Temporal quantum theory

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We propose a framework for temporal quantum theories for the purpose of describing states and observables associated with extended regions of space-time quantum mechanically. The proposal is motivated by Isham's history theories. We discuss its relation to Isham's history theories and to standard quantum mechanics. We generalize the Isham-Linden information entropy to the present context. [S1050-2947(99)07902-0]

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I. INTRODUCTION

Standard nonrelativistic quantum mechanics is based on notions of states and observables at fixed instants of time. In Hilbert space quantum mechanics the states at some fixed time are represented by positive trace class operators with trace one on some Hilbert space $\mathfrak{H}$ and the observables are identified with self-adjoint operators on $\mathfrak{H}$. The time evolution is governed by a semigroup $\{U(t',t)\}$ of unitary operators on $\mathfrak{H}$.

It has been felt by several authors that the notions of observables and states at a fixed time slice are idealizations and might be inappropriate when it comes to describing relativistic situations quantum mechanically. For instance, in the algebraic approach to quantum field theory, the theory is intrinsically characterized by associating with every open region $\mathcal{O}$ of space-time an algebra $\mathcal{A}(\mathcal{O})$ of operators on some Hilbert space $\mathcal{O}\to\mathcal{A}(\mathcal{O})$ [1]. Hegerfeldt's works [2] about localization observables are another example of papers studying observables associated with bounded regions of space-time. However, the notion of an observable associated with an extended region in space-time is foreign to the conceptual framework and formalism of standard quantum mechanics. Therefore, *a priori* it is not clear whether and, if so, how the formalism of quantum mechanics has to be changed to include such space-time observables associated with extended regions.

A downright investigation to derive the possible structure of a space-time quantum theory (and in particular the possible notions of temporal state and observable) from the mathematical structure of standard quantum mechanics has been undertaken by Isham [3], who laid down a set of axioms for *history quantum theories*. With his history quantum theories Isham pointed out an intrinsically quantum-mechanical formalism dealing with space-time observables and states. The main paradigm of the approach is to describe space-time observables and states by operators on certain tensor product Hilbert spaces. This is very natural and works fine as long as we consider finite-dimensional Hilbert spaces and a discrete set of time points, but when one takes into account infinite-dimensional Hilbert spaces or infinitely many time points or continuous time, the tensor product paradigm is quite unnatural from a mathematical point of view. In particular, the decoherence functional, which represents the state in Isham's approach, is in general a mathematically unsatisfactorily behaving object (this will be discussed in more detail in Sec. II below). Another problem of this approach is that the only presently known concrete example for a history quantum theory is standard quantum mechanics over a finite-dimensional Hilbert space.

In the present paper we take a fresh look at the problem. The target of the present investigation is to understand the physical significance of the difficulties within the mathematical framework in Isham's program more properly and to put forth a different, mathematically more natural framework for space-time quantum theories that on the one hand, is broad enough to embrace Isham's approach in the finite-dimensional case, but, on the other hand, goes significantly beyond it.

The paper is organized as follows. In Sec. II we give an account of Isham's program. In Sec. III we put forth our mathematical framework for space-time quantum mechanics and show that our framework contains all known examples of well-defined general history quantum theories as a subclass. We shall also discuss in which sense the framework covers standard quantum mechanics in the infinite-dimensional case. In Sec. IV we will discuss the definition of an information entropy for space-time quantum theory that is a generalization of the Isham-Linden information entropy to our approach. Throughout this work we use the following notation. $\mathfrak{H}$ always denotes the single-time Hilbert space in ordinary quantum mechanics and $\mathfrak{K}$ always denotes the "proposition" Hilbert space introduced in Sec. III. $\mathcal{H}$ or $\mathcal{V}$ denote general Hilbert spaces or tensor product Hilbert spaces. The set of bounded operators on some Hilbert space $\mathcal{H}$ is denoted by $\mathcal{B}(\mathcal{H})$, the set of compact operators on $\mathcal{H}$ by $\mathcal{K}(\mathcal{H})$, and the set of projection operators on some Hilbert space $\mathcal{H}$ by $\mathcal{P}(\mathcal{H})$. We adopt the convention that all inner products and sesquilinear forms on Hilbert spaces are linear in their second variable and conjugate linear in the first variable.

II. HISTORY QUANTUM THEORIES

A. Generalities

In the mathematical formulation of standard quantum mechanics every quantum-mechanical system is characterized

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by some Hilbert space $\mathfrak{H}$ and some semigroup of unitary time evolution operators acting on $\mathfrak{H}$. The possible states of the quantum-mechanical system at some fixed instant of time are identified with the trace class operators on $\mathfrak{H}$ and the possible observables are identified with the self-adjoint operators on $\mathfrak{H}$. It is well known that according to the spectral theorem, every observable can be disintegrated into so-called yes-no observables represented by projection operators on $\mathfrak{H}$. The projection operators represent the elementary propositions about the system. The quantum-mechanical probability for the proposition represented by the projection P is in the state ρ then given by $\mathrm{tr}_{\mathfrak{H}}(P\rho)$.

As already anticipated above, in the history approach one aims at including space-time observables in the quantum-mechanical formalism. In a first moderate step one considers a finite sequence of projection operators $P_{t_1}, \ldots, P_{t_n}$ associated with times $t_1, \ldots, t_n$ that corresponds to a time sequence of propositions. Such a sequence is called a homogeneous history. The quantum-mechanical probability of a history $h \simeq \{P_{t_1}, \ldots, P_{t_n}\}$ is given by

$$\mathrm{tr}_{\mathfrak{H}}(P_{t_n} \cdots P_{t_1} \rho P_{t_1} \cdots P_{t_n})$$

(notice that we work in the Heisenberg picture and suppress for notational simplicity the time dependence of the operators). Slightly abstracting from this expression one defines the *decoherence functional* d_ρ on pairs of homogeneous histories by [4,5]

$$d_\rho(h,k) := \mathrm{tr}_{\mathfrak{H}}(P_{t_n} \cdots P_{t_1} \rho Q_{t_1} \cdots Q_{t_n}), \tag{1}$$

where $h \simeq \{P_{t_1}, \ldots, P_{t_n}\}$ and $k \simeq \{Q_{t_1}, \ldots, Q_{t_n}\}$. Histories that differ from each other only by the insertion or the omission of the unit operator at intermediate times are physically equivalent and identified with each other.

The next major step in the construction of a general history theory is to embed the set of all (equivalence classes of) homogeneous histories injectively into a larger space $\mathcal{V}$, which is endowed with a partially defined sum, such that the decoherence functional d_ρ can be extended unambiguously to a bounded biadditive functional $D_\rho : \mathcal{V} \times \mathcal{V} \to \mathbb{C}$ subject to the further conditions (i) $D_\rho(u,v) = D_\rho(v,u)^*$ for all $u,v \in \mathcal{V}$ and (ii) $D_\rho(u,u) \geqslant 0$ for all $u \in \mathcal{V}$. [In the previous literature about general history quantum theories the space $\mathcal{V}$ carried usually some additional structure, e.g., that of a lattice or a unital *-algebra. In the history approach put forth by Gell-Mann and Hartle (see [6–9] and below) the embedding of the homogeneous histories into the larger space of so-called *class operators* is not injective.] The homogeneous histories are identified with their images in $\mathcal{V}$, which are also called homogeneous histories. The elements in $\mathcal{V}$ that are not an image of some homogeneous history are called *inhomogeneous histories*. All elements in $\mathcal{V}$ are interpreted as the general (measurable) space-time propositions in the theory.

A subset $\mathcal{V}_0$ of $\mathcal{V}$ is called a *consistent set of histories* if D_ρ induces a probability measure $p : \mathcal{V}_0 \to \mathbb{C}$, $p(v_0) := D_\rho(v_0, v_0)$ on $\mathcal{V}_0$. The consistent subsets of $\mathcal{V}$ are exactly those subsets $\mathcal{V}_0$ that can be endowed with a Boolean structure and satisfy $\mathrm{Re} D_\rho(u_0, v_0) = 0$ for all mutually disjoint $u_0, v_0 \in \mathcal{V}_0$. The quantum character of the theory and the principle of complementarity exhibit themselves in the fact that there are several mutually inconsistent consistent subsets of $\mathcal{V}$ [5].

B. Isham's history theories

In the history approach developed by Isham and coworkers [3,10–18] and other authors [19–28] the crucial observation was that homogeneous histories as above can be mathematically conveniently described by using a tensor product formalism. Correspondingly, the homogeneous history $h \simeq \{P_{t_1}, \ldots, P_{t_m}\}$ is mapped to the projection operator $P_{t_1} \otimes \cdots \otimes P_{t_m}$ on the tensor product Hilbert space $\otimes_{t_i \in \{t_1, \ldots, t_m\}} \mathfrak{H}_{t_i}$ where $\mathfrak{H}_{t_i} = \mathfrak{H}$ for all i. It is mathematically convenient to postulate that the space of all histories is given by $\mathcal{P}(\otimes_{t_i \in \{t_1, \ldots, t_n\}} \mathfrak{H}_{t_i})$. Some of the *inhomogeneous* elements within $\mathcal{P}(\otimes_{t_i \in \{t_1, \ldots, t_n\}} \mathfrak{H}_{t_i})$ can be straightforwardly interpreted as coarse grainings of homogeneous histories (in the proposition picture these histories represent composed propositions such as "h_1 or h_2 are true," etc.). However, if the single-time Hilbert space $\mathfrak{H}$ is infinite dimensional there are always some elements in $\mathcal{P}(\otimes_{t_i \in \{t_1, \ldots, t_n\}} \mathfrak{H}_{t_i})$ that admit no physical interpretation as coarse graining of homogeneous histories.

