Coherent states for the Kepler motion

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(Received 11 May 1998)

The dynamics of the electron wave function in a hydrogen atom is mapped onto that of a two-dimensional harmonic oscillator. Based on the harmonic-oscillator eigenstates, a coherent state wave function for the atomic electron is proposed. [S1050-2947(99)04002-0]

PACS number(s): 31.15.−p, 03.65.Ge, 03.65.Bz

Since quantum mechanics emerged over seven decades ago, the classical limit in its framework has been one of the central issues [1]. Nevertheless, the classical limit of the Kepler motion, which is obviously one of the most fundamental applications of both quantum and classical mechanics, has not been fully understood. The crucial question is whether one can construct a stable wave-packet state that does not change its shape during the time evolution. This question is relevant not only to theoretical arguments but also to experimental preparations.

Recently, ten Wolde *et al.* succeeded in observing a radially localized electron wave packet by coherently exciting Rydberg states of rubidium atoms in a direct pump-probe experiment [2]. The experiment seems to open the new possibility of examining various theoretical attempts to obtain the classical limit of the quantum Kepler motion.

As is well known, the quantum harmonic oscillator can have such a wave-packet solution, i.e., the coherent state [3,4]. Therefore, it is mathematically natural to map the Kepler motion onto a system of harmonic oscillators. In this paper, using the theory of Ravndal and Toyoda on the SU(2)⊗SU(2) dynamical symmetry of the quantum Kepler motion [5], we explicitly map the dynamics of the quantum Kepler motion, i.e., the electron in a hydrogen atom, onto a two-dimensional quantum harmonic oscillator. Then we propose a coherent state that satisfies the constraint due to the extra degree of freedom and keeps its shape during the time evolution with respect to the auxiliary time variable introduced by Duru and Kleinert [6,7].

Coherent states for the Kepler motion with the auxiliary time of Duru and Kleinert have been considered by Gerry [8] and by Gerry and Kiefer [9]. In Ref. [8] the Kustaanheimo-Stiefel transformation [10] is used to introduce the ordinary boson coherent states. In Ref. [10] the SO(2,1) radial subgroup of the SO(4,2) dynamical group of the Kepler motion is used to introduce Perelomov's coherent states [11,12], in contrast to the SU(2)⊗SU(2) symmetry used in the present paper. The coherent states considered in Refs. [8] and [9] are clearly different from our coherent states given in this paper.

In the squared parabolic coordinates (μ,ν,φ) defined by

$$x=\mu\nu\cos\varphi,\quad y=\mu\nu\sin\varphi,\quad z=\tfrac{1}{2}(\mu^2-\nu^2),\tag{1}$$

the Schrödinger equation for the electron wave function in a hydrogen atom can be written as

$$i\hbar\frac{\partial}{\partial t}\psi(\mu,\nu,\varphi,t)=H\left(\mu,\frac{\partial}{\partial\mu},\nu,\frac{\partial}{\partial\nu},\frac{\partial}{\partial\varphi}\right)\psi(\mu,\nu,\varphi,t),\tag{2}$$

where the Hamiltonian operator is [5]

$$\begin{aligned}H&\left(\mu,\frac{\partial}{\partial\mu},\nu,\frac{\partial}{\partial\nu},\frac{\partial}{\partial\varphi}\right)\\&=\frac{-\hbar^2}{2m}\frac{1}{\mu^2+\nu^2}\\&\quad\times\left(\frac{\partial^2}{\partial\mu^2}+\frac{1}{\mu}\frac{\partial}{\partial\mu}+\frac{1}{\mu^2}\frac{\partial^2}{\partial\varphi^2}\right)-\frac{\hbar^2}{2m}\frac{1}{\mu^2+\nu^2}\\&\quad\times\left(\frac{\partial^2}{\partial\nu^2}+\frac{1}{\nu}\frac{\partial}{\partial\nu}+\frac{1}{\nu^2}\frac{\partial^2}{\partial\varphi^2}\right)-\frac{2e^2}{\mu^2+\nu^2}+E_0.\end{aligned}\tag{3}$$

