Bound entanglement and teleportation

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Recently M. Horodecki, P. Horodecki, and R. Horodecki [Phys. Rev. Lett. **80**, 5239 (1998)] have introduced a set of density matrices of two spin-1 particles from which it is not possible to distill any maximally entangled states, even though the density matrices are entangled. Thus these density matrices do not allow reliable teleportation. However, it might nevertheless be the case that these states can be used for teleportation, not reliably, but still with fidelity greater than that which may be achieved with a classical scheme. We show that, at least for some of these density matrices, teleportation cannot be achieved with better than classical fidelity. [S1050-2947(99)03301-6]

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Nonlocality, discovered by Bell more than 30 years ago, has recently shown itself to have many manifestations: teleportation [1], distillability [2], reduction of communication complexity [3], etc. It is not clear yet what the relations between these different manifestations are [4].

In two very interesting papers [5,6] a set of density matrices of two spin-1 particles were introduced from which it is not possible to distill any states of which are maximally entangled even though the density matrices are entangled. This is very surprising since in the case of spin-1/2 particles any entangled density matrix is distillable. Distillability, however is just one aspect of nonlocality. Thus, although this aspect of nonlocality is not active, the Horodecki density matrices may actively manifest other forms of nonlocality. For example, bound entanglement may be pumped into a single pair of free entangled particles [7]. Here we investigate teleportation.

Since one cannot distill pure singlets from Horodecki density matrices, these density matrices do not allow reliable teleportation. However, it might nevertheless be the case that these states can be used for teleportation, not reliably, but still with fidelity greater than that which may be achieved with a classical scheme. This is the general question we investigate here.

First let us consider the case of spin-1/2 particles. Since one cannot distill states formally equivalent to spin-1/2 singlets from the Horodecki density matrices, these density matrices cannot be used for reliable teleportation of spin-1/2 states. Furthermore, it is most probable that these density matrices cannot be used to teleport spin-1/2 states at all with better than classical fidelity. The reason is the following. Suppose Alice produces locally a spin-1/2 singlet and teleports one of the spins to Bob. If Alice and Bob share a maximally entangled state of two spin-1 particles, then at then end of the process this state is destroyed and it is replaced by a state that is equivalent to a maximally entangled state of two spin-1/2 particles (i.e., the state originally held by Alice).

If, however, Alice and Bob were to share a Horodecki density matrix, at the end of the process, the original Horodecki matrix is destroyed and now Alice and Bob share a pair of particles whose state is equivalent to one of spin-1/2 particles, but not a faithful copy of Alice's original state. Presumably, however, if the teleportation works better than any classical scheme, we expect that this state is still entangled. This, however, cannot happen because from any entangled state of spin-1/2 particles one can distill singlets. Thus the procedure would be tantamount to distilling singlets from the Horodecki matrices. We thus expect that Horodecki density matrices cannot be used to teleport spin-1/2 states with better than classical fidelity.

However, the above discussion leaves open the question of whether spin-1 states can be teleported with better than classical fidelity using Horodecki density matrices. In particular the above argument does not rule out this possibility for the following reason. Suppose now that Alice were to prepare a maximally entangled state of two spin-1 particles and teleport the state of one of them to Bob using a Horodecki pair. At the end of the process the Horodecki matrix is again destroyed and Alice and Bob now share some state of two spin-1 particles. As before this state will not be a faithful copy of the original singlet prepared by Alice; however, if the teleportation works better than any classical scheme, we expect that this state is still entangled. However, now there is the possibility that this state is a state of Horodecki-type *bound* entanglement that does not allow distillation and thus leads to no contradiction of the Horodeckis' general arguments.