Again, histories that differ from each other only by the insertion or the omission of the unit operator at intermediate times are physically equivalent and identified with each other. We shall refer to this natural equivalence relation between histories as the *canonical equivalence* and use the symbol $\sim_c$ in the following. For every history h defined on $\otimes_{t_i \in \{t_1, \ldots, t_n\}} \mathfrak{H}_{t_i}$, where $\mathfrak{H}_{t_i} = \mathfrak{H}$ for all i, consider its equivalence class $\varepsilon(h)$ of histories. We say that a finite set of time points $s = \{t_1, \ldots, t_m\}$ is the *support of* h if (i) there is an element in $\varepsilon(h)$ defined on $\otimes_{t_i \in s} \mathfrak{H}_{t_i}$ and (ii) for every proper subset s' of s there is no element in $\varepsilon(h)$ defined on $\otimes_{t_i \in s'} \mathfrak{H}_{t_i}$. Mathematically speaking, the system $\{\mathcal{P}(\mathfrak{H}_{t_1} \otimes \cdots \otimes \mathfrak{H}_{t_n}) | \{t_1, \ldots, t_n\} \subset \mathbb{R}\}$ of sets of projection operators associated with all possible finite sets of time points forms a *directed* or *inductive system*. The space of all histories is identified with the disjoint union over the sets of *all* projections on all finite tensor product Hilbert spaces of the form $\otimes_{t_i \in \{t_1, \ldots, t_n\}} \mathfrak{H}_{t_i}$ modulo the physical equivalence $\sim_c$ or, mathematically more precise, with the *direct* or *inductive limit* of the directed system of histories

$$\{\mathcal{P}(\mathfrak{H}_{t_1} \otimes \cdots \otimes \mathfrak{H}_{t_n}) | \{t_1, \ldots, t_n\} \subset \mathbb{R}\}$$

(a proof for the existence of this direct limit can be found, for instance, in [19]).

In an important paper Isham, Linden, and Schreckenberg [14] showed that if the single-time Hilbert space $\mathfrak{H}$ is finite dimensional, the decoherence functional d_ρ defined on pairs of homogeneous histories can be unambiguously extended to a bounded biadditive functional D_ρ on the space of all histories. Moreover, they showed that for fixed n and ρ there exists a trace class operator $\mathfrak{X}_\rho$ on $\otimes_{2n} \mathfrak{H}$ such that D_ρ can be written as

$$D_\rho(u,v) = \mathrm{tr}_{\otimes_{2n} \mathfrak{H}}(u \otimes v \mathfrak{X}_\rho)$$

for all $u,v \in \mathcal{P}(\mathfrak{H}_{t_1} \otimes \cdots \otimes \mathfrak{H}_{t_n})$.

Abstracting from this result the properties of general decoherence functionals, Isham [3] arrived at an axiomatic characterization of general history quantum theories according to which a general history quantum theory is given by its space $\mathcal{U}$ of histories and by its space $\mathcal{D}$ of decoherence functionals. The histories in $\mathcal{U}$ and the decoherence functionals in $\mathcal{D}$ represent the (measurable) propositions and the states in the theory, respectively. The space of histories $\mathcal{U}$ is required to have a partial sum defined on it and to contain a unit **1**. Every decoherence functional $d \in \mathcal{D}$ is required to be bounded and additive in both arguments and has to satisfy (i) $d(\mathbf{1},\mathbf{1}) = 1$, (ii) $d(x,x) \geqslant 0$, and (iii) $d(x,y) = d(y,x)^*$ for all histories $x,y \in \mathcal{U}$.

A choice for the space of histories in a general history quantum theory suggesting itself is the set of projection operators $\mathcal{P}(\mathcal{H})$ on some Hilbert space $\mathcal{H}$ or, slightly more general, the set of projection operators $\mathcal{P}(\mathcal{A})$ in a von Neumann algebra $\mathcal{A}$. In the case that the space of histories is given by $\mathcal{P}(\mathcal{H})$ for some finite-dimensional Hilbert space $\mathcal{H}$ Isham, Linden, and Schreckenberg [14] showed that for every bounded decoherence functional d on $\mathcal{P}(\mathcal{H})$ there exists a trace class operator $\mathfrak{X}_d$ on $\mathcal{H} \otimes \mathcal{H}$ such that

$$d(x,y) = \mathrm{tr}_{\mathcal{H}\otimes\mathcal{H}}(x \otimes y \mathfrak{X}_d) \tag{2}$$

holds for all $x,y \in \mathcal{P}(\mathcal{H})$. Subsequently, Isham and coworkers studied different aspects of general history quantum theories over finite-dimensional Hilbert spaces in some detail, buoying up the fruitfulness of the tensor-product-based approach.

On the other hand, the use of tensor product spaces to describe temporarily extended objects has its limitations. Despite some interesting research and progress made recently [12,15], the incorporation of continuous histories into the approach is still a challenge and the tensor-product-based approach does not seem to be well adapted to it.

Recently the present author and Wright [24] showed that if the single-time Hilbert space $\mathfrak{H}$ in standard quantum mechanics is infinite dimensional, then no decoherence functional d_ρ [corresponding to the initial state ρ; see Eq. (1)] defined on homogeneous histories can be extended to a bounded or even to a finitely valued functional on the space of ''all histories'' in Isham's approach. In [24] an example for an element $h_\infty \in \mathcal{P}(\mathfrak{H}_{t_1} \otimes \cdots \otimes \mathfrak{H}_{t_n})$ was constructed such that no decoherence functional d_ρ assumes a finite value at h_∞ if extended (see also Appendix B). One possible way out of this dilemma is to allow for decoherence functionals assuming values in the Riemann sphere $\mathbb{C} \cup \{\infty\}$. Histories h with $d(h,h) = \infty$ are then called *singular* histories for the decoherence functional d. Histories $h \in \mathcal{P}(\mathfrak{H}_{t_1} \otimes \cdots \otimes \mathfrak{H}_{t_n})$ with $d_\rho(h,h) > 1$ or $d_\rho(h,h) = \infty$ are in no consistent set and represent no physical propositions in the state d_ρ. Adopting this point of view, one could simply forget about the singular histories. However, in [24] it has been shown that there are certain histories in $\mathcal{P}(\mathfrak{H}_{t_1} \otimes \cdots \otimes \mathfrak{H}_{t_n})$ that are singular for every decoherence functional d_ρ. Thus these singular histories will be in no consistent set of histories for all states d_ρ. This result indicates that in the infinite-dimensional case the space $\mathcal{P}(\mathfrak{H}_{t_1} \otimes \cdots \otimes \mathfrak{H}_{t_n})$ contains unphysical elements that represent no physical histories at all. We shall propose an alternative mathematical framework for the description of temporal quantum theories and we shall show that the decoherence functional d_ρ of the history version of standard quantum mechanics (when cast into the present framework) is a finitely valued functional.

For the convenience of the reader and for later reference we cite the following representation of the standard decoherence functional as an in general infinite sum. For n-time homogeneous histories p and q and for a finite-dimensional or infinite-dimensional single-time Hilbert space $\mathfrak{H}$ [14,24] we have

$$d_\rho(p,q) = \sum_{j_1,\ldots,j_{2n}} \omega_{j_1} \langle \tilde{\epsilon}_{j_1,\ldots,j_{2n}}, (p \otimes q)\epsilon_{j_1,\ldots,j_{2n}} \rangle, \tag{3}$$

where we have introduced the abbreviations

$$\epsilon_{j_1,\ldots,j_{2n}} := \psi_{j_1} \otimes e^{2n}_{j_{2n}} \otimes \cdots \otimes e^{n+2}_{j_{n+2}} \otimes e^2_{j_2} \otimes \cdots \otimes e^{n+1}_{j_{n+1}},$$

$$\tilde{\epsilon}_{j_1,\ldots,j_{2n}} := e^{2n}_{j_{2n}} \otimes \cdots \otimes e^{n+1}_{j_{n+1}} \otimes \psi_{j_1} \otimes e^2_{j_2} \otimes \cdots \otimes e^n_{j_n}$$

and $\{e^k_{j_k}\}$ are orthonormal bases of $\mathfrak{H}$ for all $2 \leqslant k \leqslant 2n$, where $\rho = \Sigma_i \omega_i P_{\psi_i}$ denotes the spectral resolution of ρ, and P_{ψ_i} denotes the projection operator onto the subspace of $\mathfrak{H}$ spanned by ψ_i, and $\omega_i \geqslant 0$ for all i. We shall always assume that the orthonormal system $\{\psi_i\}$ has been extended to an orthonormal basis of $\mathfrak{H}$.