The added positive constant E_0 is to be fixed depending on the initial condition. For simplicity we neglect the spin variables. Keeping the spatial coordinates unchanged, we change the time variable from t to τ, as is defined by

$$\tau=\frac{t}{\mu^2+\nu^2}.\tag{4}$$

It should be noted that this τ is essentially the same as the auxiliary time used in the path-integral formulation of the hydrogen atom problem by Duru and Kleinert [6]. In terms of the new set of the variables, the Schrödinger equation can be expressed as

$$\begin{aligned}i\hbar&\frac{\partial}{\partial\tau}\chi(\mu,\nu,\varphi,\tau)\\&=\left[\frac{-\hbar^2}{2m}\left(\frac{\partial^2}{\partial\mu^2}+\frac{1}{\mu}\frac{\partial}{\partial\mu}+\frac{1}{\mu^2}\frac{\partial^2}{\partial\varphi^2}\right.\right.\\&\quad\left.+\frac{\partial^2}{\partial\nu^2}+\frac{1}{\nu}\frac{\partial}{\partial\nu}+\frac{1}{\nu^2}\frac{\partial^2}{\partial\varphi^2}\right)\\&\quad\left.-2e^2+(\mu^2+\nu^2)E_0\right]\chi(\mu,\nu,\varphi,\tau),\end{aligned}\tag{5}$$

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where χ denotes the wave function,

$$\psi(\mu,\nu,\varphi,t)=\psi(\mu,\nu,\varphi,(\mu^2+\nu^2)\tau)=\chi(\mu,\nu,\varphi,\tau). \quad (6)$$

Then, introducing the auxiliary variables φ_μ and φ_ν [5], we can rewrite Eq. (5) as

$$i\hbar\frac{\partial}{\partial\tau}\Psi(\mu,\varphi_\mu,\nu,\varphi_\nu,\tau) =\left[\frac{-\hbar^2}{2m}\left(\frac{\partial^2}{\partial\mu^2}+\frac{1}{\mu}\frac{\partial}{\partial\mu}+\frac{1}{\mu^2}\frac{\partial^2}{\partial\varphi_\mu^2}-\frac{2m}{\hbar^2}\mu^2E_0\right) -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial\nu^2}+\frac{1}{\nu}\frac{\partial}{\partial\nu}+\frac{1}{\nu^2}\frac{\partial^2}{\partial\varphi_\nu^2}-\frac{2m}{\hbar^2}\nu^2E_0\right)-2e^2\right] \times\Psi(\mu,\varphi_\mu,\nu,\varphi_\nu,\tau), \quad (7)$$

which is associated with the condition

$$\frac{\partial}{\partial\varphi_\mu}\Psi(\mu,\varphi_\mu,\nu,\varphi_\nu,\tau)=\frac{\partial}{\partial\varphi_\nu}\Psi(\mu,\varphi_\mu,\nu,\varphi_\nu,\tau). \quad (8)$$

Owing to the above condition the excessive unphysical degrees of freedom emerging from the use of the auxiliary variables can be restricted. If we put $\varphi_\mu=\varphi_\nu=\varphi$, the new wave function is reduced to the original wave function

$$\Psi(\mu,\varphi_\mu,\nu,\varphi_\nu,\tau)\big|_{\varphi_\mu=\varphi_\nu=\varphi}=\chi(\mu,\nu,\varphi,\tau). \quad (9)$$

The time evolution of the wave function $\Psi(\mu,\varphi_\mu,\nu,\varphi_\nu,\tau)$ with respect to the time τ is given by the generator,

$$F=\frac{-\hbar^2}{2m}\left(\frac{\partial^2}{\partial\mu^2}+\frac{1}{\mu}\frac{\partial}{\partial\mu}+\frac{1}{\mu^2}\frac{\partial^2}{\partial\varphi_\mu^2}-\frac{2m}{\hbar^2}\mu^2E_0\right) -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial\nu^2}+\frac{1}{\nu}\frac{\partial}{\partial\nu}+\frac{1}{\nu^2}\frac{\partial^2}{\partial\varphi_\nu^2}-\frac{2m}{\hbar^2}\nu^2E_0\right)-2e^2, \quad (10)$$

which yields

$$\Psi(\mu,\varphi_\mu,\nu,\varphi_\nu,\tau)=\exp\left[\frac{-i}{\hbar}\tau F\right]\Psi(\mu,\varphi_\mu,\nu,\varphi_\nu,0). \quad (11)$$