Furthermore, another argument that might give hope to the possibility that spin-1 states can be teleported better than the classical case while spin-1/2 states cannot is that one expects that the classical fidelity to be lower for spin-1 than spin-1/2 states. This is because it is more difficult to identify, using a measurement, a state that may be anywhere in a three-dimensional Hilbert space than a state that may be anywhere in a two-dimensional Hilbert space. In this paper we show that, despite the arguments presented above, at least for some of the Horodecki density matrices, it is not possible to teleport spin-1 states with better than classical fidelity, thus confirming the remarkable nature of these density matrices.

As is customary, we imagine that Alice and Bob each have one of the pair of particles described by the density matrix ρ_a. Alice receives a particle in an unknown state ϕ and she performs a measurement of the pair of particles she has and transmits some information to Bob. The aim is to maximize the fidelity of transmission, averaged over all unknown states ϕ.

To be explicit, we write the state ϕ as

$$|\phi\rangle=(c_5+is_5c_4)|1\rangle+s_5s_4(c_3+is_3c_2)|2\rangle +s_5s_4s_3s_2e^{i\theta_1}|3\rangle, \tag{1}$$

with respect to some basis $|1\rangle$, $|2\rangle$, or $|3\rangle$, where $s_5=\sin\theta_5$, $c_5=\cos\theta_5$, etc. (for simplicity we have taken ϕ to be a vector rather than a ray). With this parametrization, the U(3) invariant measure is

$$d\mu(\phi)=\frac{1}{\pi^3}\sin^4\theta_5\sin^3\theta_4\sin^2\theta_3\sin\theta_2 d\theta_5\, d\theta_4\, d\theta_3\, d\theta_2\, d\theta_1. \tag{2}$$

We first derive a general expression for the fidelity of transmission using an arbitrary density matrix ρ shared between Alice and Bob. A convenient parametrization of ρ, the Schmidt representation, is

$$\rho=\frac{1}{9}I\otimes I+\frac{1}{6}r_i\lambda_i\otimes I+\frac{1}{6}I\otimes s_i\lambda_i+\frac{1}{4}t_{ij}\lambda_i\otimes\lambda_j, \tag{3}$$

where λ_i are the Gell-Mann matrices (see, for example, [8]) that satisfy $\mathrm{Tr}(\lambda_i)=0$ and $\mathrm{Tr}(\lambda_i\lambda_j)=2\delta_{ij}$. Thus $r_i=\mathrm{Tr}(\rho\lambda_i\otimes I)$, $s_i=\mathrm{Tr}(\rho I\otimes\lambda_i)$, and $t_{ij}=\mathrm{Tr}(\rho\lambda_i\otimes\lambda_j)$.

The fidelity of transmission is

$$F=\int d\mu(\phi)\sum_{k=1}^{9}p_k\mathrm{Tr}_3(\rho_kP_\phi), \tag{4}$$

where

$$p_k=\mathrm{Tr}_{1,2,3}(P_k\otimes U_k)(P_\phi\otimes\rho)(P_k\otimes U_k^\dagger) \tag{5}$$

is the probability of the kth outcome,

$$\rho_k=\frac{1}{p_k}\mathrm{Tr}_{1,2}(P_k\otimes U_k)(P_\phi\otimes\rho)(P_k\otimes U_k^\dagger) \tag{6}$$

is the output state,

$$P_k=\frac{1}{9}I\otimes I+\frac{1}{6}R_i^{(k)}\lambda_i\otimes I+\frac{1}{6}I\otimes S_i^{(k)}\lambda_i+\frac{1}{4}T_{ij}^{(k)}\lambda_i\otimes\lambda_j \tag{7}$$

are the projection operators corresponding to the measurement Alice makes, U_k are the unitary operators Bob performs that depend on which result Alice obtains, and

$$P_\phi=\frac{1}{3}I+\frac{1}{2}\alpha_i\lambda_i \tag{8}$$

is the projection operator of the unknown input state ϕ. The subscripts on the traces indicate the Hilbert space over which the trace is taken.