C. Gell-Mann–Hartle history theories

The main predecessor to Isham's history quantum theories was the approach put forth by Gell-Mann and Hartle [6–9]. In this approach the homogeneous histories are mapped to so-called class operators

$$h \simeq \{P_{t_1}, \ldots, P_{t_n}\} \rightarrow C(h) := P_{t_1} \cdots P_{t_n}.$$

Notice that again we use the Heisenberg picture and suppress the explicit time dependence of the operators. Class operators act on the single-time Hilbert space $\mathfrak{H}$. The inhomogeneous histories are indirectly defined by so-called *coarse graining* prescriptions. An inhomogeneous history in the Gell-Mann–Hartle approach is in general a sum of class operators that correspond to mutually exclusive homogeneous histories. While the decoherence functional d_ρ extends to inhomogeneous class operators straightforwardly (by linearity in both arguments) and the interpretation of the inhomogeneous histories is equally straightforward, the formalism essentially stays on the level of homogeneous histories and we are lacking a simple and direct characterization of the mathematical structure of the space of class operators, which makes the Gell-Mann–Hartle approach virtually incomprehensible to a rigorous mathematical investigation: Given an arbitrary operator $\mathfrak{c}$ in the unit sphere of $\mathcal{B}(\mathfrak{H})$, we have no simple criterion to decide whether or not $\mathfrak{c}$ is a class operator and if so, whether it is a homogeneous class operator or an inhomogeneous class operator. Moreover, as already mentioned above, the map of homogeneous histories to class operators is by no means injective and in general there is more than one homogeneous history corresponding to a given class operator.

III. TEMPORAL QUANTUM MECHANICS

A. General framework

In this section we put forth our framework for space-time quantum mechanics. We first state the main principles without further motivation and then proceed to show how standard quantum mechanics and Isham's general history theories over finite-dimensional Hilbert spaces fit into the scheme, which serves as an *a posteriori* motivation.

The basic ingredient in our framework of *temporal quantum theories* is a Hilbert space $\mathfrak{K}$ whose elements are interpreted as the (measurable) space-time propositions about the system. Actually, not all elements in $\mathfrak{K}$ represent physically meaningful propositions; see below. In spite of this we shall refer to the Hilbert space $\mathfrak{K}$ as the space of propositions and to the elements of $\mathfrak{K}$ simply as propositions. We assume that there exists one distinguished element in $\mathfrak{K}$, denoted by e, that represents the indifferent proposition that is always true. The trivial proposition complementary to e that is always false is identified with the zero vector in $\mathfrak{K}$.

We shall argue that the norm induced by the inner product $\langle\ ,\ \rangle$ in $\mathfrak{K}$ subsumes the *a priori* structural information about the propositions that is encoded within the propositions Hilbert space $\mathfrak{K}$. It is a quantitative measure for the fine grainedness of propositions within the descriptive scheme provided by $\mathfrak{K}$, i.e., the smaller $\langle b,b\rangle$ is, the more ''fine grained'' the proposition corresponding to $b\in\mathfrak{K}$ is. More specifically, we shall see below that the amount of information associated with a proposition $b\in\mathfrak{K}$ is given by $-\ln[p(b)/\langle b,b\rangle]$ [where $p(b)$ denotes the probability of b], which is just the difference between the information associated with the probability distribution and the structural information encoded in $\langle\ ,\ \rangle$.

It is very important not to confuse the ''temporal'' Hilbert space $\mathfrak{K}$ with the single-time Hilbert space $\mathfrak{H}$ in ordinary quantum mechanics. The elements of the single-time Hilbert space $\mathfrak{H}$ in ordinary quantum mechanics correspond to the possible pure *states* of the system. Hence the two Hilbert spaces $\mathfrak{K}$ and $\mathfrak{H}$ have *a priori* nothing to do with each other. We shall clarify the *a posteriori* relation between the temporal Hilbert space $\mathfrak{K}$ and the single-time Hilbert space $\mathfrak{H}$ below.

We shall see that in Isham's abstract history quantum theories the propositions Hilbert space $\mathfrak{K}$ in general depends both on the single-time Hilbert space $\mathfrak{H}$ and on the quantum state given by some decoherence functional d. Physically this reflects the fact that the global propositions one may sensible ask about the system can change when the global state d of the system is changed. We shall see that essentially the Hilbert space $\mathfrak{K}$ corresponding to a decoherence functional d is constructed from a larger space of propositions by omitting some propositions with vanishing probability in the state d.

The quantum-mechanical temporal states of the system are given by self-adjoint bounded operators T on $\mathfrak{K}$ such that $\langle e,Te\rangle=1$, where $\langle\ ,\ \rangle$ denotes the inner product in $\mathfrak{K}$. We denote the set of all such operators by $\mathcal{W}_e(\mathfrak{K})$. Wright showed that one can associate a bounded operator T with every bounded decoherence functional in a general history quantum theory (see [25] and below). Thus we shall refer to the state operator T also as the *Wright operator* of the system.

Since the bounded, self-adjoint operators on $\mathfrak{K}$ are in one-to-one correspondence with bounded sesquilinear forms on $\mathfrak{K}$, we can alternatively define the states as bounded sesquilinear forms s on $\mathfrak{K}$ satisfying $s(e,e)=1$.

We propose that the expression for the probability functional on $\mathfrak{K}$ in the state given by the operator $T\in\mathcal{W}_e(\mathfrak{K})$ is given by

$$p_T(x)=\langle x,Tx\rangle \tag{4}$$

for all $x\in\mathfrak{K}$. This proposal is motivated by the history formulation of standard quantum mechanics, discussed in Sec. III B. We shall see there that we can find for every fixed temporal support s a T with $\langle e,Te\rangle=1$ such that p_T can be interpreted as the probability for propositions with temporal support s. As will also become clear below, in general, however, the probabilities associated with propositions corresponding to different temporal supports cannot be properly normalized. Only the probability density (i.e., the probability per quantum degree of freedom) and the probabilities for fixed temporal supports can be brought into the form (4) for some appropriate T. This substantiates our *proposal* that in a general theory without an *a priori* space-time the Wright operator generates the probability densities via Eq. (4). Every subsystem of the quantum system in question is characterized by some subset of all quantum degrees of freedom and the probabilities for propositions corresponding to some subsystem can be obtained from Eq. (4) by multiplying with a suitable normalization factor (representing the number of quantum degrees of freedom corresponding to the subsystem in question).

From the definition of the ''probability'' functional $p_T:\mathfrak{K}\to\mathbb{R}$ it is obvious that p_T is not necessarily positive definite and defines no linear functional on all of $\mathfrak{K}$. Thus it is useful to adopt a consistent-histories-type point of view [5]. All propositions $y\in\mathfrak{K}$ with either $p_T(y)<0$ or $p_T(y)>1$ are assumed to be physically meaningless in the state $T\in\mathcal{W}_e(\mathfrak{K})$. We say that a set $\mathcal{C}:=\{x_i|i\in I,x_i\in\mathfrak{K}\}$ is *consistent* in the state $T\in\mathcal{W}_e(\mathfrak{K})$ if (i) $x_i\perp x_j$ for $i\neq j$, (ii) $\Sigma_i x_i=e$, (iii) $0<p_T(x_i)\leqslant 1$ for all i, and (iv) $\Sigma_i p_T(x_i)=p_T(\Sigma_i x_i)=p_T(e)=1$. Consistent sets of propositions represent the analog of sets of commuting, mutually exclusive yes-no observables in standard quantum mechanics and the existence of several mutually inconsistent (complementary) consistent sets of propositions reflects the quantum character of the theory. The space of *all* (measurable) propositions in our approach carries the structure of a Hilbert space but no lattice theoretical structure. Only the consistent subsets of $\mathfrak{K}$ carry the structure of a Boolean algebra. It is a virtue of the consistent-histories philosophy that it allows us to consider spaces of propositions without any lattice theoretical structure on it and our unifying mathematical treatment of temporal quantum theories relies heavily upon this feature of the consistent-histories philosophy. Consistent sets of propositions are also called *windows* or *frameworks* for the description of a quantum system. A *refinement* $W_2:=\{y_j\}_{j\in J}$ of a consistent set $W_1:=\{x_i\}_{i\in I}$ for T is a consistent set for T such that each element $x_i\in W_1$ can be written as a finite sum of

elements in W_2. A consistent set is said to be *maximally refined* if it has no consistent refinement.

We call $\mathfrak{K}$ the space of propositions about the system. However, notice that there are in general many elements in $\mathfrak{K}$ that are in no consistent set for some state T. The Hilbert space $\mathfrak{K}$ serves as a mathematically nice space into which the propositions are embedded. The example of standard quantum mechanics discussed in Sec. III B will clarify this point. The consistency condition for every physical state singles out the elements that can be interpreted as physically meaningful propositions in the respective state of the system.