The operators $\partial/\partial\varphi_\mu$, $\partial/\partial\varphi_\nu$, and the generator F can be simultaneously diagonalized because they commute with each other. To perform the diagonalization we introduce the operators [5]

$$A_\pm=\frac{1}{2\sqrt{m\hbar\omega}}\left\{-i\hbar\left(\frac{\partial}{\partial\xi_\mu}\pm i\frac{\partial}{\partial\eta_\mu}\right)-im\omega(\xi_\mu\pm i\eta_\mu)\right\}, \quad (12a)$$

$$A_\pm^\dagger=\frac{1}{2\sqrt{m\hbar\omega}}\left\{-i\hbar\left(\frac{\partial}{\partial\xi_\mu}\mp i\frac{\partial}{\partial\eta_\mu}\right)+im\omega(\xi_\mu\mp i\eta_\mu)\right\}, \quad (12b)$$

$$B_\pm=\frac{1}{2\sqrt{m\hbar\omega}}\left\{-i\hbar\left(\frac{\partial}{\partial\xi_\nu}\pm i\frac{\partial}{\partial\eta_\nu}\right)-im\omega(\xi_\nu\pm i\eta_\nu)\right\}, \quad (12c)$$

$$B_\pm^\dagger=\frac{1}{2\sqrt{m\hbar\omega}}\left\{-i\hbar\left(\frac{\partial}{\partial\xi_\nu}\mp i\frac{\partial}{\partial\eta_\nu}\right)+im\omega(\xi_\nu\mp i\eta_\nu)\right\}, \quad (12d)$$

where the new variables are defined by

$$\xi_\mu=\mu\cos\varphi_\mu,\quad \eta_\mu=\mu\sin\varphi_\mu,\quad \xi_\nu=\nu\cos\varphi_\nu,\quad \eta_\nu=\nu\sin\varphi_\nu, \quad (13)$$

and the parameter ω denotes

$$\omega=\sqrt{2E_0/m}. \quad (14)$$

The only nonvanishing commutators between the operators defined by Eqs. (12a)–(12d) are

$$[A_+,A_+^\dagger]=[A_-,A_-^\dagger]=[B_+,B_+^\dagger]=[B_-,B_-^\dagger]=1, \quad (15)$$

which show that they are the creation and annihilation operators for the harmonic oscillators. In terms of these operators, the generator F can be expressed as

$$F=\hbar\omega\{A_+^\dagger A_++A_-^\dagger A_-+B_+^\dagger B_++B_-^\dagger B_-+2\}-2e^2. \quad (16)$$

The operators $\partial/\partial\varphi_\mu$ and $\partial/\partial\varphi_\nu$ are also expressed as

$$i\frac{\partial}{\partial\varphi_\mu}=A_+^\dagger A_+-A_-^\dagger A_- \quad\text{and}\quad i\frac{\partial}{\partial\varphi_\nu}=B_+^\dagger B_+-B_-^\dagger B_-. \quad (17)$$

Thus we can rewrite Eq. (8) as the condition for the physically allowed states in terms of these operators:

$$(A_+^\dagger A_+-A_-^\dagger A_--B_+^\dagger B_++B_-^\dagger B_-)|\text{physical}\rangle=0. \quad (18)$$

The relevant eigenstates of the number operators $A_\pm^\dagger A_\pm$ and $B_\pm^\dagger B_\pm$ are defined by

$$A_\pm^\dagger A_\pm|m_+,m_-,n_+,n_-\rangle=|m_+,m_-,n_+,n_-\rangle m_\pm \quad (19)$$

and

$$B_\pm^\dagger B_\pm|m_+,m_-,n_+,n_-\rangle=|m_+,m_-,n_+,n_-\rangle n_\pm, \quad (20)$$

where the eigenvalues m_+, m_-, n_+, and n_- are nonnegative integers.

Noting that any linear superposition of the eigenstates $|m_+,m_-,m_++n_--m_-,n_-\rangle$ satisfies the physical state condition (18), we define a coherent state

$$|\Psi(\alpha,\beta;\gamma)\rangle=\exp[-\tfrac{1}{2}(|\alpha|^2+|\beta|^2+|\gamma|^2)] \times\sum_{m_+=0}^{\infty}\sum_{n_-=0}^{\infty}\sum_{m_-=0}^{\infty}\frac{\alpha^{m_+}}{\sqrt{m_+!}}\frac{\beta^{n_-}}{\sqrt{n_-!}}\frac{\gamma^{m_-}}{\sqrt{m_-!}} \times|m_+,m_-,m_++n_--m_-,n_-\rangle, \quad (21)$$

