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Measurable characteristics of a nonrelativistic quantum particle

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We analyze the connection between the path integral and operator approaches to the quantum measurement problem. In general, an act of measurement is shown to destroy interference between components of the wave function related to the particle's histories. Classically, there exist a class of meters suitable for measuring the value of a given dynamical variable $\mathcal{F}$. Quantally, different meters produce different results. The standard von Neumann measurement is one particular case. Rearranging Feynman paths according to the value of (time average of) $\mathcal{F}$ defines a different type of meter. The two methods disagree if the duration of the measurement is very short. Possible ways to measure the particle's momentum are studied in detail. The semiclassical limit of a measurement is analyzed. [S1050-2947(99)02102-2]

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I. INTRODUCTION

Feynman wrote that ''any other situation in quantum mechanics, it turns out, can always be explained by saying, 'You remember the case of the experiment with the two holes? It's the same thing' '' [1]. In Feynman's quantum mechanics [2], quantum histories (Feynman paths for a single structureless particle) interfere to produce the wave function $\Psi(x,t)$ at location x at time t. In the presence of such interference, nothing is known about the particle's history except that its position is x at the time t. One learns about the particle's past by making different histories distinguishable, for example, by setting up a meter which distinguishes between the paths going through different holes in the double-slit experiment. The price of such information is that the interference pattern on the screen is destroyed, or, more generally that the probability to find the particle in x at t is no longer equal to $|\Psi(x,t)|^2$. In this approach an act of measurement is, therefore, the destruction of interference between particle's histories.

At first glance, the problem of determining the value of a function $\mathcal{F}(p,x)$ of the particle's momentum p and coordinate x, appears different. Measurement of $\mathcal{F}(p,x)$ requires constructing a Hermitian operator $\hat{\mathcal{F}}(p,x)$ and expanding the wavefunction $\Psi(x,t)$ in the eigenstates of $\hat{\mathcal{F}}(p,x)$ [3],

$$\hat{\mathcal{F}}(p,x)\phi_i(x)=\mathcal{F}_i\phi_i(x),$$

i.e.,

$$\Psi(x,t)=\sum_i c_i\phi_i(x). \tag{1.1}$$

The probability that $\mathcal{F}(p,x)$ has the value $\mathcal{F}_i$ is then given by $|c_i|^2$. As shown by von Neumann, the probability distribution $|c_i|^2$ can be measured, at least in principle, by coupling the particle to an external degree of freedom [4].

This straightforward recipe fails, however, if the measured quantity is defined over a certain duration rather than at a single instant in time. One such example is the amount of time a quantum particle spends in a given region of space [5]. However, even the simple case of the linear momentum, $\mathcal{F}(p,x)=p\equiv m\dot{x}$, needs clarification. Classically, determination of the particle's velocity $\dot{x}$ requires evaluating its position not once, but at *two*, however close, moments of time. Unlike the classical trajectories, quantum (Feynman) paths are highly irregular [2]. One might think, therefore, that defining a quantum particle's momentum at any given time is difficult or impossible. However, according to Eq. (1.1), the corresponding probability distribution is readily given in terms of the plane-wave expansion of the wavefunction $\Psi(x,t)$ at *one* given time [3,4].

The main purpose of this paper is to establish a general relation between the Feynman path integral and operator approaches to the quantum measurement problem. In particular, we shall demonstrate that a measurement of a dynamical quantity $\mathcal{F}(p,x)$ can be understood as distinguishing between interfering alternatives related to the particle's histories. Previous work on the connection between histories and quantum observables was done by Aharonov and co-workers [6–9], Griffiths [10], Gell-Mann and Hartle [11], and Yamada and co-workers [12,13]. The relation between restricted path integrals and operators was studied in Ref. [14]. The rest of the paper is organized as follows. In Sec. II we introduce a class of equivalent meters for a classical dynamical variable $\mathcal{F}(p,x)$. In Sec. III, we show that quantally the action of a meter can be described as additional weighting of Feynman paths in the particle's path integral. We also demonstrate that meters that give the same result in the classical limit may differ in the full quantum case. In Sec. IV, we use the value of $\mathcal{F}(p,x)$, f, as an independent variable, and obtain the von Neumann approach as a particular case. In Sec. V, we analyze the case of the particle's momentum. In Sec. VI, we return the (semi)classical limit of a quantum measurement and study the occurrence of complex valued quantities for classically forbidden transitions. Section VII contains our conclusions.