The framework for temporal quantum theories introduced above must not be considered as a fixed, rigid set of axioms but should be viewed as a set of *cum grano salis* working hypotheses that might be in need for change in the future. In fact, we shall see that the history version of standard quantum mechanics over infinite-dimensional Hilbert spaces fits only into the framework when one allows for elements of infinite norm in the Hilbert space $\mathfrak{K}$. We shall call such Hilbert spaces *improper Hilbert spaces*; see Appendix A. (We mention that there is an alternative formulation of the history version of standard quantum mechanics over an infinite-dimensional Hilbert space as a temporal quantum theory obtained in [24] in which all information about probabilities and the quantum state is thrown into the propositions Hilbert space $\mathfrak{K}$.) At the basis of the present investigation is the (tacit) assumption that essential features of a mathematical framework for temporal quantum mechanics can be read off a temporal reformulation of ordinary quantum mechanics. Infinite-dimensional Hilbert spaces are needed in ordinary quantum mechanics to describe observables with a continuous spectrum such as position and momentum observables. Thus it can be argued that the mathematical difficulties in the history version of quantum mechanics over infinite-dimensional Hilbert spaces are connected with the fact that standard quantum mechanics involves the concept of an underlying space-time continuum (or involves other observables with continuous spectrum). An appealing idea put forward by many authors is that in a quantum theory of space-time observables the underlying concept of space-time should, in one way or another, be of a discrete nature. Moreover, the form of the canonical decoherence functional in standard quantum mechanics is based on the idealized notions of states and observables at a fixed time instant and is intrinsically nonrelativistic (in that the prescription for the computation of probabilities involves a series of *pro forma* global reductions of the state). Accordingly, we cannot expect that the mathematical structure of a quantum theory of space-time events can be fully derived from standard quantum mechanics. In particular we feel that the appearance of ''infinitely'' coarse grained histories in the history version of standard quantum mechanics is a reflection of the fact that the theory is based on overidealized notions such as observables and states at some instant of time. Therefore, arguably, it is inappropriate to base our mathematical framework for temporal quantum theories upon the concept of improper Hilbert space.

B. Examples

1. Isham's history quantum theories

As a first example we consider Isham's abstract history quantum theories over some finite- or infinite-dimensional Hilbert space $\mathcal{V}$ (with dimension greater than 2). In this approach the space of histories is identified with the set $\mathcal{P}(\mathcal{V})$ of projections on $\mathcal{V}$ and the state is given by some bounded decoherence functional d, i.e., by some bounded biorthoadditive functional $d:\mathcal{P}(\mathcal{V})\times\mathcal{P}(\mathcal{V})\to\mathbb{C}$ satisfying (i) $d(1,1)=1$, (ii) $d(p,q)=d(q,p)^*$, and (iii) $d(p,p)\geq 0$ for all $p,q\in\mathcal{P}(\mathcal{V})$.

We appeal now to an important result of Wright [25], Corollary 4, according to which there exists a Hilbert space $\mathfrak{K}$, a self-adjoint bounded operator T on $\mathfrak{K}$, and a map $x\to[x]$ from $\mathcal{B}(\mathcal{V})$ into a dense subspace of $\mathfrak{K}$ such that $D:\mathcal{B}(\mathcal{V})\times\mathcal{B}(\mathcal{V})\to\mathbb{C}, D(x,y)=\langle[x],T[y]\rangle$ is an extension of d. With $e:=[1]$ it follows that $\langle e,Te\rangle=1$. Thus Wright's result implies that a general history quantum theory can always be brought into the form of a space-time quantum theory. If $\mathcal{V}$ is finite dimensional, then it is possible to show that the Hilbert space $\mathfrak{K}$ can be chosen independently of d and may be identified with $\mathcal{B}(\mathcal{V})$; see Remark (v) in Sec. III in [25].

Wright's result depends crucially on the fact that $D:\mathcal{B}(\mathcal{V})\times\mathcal{B}(\mathcal{V})\to\mathbb{C}$ satisfies a Haagerup-Pisier-Grothendieck inequality, i.e., that there exists a positive linear functional ϕ on $\mathcal{B}(\mathcal{V})$ with $\phi(1)=1$ and a constant $C>0$ such that

$$|D(x,y)|^2\leq C\,\phi(xx^\dagger+x^\dagger x)\,\phi(yy^\dagger+y^\dagger y)$$

for all $x,y\in\mathcal{B}(\mathcal{V})$. The semi-inner product on $\mathcal{B}(\mathcal{V})$ is then constructed from ϕ as $\langle y,x\rangle_\phi=\frac{1}{2}\phi(xy^\dagger+y^\dagger x)$. Let N_ϕ be the corresponding null space; then $\mathfrak{K}$ is chosen as the completion of $\mathcal{B}(\mathcal{V})/N_\phi$ with respect to the inner product induced by ϕ. The functional ϕ is not unique. For every ϕ there exists a trace class operator τ_ϕ on $\mathcal{V}$ such that $\phi(x)=\mathrm{tr}_{\mathcal{V}}(x\tau_\phi)$ for all $x\in\mathcal{B}(\mathcal{V})$. Thus the Hilbert space $\mathfrak{K}$ depends on d. As just explained, the space $\mathfrak{K}$ is always (the completion of a space) of the form $\mathcal{B}(\mathcal{V})/N$, where N is a set of elements of $\mathcal{B}(\mathcal{V})$ with $D(n,n)=0$ for $n\in N$. Loosely speaking, one may think of $\mathfrak{K}$ as a subspace of $\mathcal{B}(\mathcal{V})$ in which some unphysical elements with vanishing probability have been dismissed. The different choices of ϕ correspond to different null spaces N_ϕ. The probabilities for physical propositions do not change for different choices of ϕ, but the number of unphysical propositions with probability zero in $\mathfrak{K}$ changes. That is, changing ϕ amounts to changing the number of redundant propositions in $\mathfrak{K}$. Wright's result also applies to arbitrary von Neumann algebras with no type I_2 direct summand.

2. The history version of standard quantum mechanics

As before, we denote the single-time Hilbert space by $\mathfrak{H}$ and the decoherence functional associated with the state ρ by d_ρ. Consider first the case that $\mathfrak{H}$ is finite dimensional. As discussed above, the homogeneous histories associated with the times $\{t_1,\ldots,t_n\}$ are identified with homogeneous projection operators of the form $P_{t_1}\otimes\cdots\otimes P_{t_n}$ on $\mathfrak{H}_{t_1}\otimes\cdots\otimes\mathfrak{H}_{t_n}$. The space of all histories is identified with the direct limit of the directed system of histories

$$\{\mathcal{P}(\mathfrak{H}_{t_1}\otimes\cdots\otimes\mathfrak{H}_{t_n})|\{t_1,\ldots,t_n\}\subset\mathbb{R}\},$$

as discussed earlier. Consider some fixed n and some fixed set of times $\{t_1,\ldots,t_n\}$ and write $\mathcal{V}_{t_1,\ldots,t_n}:=\mathfrak{H}_{t_1}\otimes\cdots\otimes\mathfrak{H}_{t_n}$. Consider the restriction of d_ρ to $\mathcal{P}(\mathcal{V}_{t_1,\ldots,t_n})$. Isham, Linden, and Schreckenberg have shown that there exists a trace class operator $\mathfrak{X}_d$ on $\mathcal{V}_{t_1,\ldots,t_n}\otimes\mathcal{V}_{t_1,\ldots,t_n}$ such that

$$d_\rho(p,q)=\mathrm{tr}_{\mathcal{V}_{t_1,\ldots,t_n}\otimes\mathcal{V}_{t_1,\ldots,t_n}}(p\otimes q\mathfrak{X}_d)$$

for all $p,q\in\mathcal{P}(\mathcal{V}_{t_1,\ldots,t_n})$. From this it is obvious that d_ρ can be extended to a bounded functional on all of $\mathcal{B}(\mathcal{V}_{t_1,\ldots,t_n})$. For what follows it is convenient to introduce the *density*

$$\delta_\rho(p,q):=\frac{\mathrm{tr}_{\mathcal{V}_{t_1,\ldots,t_n}\otimes\mathcal{V}_{t_1,\ldots,t_n}}(p\otimes q\mathfrak{X}_d)}{\mathrm{tr}_{\mathcal{V}_{t_1,\ldots,t_n}}(1)}$$

for all $p,q\in\mathcal{P}(\mathcal{V}_{t_1,\ldots,t_n})$. The quantity $\delta_\rho(p,p)$ is then a probability per quantum (space-time) degree of freedom.

One can look upon d_ρ and the space of all histories from a slightly different perspective. To this end consider the directed system $\{\mathcal{B}(\mathfrak{H}_{t_1}\otimes\cdots\otimes\mathfrak{H}_{t_n})|\{t_1,\ldots,t_n\}\subset\mathbb{R}\}$ and its direct limit, which we denote by $\mathcal{B}$ (the existence of this direct limit as a C^*-algebra follows, e.g., from Proposition 11.4.1 in [29]). Consider the map π that maps every homogeneous bounded operator $b:=b_{t_1}\otimes\cdots\otimes b_{t_n}$ on $\mathcal{B}(\mathcal{V}_{t_1,\ldots,t_n})$ to $\pi(b):=b_{t_1}\cdots b_{t_n}\in\mathcal{B}(\mathfrak{H})$. From Proposition 11.1.8 (ii) in [29] it follows that π can be uniquely extended to a linear map from $\mathcal{B}$ to $\mathcal{B}(\mathfrak{H})$, which we will also denote by π. Every d_ρ can be extended to a sesquilinear form D_ρ on $\mathcal{B}$ such that this extension can then be written as

$$D_\rho:\mathcal{B}\times\mathcal{B}\to\mathbb{C},D_\rho(b_1,b_2):=\mathrm{tr}_{\mathfrak{H}}[\pi(b_1)^\dagger\rho\pi(b_2)].$$