where the complex numbers α, β, and γ are to be specified according to the initial condition. It should be remarked that the products $\alpha\gamma$ and $\beta\gamma$ are the eigenvalues of the operators A_-A_+ and A_-B_- such that

$$A_-A_+|\Psi(\alpha,\beta;\gamma)\rangle=|\Psi(\alpha,\beta;\gamma)\rangle\alpha\gamma \tag{22}$$

and

$$A_-B_-|\Psi(\alpha,\beta;\gamma)\rangle=|\Psi(\alpha,\beta;\gamma)\rangle\beta\gamma, \tag{23}$$

which can be straightforwardly obtained from Eq. (21).

By virtue of the physical state condition (18), the time evolution of the coherent state with respect to the time τ is given by

$$\begin{aligned}\exp\left(-\frac{i}{\hbar}\tau F\right)|\Psi(\alpha,\beta;\gamma)\rangle &=\exp\left(-\frac{i}{\hbar}\tau\{\hbar\omega(A_+^\dagger A_+ + A_-^\dagger A_- + B_+^\dagger B_+ + B_-^\dagger B_- +2)-2e^2\}\right)|\Psi(\alpha,\beta;\gamma)\rangle\\ &=\exp\left(-\frac{i}{\hbar}\tau\{2\hbar\omega(A_+^\dagger A_+ + B_-^\dagger B_- +1)-2e^2\}\right)\\ &\quad\times|\Psi(\alpha,\beta;\gamma)\rangle.\end{aligned} \tag{24}$$

Hence the dynamics is equivalent to that of the two-dimensional harmonic oscillator.

Furthermore, substituting the explicit form (21) into Eq. (24), we obtain

$$\begin{aligned}&\exp\left(-\frac{i}{\hbar}\tau\{2\hbar\omega(A_+^\dagger A_+ + B_-^\dagger B_- +1)-2e^2\}\right)|\Psi(\alpha,\beta;\gamma)\rangle\\ &=\exp\left(-\frac{1}{2}\{|\alpha|^2+|\beta|^2+|\gamma|^2\}\right)\sum_{m_+=0}^{\infty}\sum_{m_-=0}^{\infty}\sum_{n_-=0}^{\infty}\frac{\alpha^{m_+}\beta^{n_-}\gamma^{m_-}}{\sqrt{m_+!n_-!m_-!}}\exp(-i2\omega\tau m_+)\exp(-i2\omega\tau n_-)\\ &\quad\times\exp\left[i2\left(\frac{e^2}{\hbar}-\omega\right)\tau\right]|m_+,m_-,m_++n_--m_-,n_-\rangle\\ &=\exp(-\tfrac{1}{2}\{|\alpha(\tau)|^2+|\beta(\tau)|^2+|\gamma|^2\})\\ &\quad\times\sum_{m_+=0}^{\infty}\sum_{m_-=0}^{\infty}\sum_{n_-=0}^{\infty}\frac{\alpha(\tau)^{m_+}\beta(\tau)^{n_-}\gamma^{m_-}}{\sqrt{m_+!n_-!m_-!}}|m_+,m_-,m_++n_--m_-,n_-\rangle\exp\left[i2\left(\frac{e^2}{\hbar}-\omega\right)\tau\right]\\ &=|\Psi(\alpha(\tau),\beta(\tau);\gamma)\rangle\exp\left[i2\left(\frac{e^2}{\hbar}-\omega\right)\tau\right],\end{aligned} \tag{25}$$

where the time-dependent eigenvalues,

$$\alpha e^{-i2\omega\tau}=\alpha(\tau) \tag{26a}$$

and

$$\beta e^{-i2\omega\tau}=\beta(\tau), \tag{26b}$$

have been defined [13]. Equation (25) shows the modulus of the wave function does not change its shape during the time evolution with respect to the auxiliary time variable τ. As long as the modulus of the wave function is concerned, the τ dependence appears only through the complex eigenvalues, which correspond to the classical orbit. If the corresponding classical orbit is elliptic, the complex eigenvalues change with respect to the original time t. Nevertheless, the change is periodic so that the change of the modulus of the wave function is also periodic and its minimum wave-packet property is periodically retained.