II. CLASSICAL MEASUREMENTS AND METERS

Consider the value of a classical dynamical variable $\mathcal{F}(p,x)$ at a given time t. Further we shall want to extend the analysis to the quantum case where Feynman paths are known to be highly irregular [2], so we start by averaging $\mathcal{F}(p,x)$ over a time interval $[t,t+T]$,

$$\langle\mathcal{F}\rangle_T = T^{-1}\int_{t-T}^{t}\mathcal{F}(p,x)dt. \tag{2.1}$$

The instantaneous value $\mathcal{F}(p(t),x(t))$ (if it exists) can then be obtained from Eq. (2.1) by taking the limit $T\to 0$. Therefore, we need to measure $\langle\mathcal{F}\rangle_T$ along a *classical trajectory* of a particle described by a Hamiltonian $H_0(p,x)$. This can be done by coupling the particle to a classical ''meter'' whose coordinate and momentum we denote f and λ, respectively. We shall switch the meter on at $t-T$ and then read it at the time t. Provided the Hamiltonian $H(p,x,\lambda)$ describing both the particle and the meter does not depend on f, the equations of motion are ($\partial_x \equiv \partial/\partial x$)

$$\dot{x} = \partial_p H(p,x,\lambda), \tag{2.2a}$$

$$\dot{p} = -\partial_x H(p,x,\lambda), \tag{2.2b}$$

$$\dot{f} = \partial_\lambda H(p,x,\lambda), \tag{2.2c}$$

$$\dot{\lambda} = 0, \tag{2.2d}$$

so that the momentum λ is conserved, $\lambda=\text{const}$. We take $f(t-T)=0$ and then run the meter until $t=t$. To ensure that the meter does not perturb the particle's motion, we choose

$$\lambda = 0, \tag{2.3}$$

$$H(p,x,0)=H_0(p,x). \tag{2.4}$$

Also, the meter must measure the time average of $\mathcal{F}(p,x)$, so we define

$$\frac{\partial H(p,x,\lambda)}{\partial\lambda} = T^{-1}\mathcal{F}(p,x). \tag{2.5}$$

With these assumptions Eqs. (2.2) become

$$\dot{x} = \partial_p H_0(p,x), \tag{2.6a}$$

$$\dot{p} = -\partial_x H_0(p,x), \tag{2.6b}$$

$$f(t) = T^{-1}\int_{t-T}^{t}\mathcal{F}(p,x)dt, \tag{2.6c}$$

where $p(t)$ and $x(t)$ are evaluated along the unperturbed particle's trajectory as determined by the Hamiltonian $H_0(p,x)$. We can, therefore, use the meter's position f as a pointer to read the value $\langle\mathcal{F}\rangle_T$ directly, for example, from a suitably calibrated scale. We note next that Eqs. (2.4) and (2.5) do not define the meter uniquely. Indeed, expanding $H(p,x,\lambda)$ in powers of λ,

$$H(p,x,\lambda)=H_0(p,x)+T^{-1}\mathcal{F}(p,x)\lambda+\sum_{n=2}^{\infty}H^{(n)}(p,x,0)\lambda^n, \tag{2.7}$$

we note that because the terms $H^{(n)}$, $n>1$, in the sum vanish for $\lambda=0$, they do not affect the work of the meter and can be chosen arbitrarily. In other words, all meters described by Eq. (2.7) are equivalent in the classical limit. Finally, the instantaneous value of $\mathcal{F}(p,x)$ can be measured by choosing $T\to 0$,

$$\lim_{T\to 0}\langle\mathcal{F}\rangle_T = \mathcal{F}(p,x)|_t. \tag{2.8}$$

In this limit the meter strongly interacts with the particle over a short period of time.

III. QUANTUM MEASUREMENTS AND METERS

Next we consider the case when both the particle and meter are to be described quantum mechanically. As in the classical case, we shall use the position of the meter f to obtain information about $\langle\mathcal{F}\rangle_T$. From our brief classical analysis we may conclude, first, that a quantum measurement will perturb the particle's motion. Indeed, for an accurate measurement we need the meter's position to be well defined. On the other hand, because of the uncertainty relation $\Delta f\Delta\lambda > \hbar$, we can no longer make the momentum of the meter λ zero as required by Eq. (2.3). Second, meters which give the same result in the classical limit will not necessarily be equivalent in the quantum case. It is readily seen that for $\lambda\neq 0$, different Hamiltonians in Eq. (2.7) would affect particle's motion differently.