The corresponding extension of δ_ρ will be denoted by Δ_ρ. Consider some fixed set of time points $\{t_1,\ldots,t_n\}$. We define an inner product on $\mathcal{B}$ by

$$\langle b_1,b_2\rangle:=\frac{\mathrm{tr}_{\mathfrak{H}_{t_1}\otimes\cdots\otimes\mathfrak{H}_{t_n}}(b_1^\dagger b_2)}{\mathrm{tr}_{\mathfrak{H}_{t_1}\otimes\cdots\otimes\mathfrak{H}_{t_n}}(1)}$$

for all $b_1,b_2\in\mathcal{B}$, where $\{t_1,\ldots,t_n\}$ denotes the support of $b_1^\dagger b_2$. We shall denote the norm induced by the inner product $\langle\,,\,\rangle$ by $\|\;\|_2$ for reasons to become clear below. The factor $\mathrm{tr}_{\mathfrak{H}_{t_1}\otimes\cdots\otimes\mathfrak{H}_{t_n}}(1)$ in needed to ensure the additivity of $\langle\,,\,\rangle$ on all of $\mathcal{B}$. Denote by $\mathfrak{K}$ the Hilbert space completion of $\mathcal{B}$ with respect to $\langle\,,\,\rangle$. Notice that although $\mathfrak{H}$ is finite dimensional, $\mathcal{B}$ and $\mathfrak{K}$ are infinite dimensional. Since the trace is a bounded linear functional on $\mathcal{B}(\mathfrak{H}_{t_1}\otimes\cdots\otimes\mathfrak{H}_{t_n})$ and since D_ρ is bounded with respect to the ordinary operator norm, it follows that Δ_ρ extends uniquely to a bounded sesquilinear form on $\mathfrak{K}$. So there exists a bounded, self-adjoint operator τ_ρ in $\mathcal{B}(\mathfrak{K})$ such that $\Delta_\rho(x,y)=\langle x,\tau_\rho y\rangle$ for all $x,y\in\mathfrak{K}$. Let $b\in\mathcal{B}$ and let $\{t_1,\ldots,t_m\}$ denote the temporal support of b. Define

$$T_\rho b:=(\dim\mathfrak{H})^m\tau_\rho b.$$

Then T_ρ is an unbounded operator on $\mathfrak{K}$ whose (dense) domain of definition is $\mathcal{B}$. T_ρ is not self-adjoint on all of $\mathcal{B}$. However, the restriction of T_ρ to a subset of $\mathcal{B}$ containing only elements with fixed support is bounded and self-adjoint. Then the sesquilinear form D_ρ obviously satisfies $D_\rho(b,b)=\langle b,T_\rho b\rangle$ for all $b\in\mathcal{B}$. Since sesquilinear forms are uniquely determined by their quadratic forms we find $D_\rho(b_1,b_2)=\langle b_1,T_\rho b_2\rangle$ for all $b_1,b_2\in\mathcal{B}$ for which the supports of b_1 and b_2 are equal. Let, finally, $e:=1$ denote the indifferent proposition that is always true; then T_ρ satisfies $\langle e,T_\rho e\rangle=1$.

This shows that the history version of standard quantum mechanics over a finite-dimensional single-time Hilbert space $\mathfrak{H}$ can indeed be brought into the form of a space-time quantum theory as formulated in Sec. III A. Here the propositions Hilbert space $\mathfrak{K}$ is independent of the initial quantum state ρ and only the temporal quantum state T_ρ depends on ρ.

In the operator formulation of the history version of quantum mechanics propositions are identified with projection operators and *consistent sets* of propositions are defined with the help of d_ρ as those exhaustive sets $\mathcal{C}$ of mutually perpendicular projections for which $\mathrm{Re}\,d_\rho(p,q)=0$ for all $p,q\in\mathcal{C}$. Comparing this with our definition of consistent sets of propositions in the propositions Hilbert space $\mathfrak{K}$ given in Sec. III A, we see that, in the case of finite-dimensional standard quantum mechanics, every consistent set of propositions in $\mathfrak{K}$ corresponds, when pulled back to the standard operator formulation, to a consistent set of projection operators.

Given now some proposition $\overline{p}\in\mathfrak{K}$ corresponding, on the operator level, to some projection operator p, the larger the space on which p projects is, the larger the norm of the image $\overline{p}$ of p in $\mathfrak{K}$ is. This substantiates our physical interpretation of $\|b\|_2^2=\langle b,b\rangle$ as a quantitative measure of how coarse grained the proposition $b\in\mathfrak{K}$ is.

We notice in passing that for any $p\in\mathbb{R}$ with $p\geqslant 1$ we can define a norm on $\mathcal{B}$ by

$$\|b\|_p:=\left(\frac{\mathrm{tr}_{\mathfrak{H}_{t_1}\otimes\cdots\otimes\mathfrak{H}_{t_n}}[(b^\dagger b)^{p/2}]}{\mathrm{tr}_{\mathfrak{H}_{t_1}\otimes\cdots\otimes\mathfrak{H}_{t_n}}(1)}\right)^{1/p} \tag{5}$$

for all $b\in\mathcal{B}$, where $\{t_1,\ldots,t_n\}$ denotes the support of b. For a proof see for instance, [30], Sec. V.6. The norm $\|\;\|_2$ induced by the inner product $\langle\,,\,\rangle$ obviously corresponds to $p=2$. (Notice that $\|\;\|_p$ defines no crossnorm on $\mathcal{B}$.)

Next consider the case that the single-time Hilbert space $\mathfrak{H}$ is infinite dimensional. To simplify notation we assume in the following that $\mathfrak{H}$ is separable. The extension of our results to nonseparable Hilbert spaces is obvious. We proceed in analogy with the finite-dimensional case. The algebraic tensor product of $\mathcal{B}(\mathfrak{H}_{t_1}),\ldots,\mathcal{B}(\mathfrak{H}_{t_n})$ is the set of all finite linear combinations of homogeneous operators

$$b_1\otimes\cdots\otimes b_n,$$

where $b_i\in\mathcal{B}(\mathfrak{H}_{t_i})$ and is denoted by

$$\mathcal{B}(\mathfrak{H}_{t_1})\otimes_{alg}\cdots\otimes_{alg}\mathcal{B}(\mathfrak{H}_{t_n}).$$

Consider the directed system

$$\{\mathcal{B}(\mathfrak{H}_{t_1})\otimes_{alg}\cdots\otimes_{alg}\mathcal{B}(\mathfrak{H}_{t_n})|\{t_1,\ldots,t_n\}\subset\mathbb{R}\}$$

and its direct limit, which we denote by $\mathcal{B}_{alg}$ (its existence as a C^*-algebra follows again by Proposition 11.4.1 in [29]). Define the map π on homogeneous elements of $\mathcal{B}_{alg}$ as in the finite-dimensional case by $\pi(b_1\otimes\cdots\otimes b_n)=b_1\cdots b_n$; then it follows by Proposition 11.1.8 in [29] that π can be uniquely extended to a linear map on $\mathcal{B}_{alg}$. Again we denote the extension of π also by π (slightly abusing the notation). As in the finite-dimensional case the decoherence functional d_ρ associated with ρ can be extended to a sesquilinear form defined on all of $\mathcal{B}_{alg}$ and the extension D_ρ of d_ρ can be written as

$$D_\rho:\mathcal{B}_{alg}\times\mathcal{B}_{alg}\rightarrow\mathbb{C},\quad D_\rho(b_1,b_2)=\mathrm{tr}_{\mathfrak{H}}[\pi(b_1)^\dagger\rho\pi(b_2)].$$

We remark that the representation equation (3) is also valid for D_ρ on all of $\mathcal{B}_{alg}$ (this follows by linearity).

Let $b_1,b_2\in\mathcal{B}_{alg}$; then we define

$$\langle b_1,b_2\rangle=\mathrm{tr}_{\mathfrak{H}_{t_1}\otimes\cdots\otimes\mathfrak{H}_{t_n}}(b_1^\dagger b_2),$$

where $\{t_1,\ldots,t_n\}$ is the support of $b_1^\dagger b_2$. This expression is not well defined for arbitrary b_1 and b_2. If it is not well defined, then we formally set $\langle b_1,b_2\rangle:=\infty$. It is clear that the elements $b\in\mathcal{B}_{alg}$ with finite norm $\|b\|^2=\langle b,b\rangle<\infty$ are exactly the Hilbert-Schmidt operators in $\mathcal{B}_{alg}$. In particular, $\|b\|=0$ for $b\in\mathcal{B}_{alg}$ implies that $b=0$. It is well known that every trace class operator $\tau\in\mathcal{B}_{alg}$ satisfies $\mathrm{tr}[(\tau^\dagger\tau)^{1/2}]<\infty$.