Now

$$p_k\rho_k=\left(\frac{1}{27}+\frac{1}{18}r_qS_q^{(k)}+\frac{1}{18}\alpha_qR_q^{(k)}+\frac{1}{12}\alpha_pr_qT_{pq}^{(k)}\right)I +\left(\frac{1}{18}s_qO_{qj}^{(k)}+\frac{1}{12}t_{iq}S_i^{(k)}O_{qj}^{(k)} +\frac{1}{12}\alpha_iR_i^{(k)}s_qO_{qj}^{(k)}+\frac{1}{8}\alpha_pT_{pi}^{(k)}t_{iq}O_{qj}^{(k)}\right)\lambda_j, \tag{9}$$

where the orthogonal matrix $O^{(k)}$ is that induced by conjugation by the unitary matrix U_k:

$$U^{(k)}x_j\lambda_jU^{(k)\dagger}=x_iO_{ij}^{(k)}\lambda_j. \tag{10}$$

Thus

$$\mathrm{Tr}_3(p_k\rho_kP_\phi)=\left(\frac{1}{27}+\frac{1}{18}r_qS_q^{(k)}+\frac{1}{18}\alpha_qR_q^{(k)}+\frac{1}{12}\alpha_pr_qT_{pq}^{(k)}\right) +\left(\frac{1}{18}s_qO_{qj}^{(k)}+\frac{1}{12}t_{iq}S_i^{(k)}O_{qj}^{(k)} +\frac{1}{12}\alpha_iR_i^{(k)}s_qO_{qj}^{(k)}+\frac{1}{8}\alpha_pT_{pi}^{(k)}t_{iq}O_{qj}^{(k)}\right)\alpha_j. \tag{11}$$

We may now perform integration over α in the expression for the fidelity using

$$\int d\alpha\,\alpha_iM_{ij}\alpha_j=\int d\mu(\phi)\langle\phi|\lambda_i|\phi\rangle M_{ij}\langle\phi|\lambda_j|\phi\rangle =\frac{1}{6}\mathrm{Tr}(M) \tag{12}$$

and

$$\int d\alpha\,\alpha_i=\int d\mu(\phi)\langle\phi|\lambda_i|\phi\rangle=0. \tag{13}$$

Thus

$$F=\sum_k\left(\frac{1}{27}+\frac{1}{18}r_qS_q^{(k)}+\frac{1}{72}R_p^{(k)}s_qO_{qp}^{(k)}+\frac{1}{48}T_{pi}^{(k)}t_{iq}O_{qp}^{(k)}\right). \tag{14}$$

We now put a bound on the fidelity by considering the maximum value of the summand. Let us call P^{max} (with Schmidt components R^{max}, S^{max}, and T^{max}) the projection operator that maximizes the summand in Eq. (14). Without loss of generality we may take the orthogonal matrix O to be the identity. Thus

$$F\leqslant\left(\frac{1}{3}+\frac{1}{2}r_qS_q^{max}+\frac{1}{8}R_q^{max}s_q+\frac{3}{16}T_{pi}^{max}t_{ip}\right). \tag{15}$$

Let us now denote by $\hat{P}^{max}$ the projection operator defined by

$$\hat{P}^{max}=NP^{max}N, \tag{16}$$

where N is the interchange operator that we define by its action on basis vectors

$$Ne_i\otimes e_j=e_j\otimes e_i. \tag{17}$$

We may then use the fact that

$$\mathrm{Tr}(\rho\hat{P}^{max})=\frac{1}{9}+\frac{1}{6}r_iS_i^{max}+\frac{1}{6}s_iR_i^{max}+\frac{1}{4}t_{ij}T_{ji}^{max} \tag{18}$$

to rewrite the bound on the fidelity as

$$F\leqslant\left(\frac{1}{4}+\frac{3}{8}r_qS_q^{max}+\frac{3}{4}\mathrm{Tr}(\rho\hat{P}^{max})\right). \tag{19}$$

