while a double quantized vortex can disintegrate into unit vortices under external perturbation. These results are consistent with ours presented in this paper.

Finally, as an example, let us consider a ^{23}Na condensate (scattering length $a\approx 3$ nm) in a harmonic trap with $\omega_\perp = 2\pi\times 10$ Hz and $\omega_z = 2\pi\times 200$ Hz, in units of $\hbar\omega_\perp \xi_\perp^2$, $U\approx 0.02$. The plotted range of NU from 0 to 4000 in Fig. 1 corresponds to a particle number ranging from 0 to 2×10^5, well within the capability of current experiments. Recently, several methods on how to generate vortex states in alkali-metal atomic BECs have been proposed [26]. With current technology and fast progress on this field, our study on vortex stability should be experimentally testable in the near future.

ACKNOWLEDGMENTS

This research was supported by NSF Grants Nos. PHY-9415583 and PHY-9457897 and the David and Lucile Packard Foundation. N.P.B. thanks Professor D. S. Rokhsar for comments regarding Ref. [9] and Professor Gora Schlyapnikov for useful discussions.

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