We interpret $\langle\,,\rangle$ as an improper inner product on $\mathcal{B}_{alg}$ (see Appendix A). This implies in particular that additivity is only required in the finite sectors of $\mathcal{B}_{alg}$. Thus the factor $\mathrm{tr}(1)$ appearing in the definition of $\langle\,,\rangle$ in the finite-dimensional case is not only not well defined but also not needed to ensure additivity in the finite sectors of $\mathcal{B}_{alg}$. The completion of $\mathcal{B}_{alg}$ with respect to $\langle\,,\rangle$ is an improper Hilbert space that we denote by $\mathfrak{K}$. We interpret $\mathfrak{K}$ as in the finite-dimensional case as our propositions Hilbert space. (The main reason for completing $\mathcal{B}_{alg}$ here is to get a mathematically nicer space of propositions.) We find that the decoherence functional d_ρ associated with ρ can be uniquely extended to a bounded sesquilinear form $\hat{D}_\rho$ on $\mathfrak{K}$. To see that D_ρ is indeed bounded with respect to the norm induced by the inner product $\langle\,,\rangle$, recall the Cauchy-Schwarz inequality for D_ρ,

$$|D_\rho(b_1,b_2)|^2\leqslant D_\rho(b_1,b_1)D_\rho(b_2,b_2),$$

for all $b_1,b_2\in\mathcal{B}_{alg}$. When $b\in\mathcal{B}_{alg}$ is a Hilbert-Schmidt operator with n-time support, it follows that there is a constant $C>0$ such that

$$|D_\rho(b,b)|\leqslant C\|b\|_{HS}^2,$$

where $\|\,\|_{HS}$ denotes the Hilbert-Schmidt norm. To see this, recall the representation Eq. (3) and apply the Cauchy-Schwarz inequality

$$\begin{aligned}|D_\rho(b,b)|&\leqslant\sum_{j_1,\ldots,j_{2n}=1}^{\infty}\omega_{j_1}|\langle e^{2n}_{j_{2n}}\otimes\cdots\otimes e^{n+1}_{j_{n+1}}\otimes\psi_{j_1}\otimes\cdots\otimes e^n_{j_n},(b^\dagger\otimes b)(\psi_{j_1}\otimes e^{2n}_{j_{2n}}\otimes\cdots\otimes e^{n+2}_{j_{n+2}}\otimes e^2_{j_2}\otimes\cdots\otimes e^{n+1}_{j_{n+1}})\rangle|\\
&\leqslant\sum_{j_1,\cdots,j_{2n}=1}^{\infty}|\langle e^{2n}_{j_{2n}}\otimes\cdots\otimes e^{n+1}_{j_{n+1}}\otimes\psi_{j_1}\otimes\cdots\otimes e^n_{j_n},(b^\dagger\otimes b)(\psi_{j_1}\otimes e^{2n}_{j_{2n}}\otimes\cdots\otimes e^{n+2}_{j_{n+2}}\otimes e^2_{j_2}\otimes\cdots\otimes e^{n+1}_{j_{n+1}})\rangle|\\
&\leqslant\sum_{j_1,\cdots,j_{2n}=1}^{\infty}|\langle e^{2n}_{j_{2n}}\otimes\cdots\otimes e^{n+1}_{j_{n+1}}\otimes\psi_{j_1}\otimes\cdots\otimes e^n_{j_n},(b^\dagger b\otimes bb^\dagger)(e^{2n}_{j_{2n}}\otimes\cdots\otimes e^{n+1}_{j_{n+1}}\otimes\psi_{j_1}\otimes\cdots\otimes e^n_{j_n})\rangle|^{1/2}\\
&=\sum_{j_1,\cdots,j_n=1}^{\infty}|\langle\psi_{j_1}\otimes\cdots\otimes e^n_{j_n},(b^\dagger b)(\psi_{j_1}\otimes\cdots\otimes e^n_{j_n})\rangle|=\|b\|_{HS}^2=\|[b]^2\|_1,\end{aligned}$$

where $\|\,\|_1$ denotes the trace class norm and $[b]:=(b^\dagger b)^{1/2}$. From the definition of a Cauchy sequence (see Appendix A) it follows that for every Cauchy sequence $\{u_n\}$ there exists an N such that $n,m>N$ implies that u_n-u_m is a Hilbert-Schmidt operator and $[u_n-u_m]^2=(u_n-u_m)^\dagger(u_n-u_m)$ is a trace class operator converging to 0 in the trace class norm. Thus it follows from the above inequalities that D_ρ can be uniquely extended to a finitely valued sesquilinear form $\hat{D}_\rho$ on $\mathfrak{K}$.

We denote the subset of $\mathfrak{K}$ of all elements with finite norm by $\mathfrak{K}_{fin}$. The space $\mathfrak{K}_{fin}$ is a union of proper Hilbert spaces (the Hilbert spaces of Hilbert-Schmidt operators with fixed temporal support). Consequently there exists a bounded operator $\hat{T}_{\rho,i}$ on each Hilbert space $\mathfrak{K}_i\subset\mathfrak{K}_{fin}$ such that $\hat{D}_\rho(x,x)=\langle x,\hat{T}_{\rho,i}x\rangle$ for all $x\in\mathfrak{K}_i$. The sesquilinear form $\hat{D}_\rho$ also satisfies $\hat{D}_\rho(e,e)=1$, where $e:=1$ again denotes the indifferent proposition that is always true. Summarizing, we have shown that also the history version of standard quantum mechanics over an infinite-dimensional Hilbert space can be brought into the form of a temporal quantum theory.

The reader might wonder whether the history version of standard quantum mechanics can be brought into the form of a temporal quantum theory with a proper propositions Hilbert space. The answer is yes with the restriction that the sesquilinear forms (which are the states in the present framework) are then either only defined on a dense subset of the propositions Hilbert space or coincide with the inner product of the propositions Hilbert space. The latter natural representation of the standard decoherence functional (in infinite dimensions) as an inner product of a Hilbert space has been derived in [24]. In this formulation all probabilistic information is completely encoded within the inner product of the propositions Hilbert space and there is no additional notion of a state. Accordingly, the information entropy to be defined in Sec. IV is always zero for this representation. Otherwise, every positive linear functional ϕ with $\phi(1)=1$ induces a semi-inner product on $\mathcal{B}_{alg}$ by $\langle x,y\rangle_\phi=\frac{1}{2}\phi(y^\dagger x+xy^\dagger)$. It has been shown in [24] that D_ρ is not bounded with respect to the norm induced by $\langle\ ,\ \rangle_\phi$ (see also Appendix B) and that D_ρ is unbounded with respect to any C^*-norm on $\mathcal{B}_{alg}$. Thus D_ρ cannot be extended to the Hilbert space completion of $\mathcal{B}_{alg}/N_\phi$ with respect to $\langle\ ,\ \rangle_\phi$, where N_ϕ denotes the null space of $\langle\ ,\ \rangle_\phi$. Moreover, in general, the set N_ϕ may contain physical histories with nonvanishing probability. Therefore, for this construction to make sense one has to ensure that N_ϕ contains no elements with nonvanishing probability. For details the reader is referred to [24].

IV. INFORMATION ENTROPY

In this section we study the problem of defining an information entropy within our framework of temporal quantum theories. We adopt the point of view that, loosely speaking, the information entropy measures the lack of information and is a quantitative measure of the total amount of missing information on the ultramicroscopic structure of the system.

The problem of defining an information entropy for temporal quantum theories was addressed in the framework of Isham's history quantum theories by Isham and Linden [13]. They restricted themselves, however, to history theories over finite-dimensional Hilbert spaces. They considered the case that the space of histories is given by the set $\mathcal{P}(\mathcal{H})$ of projections on some finite-dimensional Hilbert space $\mathcal{H}$ and that the state is given by some bounded decoherence functional on $\mathcal{P}(\mathcal{H})$. Recall that to every decoherence functional d there is a unique trace class operator $\mathfrak{X}_d$ on $\mathcal{H}\otimes\mathcal{H}$ such that Eq. (2) holds. They proceeded as follows. First they observed that there seems to be no straightforward simple way to generalize the expression for the information entropy in single-time quantum mechanics $I_{s-t}=-\mathrm{tr}_{\mathfrak{H}}(\rho\ln\rho)$ to history quantum theories since $\mathfrak{X}_d$ is in general neither self-adjoint nor positive. Thus they defined in a first step an information entropy with respect to a consistent set of histories (a *window*) W by replacing the decoherence functional d by another decoherence functional d_W such that d_W coincides with d on W and such that the operator $\mathfrak{X}_{d_W}$ associated with d_W is self-adjoint and positive. The information entropy with respect to d and W was defined as

$$I_{d,W}:=-\mathrm{tr}(\mathfrak{X}_{d_W}\ln\mathfrak{X}_{d_W})-\ln\dim\mathcal{H}^2.$$

The term $-\ln\dim\mathcal{H}^2$ is added to ensure that the information entropy is invariant under refinement. Isham and Linden also showed that $I_{d,W}$ decreases (or remains constant) under consistent fine graining of W. An information entropy I_d associated with d can then be defined by

$$I_d:=\min_W I_{d,W},$$

where the minimum is taken over all consistent sets W of d. There are alternative possibilities to define an information entropy; see [13]. One important feature of the information entropy $I_{d,W}$ with respect to d and the window W such defined is that its definition involves explicitly the dimension of the underlying history Hilbert space $\mathcal{H}$ and the dimension of the projections in W. Thus the definitions of $I_{d,W}$ and I_d have no straightforward finite extensions to infinite-dimensional history Hilbert spaces.