It is of some interest to see the classical analogue of the auxiliary time variable τ [14]. In the case of the classical elliptic Kepler motion, the orbit in the spherical polar coordinates (r,θ,ϕ) is given by the equation

$$r=\frac{l}{1+e\cos\phi}, \tag{27}$$

where $2l$ is the length of the latus rectum and e is the eccentricity [15]. The elliptic orbit means $0<e<1$. If we introduce a parameter a, which is called the averaged length in astronomy, the relation $l=a(1-e^2)$ holds. As is well known, the eccentric anomaly u that satisfies the relation [15]

$$\tan\frac{\phi}{2}=\sqrt{(1+e)/(1-e)}\,\tan\frac{u}{2} \tag{28}$$

maps the location of the Kepler particle on the elliptic orbit onto a point on the auxiliary circle with radius a. Obviously, the speed of the mapped point on the auxiliary circle is not constant. In order to map the nonconstant circular motion onto the uniform circular motion, we note the relation

$$\frac{d}{dt}u = \frac{1}{a\sqrt{1-e^2}}r\frac{d}{dt}\phi, \tag{29}$$

and that in the Kepler motion, the angular momentum

$$r^2\left(\frac{d}{dt}\phi\right) = \text{const} \equiv h \tag{30}$$

is conserved. Using this constant value for the classical angular momentum h, we find

$$r\frac{d}{dt}u = \frac{h}{a\sqrt{1-e^2}} = \text{const}, \tag{31}$$

which yields

$$r\frac{d}{dt} = \frac{h}{a\sqrt{1-e^2}}\frac{d}{du} = (\text{const})\frac{d}{du}. \tag{32}$$

This implies that if the eccentric anomaly is adopted as the time variable, the mapped motion on the auxiliary circle becomes uniform circular motion, which has obviously the symmetry of the two-dimensional harmonic oscillator. Because $r = \frac{1}{2}(\mu^2 + \nu^2)$ in our formulation, the auxiliary time variable τ defined by Eq. (4) corresponds to the eccentric anomaly u except a constant factor.

Concluding this paper let us make a brief remark on the classical limit of the radiation damping of the atomic electron. It has been shown that the classical limit of the charged quantized harmonic oscillator interacting with the quantized radiation field can be obtained based on the generalized Hill-Wheeler method with the von Neumann lattice coherent states [16,17]. The theory can be applied to the present formulation of the atomic electron with slight modification. The details will be discussed in a forthcoming paper.

One of us (T.T.) thanks Professor T. Akamatsu (Department of Mathematics, Tokai University) for critical comments on the change of the time variable.

[1] E. Schrödinger, Naturwissenschaften **14**, 664 (1926).
[2] A. ten Wolde, L. D. Noordam, A. Lagendijk, and H. B. van Linden van den Heuvell, Phys. Rev. Lett. **61**, 2099 (1988).
[3] *Coherent States*, edited by J. R. Klauder and B.-S. Skagerstam (World Scientific, Singapore, 1985).
[4] J. R. Klauder, J. Math. Phys. **4**, 1058 (1963).
[5] F. Ravndal and T. Toyoda, Nucl. Phys. B **3**, 312 (1967).
[6] I. H. Duru and H. Kleinert, Phys. Lett. **84B**, 185 (1979).
[7] R. Ho and A. Inomata, Phys. Rev. Lett. **48**, 231 (1982).
[8] C. C. Gerry, Phys. Rev. A **33**, 6 (1986).
[9] C. C. Gerry and J. Kiefer, Phys. Rev. A **37**, 665 (1988).
[10] P. Kustaanheimo and E. Stiefel, J. Reine Angew. Math **218**, 204 (1965).
[11] A. M. Perelomov, Commun. Math. Phys. **26**, 222 (1972); **44**, 197 (1975).
[12] A. M. Perelomov, *Generalized Coherent States and Their Applications* (Springer, Berlin, 1986).
[13] C. L. Metha and E. C. G. Sudarshan, Phys. Lett. **22**, 574 (1966).
[14] Bruce R. Johnson, Phys. Rev. A **35**, 1412 (1987).
[15] E. T. Whittaker, *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies*, 4th ed. (Cambridge University Press, Cambridge, 1937).
[16] T. Toyoda and K. Wildermuth, Phys. Rev. D **22**, 2391 (1980).
[17] T. Toyoda and K. Wildermuth, Phys. Rev. A **27**, 1790 (1983).