We now consider the specific case of the matrices ρ_a introduced in [5,6]:

$$\rho_a=\frac{1}{8a+1}\begin{bmatrix} a & 0 & 0 & 0 & a & 0 & 0 & 0 & a \\ 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & a & 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1+a}{2} & 0 & \frac{\sqrt{1-a^2}}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & 0 \\ a & 0 & 0 & 0 & a & 0 & \frac{\sqrt{1-a^2}}{2} & 0 & \frac{1+a}{2} \end{bmatrix}. \tag{20}$$

We find that all the Schmidt components r_q for this matrix are zero except r_8, which is

$$r_8=\mathrm{Tr}(\rho_a\lambda_8\otimes I)=\frac{2}{\sqrt{3}}\left(\frac{a-1}{8a+1}\right). \tag{21}$$

Thus, if we write

$$\tilde{\rho}_a=\rho_a+\frac{1}{\sqrt{3}}\left(\frac{a-1}{8a+1}\right)\lambda_8\otimes I, \tag{22}$$

then we may write the bound on the fidelity as

$$F\leqslant\left(\frac{1}{4}+\frac{3}{4}\mathrm{Tr}(\tilde{\rho}_a\hat{P}^{max})\right). \tag{23}$$

We now consider under what conditions the fidelity of teleportation can be greater than any classical procedure. One particular classical scheme that Alice and Bob could use is as follows. First Alice simply measures the unknown state ϕ using an arbitrary nondegenerate operator. Let us call the eigenvectors of this operator v_1,v_2,v_3 with associated eigenvalues μ_1,μ_2,μ_3. If Alice's outcome is μ_1 she tells Bob to guess that the unknown state was v_1 and so on (this procedure may not be the optimal classical scheme, but we will not need this in what follows). The fidelity of this procedure is

$$\sum_{i=1}^{3}\int d\mu(\phi)|\langle v_i|\phi\rangle|^4=\frac{1}{2}. \tag{24}$$

Let us now return to the fidelity of teleportation. The maximum value of the fidelity in Eq. (23) is obtained when we choose P^{max} so that $\hat{P}^{max}$ is the projector onto the maximum eigenvalue of $\tilde{\rho}_a$. If this maximum eigenvalue is less than $\frac{1}{3}$, then the fidelity of teleportation (23) is less than $\frac{1}{2}$ and therefore the density matrix ρ_a cannot be used to teleport better than the optimal classical scheme (which may have fidelity greater than $\frac{1}{2}$).

By direct calculation we find that for $a=\sqrt{3}/2$ the eigenvalues of $\tilde{\rho}_a$ are

$$\begin{gathered}\frac{1}{3}\left(\frac{2\sqrt{3}-1}{4\sqrt{3}+1}\right),\\ \frac{83}{1128}-\frac{1}{376}\sqrt{3}-\frac{1}{376}(10\,588-5786\sqrt{3})^{1/2},\\ \frac{83}{1128}-\frac{1}{376}\sqrt{3}+\frac{1}{376}(10\,588-5786\sqrt{3})^{1/2},\\ \frac{7}{141}-\frac{3}{94}\sqrt{3},\end{gathered} \tag{25}$$

$$\frac{37}{564}+\frac{29}{188}\sqrt{3},$$

$$\frac{4}{141}+\frac{5}{94}\sqrt{3};$$

the first eigenvalue occurs with multiplicity 4. All the above eigenvalues are less than $\frac{1}{3}$. Thus we have shown that $\rho_{\sqrt{3}/2}$ cannot teleport a spin-1 state with fidelity better than classical. We note that we have not limited ourselves to ''standard'' teleportation: The projectors P_k were not assumed to be maximally entangled.

Numerical evidence indicates that for a roughly in the region $4/5 < a < 1$, the maximum eigenvalue of $\tilde{\rho}_a$ is less than $\frac{1}{3}$. For small a, $\tilde{\rho}_a$ does have an eigenvalue larger than $\frac{1}{3}$, so the argument presented here is not conclusive in this case.

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