It is the purpose of this section to define a corresponding notion of information entropy for our scheme of space-time quantum theories using the techniques described by Isham and Linden. Consider a space-time quantum theory as in Sec. III A over some Hilbert space $\mathfrak{K}$ of propositions and some state given by the operator $T\in\mathcal{W}_e(\mathfrak{K})$. Since T is not positive, in general the expression $-\mathrm{tr}_{\mathcal{H}}(T\ln T)$ is not well defined. We proceed in analogy with Isham and Linden and pick some set $W=\{x_i\}_{i\in I}$ of propositions in $\mathfrak{K}$ that is consistent with respect to T. We define a positive self-adjoint operator $\tilde{T}_W$ by

$$\tilde{T}_W:=\sum_{i\in I}\frac{\langle x_i,Tx_i\rangle}{\langle x_i,x_i\rangle}P_i,$$

where P_i denotes the projection in $\mathfrak{K}$ onto the subspace spanned by x_i. The operator $\tilde{T}_W$ is again a state operator in $\mathcal{W}_e(\mathfrak{K})$, i.e., satisfies $\langle e,Te\rangle=1$. To see this, we recall that $e=\Sigma_{i\in I}x_i$. Thus $\langle e,\tilde{T}_We\rangle=\Sigma_{m,l}\langle x_m,\tilde{T}_Wx_l\rangle=\Sigma_m\langle x_m,\tilde{T}_Wx_m\rangle=\Sigma_m\langle x_m,Tx_m\rangle$, where we have used that $\langle x_m,\tilde{T}_Wx_m\rangle=\langle x_m,Tx_m\rangle$ for $x_m\in W$. Since $\{x_i\}_{i\in I}$ is a consistent set for T, it follows that $\Sigma_m\langle x_m,Tx_m\rangle=\langle e,Te\rangle=1$. Thus $\langle e,\tilde{T}_We\rangle=1$ and $\tilde{T}_W\in\mathcal{W}_e(\mathfrak{K})$. For $\tilde{T}_W$ the expression $-\mathrm{tr}_{\mathcal{H}}(\tilde{T}_W\ln\tilde{T}_W)$ is well defined and this motivates the definition of the information entropy for the state T and the window W,

$$I_{T,W}:=-\mathrm{tr}_{\mathcal{H}}(\tilde{T}_W\ln\tilde{T}_W)=-\sum_{i\in I}\langle x_i,Tx_i\rangle\ln\frac{\langle x_i,Tx_i\rangle}{\langle x_i,x_i\rangle}. \tag{6}$$

An argument as in [13] shows that $I_{T,W}$ decreases or remains constant under refinements as it should. To this end, we first notice that for $1\leqslant q<\infty$

$$a\ln\left(\frac{a}{b^q}\right)-(1+a)\ln\left(\frac{(1+a)}{(1+b)^q}\right)\geqslant 0 \tag{7}$$

for all $0\leqslant a<\infty$ and $0<b<\infty$. To see this let $f_q(a,b)\equiv a\ln(a/b^q)-(1+a)\ln[(1+a)/(1+b)^q]$. The function $b\mapsto f_q(a,b)$ assumes for every fixed $0<a<\infty$ a minimum at $b=a$. The value of this minimum satisfies $f_q(a,a)\geqslant 0$ for all $0<a<\infty$, which proves the inequality (7). Now consider a window $W_1=\{x_0,x_1,x_2,\ldots,x_n\}$ and a refinement $W_2=\{y_0,z_0,x_1,x_2,\ldots,x_n\}$ of W_1, where $x_0=y_0+z_0$. We define $a:=\langle z_0,Tz_0\rangle/\langle y_0,Ty_0\rangle$ and $b:=\langle z_0,z_0\rangle/\langle y_0,y_0\rangle$. A straightforward computation shows

$$I_{T,W_1}-I_{T,W_2}=\langle y_0,Ty_0\rangle\left[a\ln\left(\frac{a}{b}\right)-(1+a)\ln\left(\frac{(1+a)}{(1+b)}\right)\right]\geqslant 0,$$

where we have used that $\langle x_0,x_0\rangle=\langle y_0,y_0\rangle+\langle z_0,z_0\rangle$ since W_2 is a consistent set for T. Thus the information entropy for T and W decreases (or remains constant) under any refinement of the window.

The information entropy for the Wright operator T can then be defined as the minimum over all consistent sets, i.e.,

$$I_T:=\min_W I_{T,W},$$

where the minimum is over all consistent sets of T.

It is instructive to compare the expression for the information entropy $I_{T,W}$ for T and W given above with the corresponding expression for the Isham-Linden information entropy for a decoherence functional d and a window V for d in history quantum theories, which was proposed in [13]

$$I_{d,V}^{IL}:=-\sum_{\alpha_i\in V} d(\alpha_i,\alpha_i)\ln\frac{d(\alpha_i,\alpha_i)}{(\dim\alpha_i/\dim\mathcal{H})^2},$$

where $\mathcal{H}$ is the finite-dimensional Hilbert space on which the operators α_i act. The factor $\dim\mathcal{H}$ is included to ensure the invariance of the information entropy upon refinement of the consistent set. Recalling that all α_i are projections, we see that the expression for the Isham-Linden entropy can be written with the norm $\|\ \|_1$ from Eq. (5) as

$$I_{d,V}^{IL}:=-\sum_{\alpha_i\in V} d(\alpha_i,\alpha_i)\ln\frac{d(\alpha_i,\alpha_i)}{\|\alpha_i\|_1^2}$$
$$=-\sum_{\alpha_i\in V}\langle\alpha_i,T\alpha_i\rangle\ln\frac{\langle\alpha_i,T\alpha_i\rangle}{\|\alpha_i\|_1^2},$$

where T is the Wright operator associated with d. We see that for any $1\leqslant p<\infty$ there is an Isham-Linden-type information entropy given by

$$I_{d,V,p}^{IL}:=-\sum_{\alpha_i\in V} d(\alpha_i,\alpha_i)\ln\frac{d(\alpha_i,\alpha_i)}{\|\alpha_i\|_p^2}.$$

All these expressions stand *a priori* on an equal footing. However, an argument as above shows that $I_{d,V,p}^{IL}$ decreases or remains constant under refinement of the consistent set if and only if $1\leqslant p\leqslant 2$. The proof is analogous to the proof given above for the information entropy $I_{T,W}$ and makes use of the general inequality (7). Obviously, the information entropy $I_{T,W}$ defined above in Eq. (6) corresponds to $p=2$ and the Isham-Linden entropy $I_{d,V}^{IL}$ corresponds to $p=1$. The case $p=2$ is somewhat preferred since only in this case the general construction given in Sec. III applies.

In the case of the history version of standard quantum mechanics over infinite-dimensional Hilbert spaces we see that the expression for the information entropy $I_{T,W}$ might become infinite when the window W involves coarse grained propositions u with $\langle u,u\rangle=\infty$. When we recall that the information entropy is a measure for the amount of missing information, it is perhaps not too surprising that in the infinite-dimensional case (corresponding to an infinite variety of possible measurement outcomes) the missing information becomes infinite for certain windows involving ''too'' coarse grained propositions. An alternative approach is (somewhat in the spirit of the topos theoretic approach to the histories approach put forth by Isham [10]) to define the information entropy by

$$\tilde{I}_{T,W}:=\sup_{W_0} I_{T,W_0},$$

where the supremum runs over all consistent refinements W_0 of W such that I_{T,W_0} is finite. Notice, however, that $\tilde{I}_{T,W}$ might also be infinite.

V. SUMMARY

In this paper we have put forth a mathematical framework for temporal quantum theories involving observables associated with extended regions of space time. The main ingredients of the framework is a Hilbert space $\mathfrak{K}$ that contains the physical (measurable) propositions about the system. The norm of an element in $\mathfrak{K}$ is interpreted as a quantitative measure of the structural information about the corresponding proposition encoded within the space $\mathfrak{K}$ and, more specifically, as a quantitative measure of the coarse grainedness of the corresponding proposition within the descriptive scheme provided by $\mathfrak{K}$. There is one distinguished element e in $\mathfrak{K}$ identified with the completely indifferent proposition that is always true. The states are given by bounded, self-adjoint, but not necessarily positive operators T on $\mathfrak{K}$ such that $\langle e,Te\rangle=1$. The expression for the probability of a proposition $x\in\mathfrak{K}$ is given by $\langle x,Tx\rangle$ provided $x\in\mathfrak{K}$. This prescription makes sense when one adopts a consistent-histories-type point of view according to which the assignment of a probability to a proposition x is unambiguously possible only with respect to a consistent set of propositions containing x.

Our proposal is motivated by recent developments in the so-called histories approach to quantum mechanics and we have seen that the history version of standard quantum mechanics can be brought into the required form in the finite-dimensional case. In the infinite-dimensional case one has to allow for a slightly more general framework in which the propositions Hilbert space $\mathfrak{K}$ is an improper Hilbert space or, alternatively, in which the states are given by densely defined unbounded sesquilinear forms on the propositions Hilbert space.

We have also seen that Isham's general history quantum theories can be brought into the form of a temporal quantum theory. Moreover, we have defined an information entropy, generalizing the Isham-Linden information entropy for history theories.

The examples discussed in Sec. III B make clear that our approach is not in contradiction to the history approach by Isham *et al.* but rather (in a sense) a complementary formulation of temporal quantum theories. In the case of standard quantum mechanics we still can think of the space of propositions $\mathfrak{K}$ essentially as a set of operators on tensor product Hilbert spaces. In this sense our approach may, loosely speaking, be looked upon as a compromise between the formulations of history quantum theories due to Gell-Mann and Hartle, on the one hand, and Isham, on the other hand.

However, as already discussed above, the history theory due to Gell-Mann and Hartle stays essentially on the level of homogeneous histories and represents only a very modest generalization of standard quantum mechanics. Isham's abstract history quantum theories represent a much more substantial generalization of standard quantum mechanics. However, it is an open problem if and how standard quantum mechanics can be recovered from them in some appropriate limit. Specifically, it is not clear at all in which limit a Hamiltonian operator can be recovered within the framework of an abstract history theory. In contrast to these two developments, the approach developed in the present paper offers a generalization of standard quantum mechanics for which there is hope that the issue of recovering standard quantum mechanics can be successfully tackled. A possibility suggesting itself is, for example, to study propositions Hilbert spaces carrying a unitary representation of the Poincaré group in which case a Hamiltonian operator can be obtained as one of the generators of the representation. These topics will be discussed elsewhere.

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APPENDIX A: IMPROPER HILBERT SPACES

Consider a vector space $\mathfrak{V}$ equipped with an improper inner product $\langle\ ,\ \rangle_{\mathfrak{V}}:\mathfrak{V}\times\mathfrak{V}\to\mathbb{C}\cup\{\infty\}$ such that (i) $\langle w,au+bv\rangle_{\mathfrak{V}}=a\langle w,u\rangle_{\mathfrak{V}}+b\langle w,v\rangle_{\mathfrak{V}}$, (ii) $\langle u,v\rangle_{\mathfrak{V}}=\langle v,u\rangle_{\mathfrak{V}}^*$, (iii) $\langle u,u\rangle_{\mathfrak{V}}\geqslant 0$, and (iv) $\langle u,u\rangle_{\mathfrak{V}}=0$ only if $u=0$ for all $a,b\in\mathbb{C}$ and $u,v,w\in\mathfrak{V}$ whenever all expressions are finite. We denote the subspace of elements in $\mathfrak{V}$ with finite norm by $\mathfrak{V}_{fin}$. A sequence $\{u_n|n\in\mathbb{N},u_n\in\mathfrak{V}\}$ converges to $u\in\mathfrak{V}$ if $\langle u_n-u,u_n-u\rangle_{\mathfrak{V}}\to 0$. A sequence $\{u_n|n\in\mathbb{N},u_n\in\mathfrak{V}\}$ is a *Cauchy sequence* if $\langle u_n-u_m,u_n-u_m\rangle_{\mathfrak{V}}\to 0$. Notice that for any Cauchy sequence $\{u_n\}$ there is an N such that $n,m>N$ implies $u_n-u_m\in\mathfrak{V}_{fin}$. The space $\mathfrak{V}$ is said to be *complete* if every Cauchy sequence converges. An orthonormal basis is a set $\{y_i|i\in\mathbb{I},y_i\in\mathfrak{V}\}$ such that (i) $\langle y_i,y_j\rangle_{\mathfrak{V}}=\delta_{ij}$ for all i,j, (ii) $\langle u,y_i\rangle_{\mathfrak{V}}<\infty$ for all i and $u\in\mathfrak{V}$, and (iii) $\langle u,y_i\rangle=0$ for all i if and only if $u=0$. An *improper Hilbert space* is now a linear space $\mathfrak{V}$ with an improper inner product $\langle\ ,\ \rangle_{\mathfrak{V}}$ such that (i) $\mathfrak{V}$ is complete and (ii) $\mathfrak{V}$ has an orthonormal basis $\{y_i\}$. Then every element $u\in\mathfrak{V}$ can be formally expanded as $u=\Sigma_i\langle u,y_i\rangle_{\mathfrak{V}}y_i$. In contrast to ordinary Hilbert spaces, however, the sum $\|u\|=\Sigma_i|\langle u,y_i\rangle_{\mathfrak{V}}|^2$ does not converge for all $u\in\mathfrak{V}$. We do not want to develop here a theory of improper Hilbert spaces, but it is important to notice that many results of the theory of Hilbert spaces are not valid for improper Hilbert spaces. Notice, however, that there always resides some (nonunique) proper Hilbert space within an improper Hilbert space.

APPENDIX B: THE DECOHERENCE FUNCTIONAL IN STANDARD QUANTUM MECHANICS

In [24] the present author and Wright studied the analytical properties of the standard decoherence functional d_ρ associated with the initial state ρ. Among others we proved that if the single-time Hilbert space is infinite dimensional, then (i) the standard decoherence functional d_ρ defined on homogeneous histories by Eq. (1) cannot be extended to a finitely valued functional on the set of all projection operators on the tensor product Hilbert space and (ii) the extension D_ρ of d_ρ to $\mathcal{B}_{alg}$ is unbounded with respect to any C^*-norm on $\mathcal{B}_{alg}$. The latter assertion, together with Theorem 4.3.2 in [29], implies that D_ρ is also unbounded with respect to the norm induced by the inner product $\langle\ ,\ \rangle_\phi$ defined at the end of Sec. III. (Theorem 4.3.2 in [29] states that every positive linear functional ϕ on $\mathcal{B}_{alg}$ is bounded with respect to any C^*-norm on $\mathcal{B}_{alg}$.) We are not going to reproduce the general considerations undertaken in [24] here, but for the convenience of the reader we give two counterexamples showing (i) and (ii), respectively. We assume for simplicity that the single-time Hilbert space is separable.

(i) Consider the representation (3). For simplicity of notation we consider the case $n=2$,

$$D_\rho(p,q)=\sum_{j_1,\ldots,j_4=1}^{\dim\mathfrak{H}}\omega_{j_1}\langle e^4_{j_4}\otimes e^3_{j_3}\otimes\psi_{j_1}\otimes e^2_{j_2},(p\otimes q)\times(\psi_{j_1}\otimes e^4_{j_4}\otimes e^2_{j_2}\otimes e^3_{j_3})\rangle,\tag{B1}$$

for all histories $p,q\in\mathcal{P}(\mathfrak{H}_{t_1}\otimes\mathfrak{H}_{t_2})$ for which the sum converges. We assume that the single-time Hilbert space $\mathfrak{H}_t$ is separable. Now choose $e^4_j=e^3_j=e^2_j=\psi_j$ for all j. Fix i_1 and let $\varphi_i:=(1/\sqrt{2})(|\psi_i\otimes\psi_{i_1}\rangle+|\psi_{i_1}\otimes\psi_i\rangle)$ for every $i\in\mathbb{N}\setminus\{i_1\}$. Then clearly $\varphi_i\perp\varphi_j$ if $i\neq j$. Set $f_{j_1,j_2,j_3}(q)=\langle\psi_{j_1}\otimes\psi_{j_2},q(\psi_{j_2}\otimes\psi_{j_3})\rangle$; then an easy computation shows that

$$D_\rho(P_{\varphi_i},q)=\frac{1}{2}\sum_{j_2}\left[\omega_{i_1}f_{i_1,j_2,i_1}(q)+\omega_i f_{i,j_2,i}(q)\right]$$

for $i\neq i_1$, where P_{φ_i} denotes the projection operator onto the subspace spanned by φ_i. Set $P=\Sigma_{i\neq i_1}P_{\varphi_i}$; then clearly the expression (B1) for $D_\rho(P,q)$ does not converge for arbitrary q.

(ii) Consider again $n=2$ and the operator

$$h=\sum_{k_1,k_4}\frac{1}{k_1+k_4}|e^4_{k_4}\otimes\psi_{k_1}\rangle\langle\psi_{k_1}\otimes e^4_{k_4}|.$$

Then h is a compact operator in the completion of the algebraic tensor product $\mathcal{K}(\mathfrak{H}_{t_1})\otimes_{alg}\mathcal{K}(\mathfrak{H}_{t_2})$. (A Cauchy sequence $\{h_n\}$ in $\mathcal{K}(\mathfrak{H}_{t_1})\otimes_{alg}\mathcal{K}(\mathfrak{H}_{t_2})$ converging to h is given, for example, by

$$h_n = \sum_{l=2}^{n} \sum_{\substack{k_1 k_4 \\ k_1+k_4=l}} [1/(k_1+k_4)] |e^4_{k_4} \otimes \psi_{k_1}\rangle\langle \psi_{k_1} \otimes e^4_{k_4}|.$$

Then $\|h_n - h_m\| \leq \max(1/n, 1/m)$.) Moreover, the sum in Eq. (B1) for $D_\rho(h,1)$ is equal to $\Sigma_{k_1,k_4}[\omega_{k_1}/(k_1+k_4)]$ and thus is clearly divergent. This shows that the canonical extension D_ρ of d_ρ on $\mathcal{B}_{alg}$ is not bounded on $\mathcal{K}(\mathfrak{H}_{t_1}) \otimes_{alg} \mathcal{K}(\mathfrak{H}_{t_2})$ with respect to the ordinary operator norm. Since, by nuclearity, all C^*-norms on $\mathcal{K}(\mathfrak{H}_{t_1}) \otimes_{alg} \mathcal{K}(\mathfrak{H}_{t_2})$ coincide, D_ρ is unbounded with respect to any C^*-norm on $\mathcal{B}_{alg}$.

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