To obtain the transition amplitude $g(x,x',t,t-T|f)$ between the initial positions x' and $f'=0$ at $t-T$ and x and f at t, we construct the classical Lagrangian $L(\dot{x},x,\dot{f})$ corresponding to a Hamiltonian $H(p,x,\lambda)$ in Eq. (2.7):

$$L(\dot{x},x,\dot{f}) \equiv \lambda\dot{f}+L_\lambda(\dot{x},x)=\lambda\dot{f}+p\dot{x}-H(p,x,\lambda). \tag{3.1}$$

For simplicity we shall consider $V(x)$ to be time independent, so that $g(x,x',t,t-T|f)=g(x,x',T|f)$. Integrating $\exp\{i\int_{t-T}^{t}L(\dot{x},x,\dot{f})dt/\hbar\}$ over all particle paths, we obtain

$$g(x,x',T|f)=\int Dx(\cdot)A[x(\cdot),f]\exp\{iS_0[x(\cdot)]/\hbar\}, \tag{3.2}$$

$$A[x(\cdot),f]\equiv\frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}d\lambda\,\exp(i\lambda f/\hbar)\exp\{i[S_\lambda-S_0]/\hbar\}, \tag{3.3}$$

where

$$S_\lambda\equiv\int_{t-T}^{t}L_\lambda(\dot{x},x)dt \tag{3.4}$$

and S_0 is the original particle's action, $S_0\equiv S_{\lambda=0}$. Further, integrating Eq. (3.2) over all f's restores the original particle's propagator $g(x,x',T)$,

$$\int_{-\infty}^{\infty}g(x,x',T|f)df=\int Dx(\cdot)\exp\{iS_0[x(\cdot)]/\hbar\} \equiv g(x,x',T). \tag{3.5}$$

Equations (3.2)–(3.5) are the main result of the paper. As far as the particle is concerned, a particular choice of $\mathcal{F}(p,x)$ and $H(p,x,\lambda)$ leads [cf. Eq. (3.2)], quantally, to the partition of the original particle's propagator $g(x,x',T)$ into subamplitudes $g(x,x',T|f)$, each labeled by the variable f. Each subamplitude is related to the particle's history between t and $t-T$, and contains contributions from one or several Feynman paths. From Eq. (3.4) we note that different values of f are, in the language of Ref. [2], interfering alternatives, similar to the two holes in the double-slit experiment. As in the two-slit case, to determine the value of f accurately we require a meter, and have no information about f unless a meter has been introduced.

IV. QUANTUM PARTICLE IN *x* AND *f* DIMENSIONS. VON NEUMANN METER AND THE EIGENFUNCTION EXPANSION

It is now easy to construct a theory in which both x and f play the role of independent variable, and the particle is described by a wavefunction $\Psi(x,t|f)$ giving the amplitude to be restricted at x and to have $\langle\mathcal{F}\rangle_T=f$. Since different f's are interfering alternatives, a measurement to accuracy Δ requires convoluting $g(x,x',T|f')$ with a (square-integrable) apparatus function $G(f-f')$ such that it vanishes rapidly outside a vicinity Δ around $f=f'$ [14]. If at $t-T$ the particle is described by the wave function $\Psi_I(x)$, integration over initial positions gives

$$\Psi(x,t|f)=\int dx'\int df'G(f-f')g(x,x',T|f')\Psi_I(x). \tag{4.1}$$

It is easy to see that $\Psi(x,t|f)$ satisfies a Schrödinger-like equation

$$i\hbar\frac{\partial\Psi(x,t|f)}{\partial t}=\hat{\mathcal{H}}\left(-i\hbar\frac{\partial}{\partial x},x,-i\hbar\frac{\partial}{\partial f}\right)\Psi(x,t|f), \tag{4.2}$$

where $\hat{\mathcal{H}}$ is the operator obtained by replacing in $H(p,x,\lambda)$ given by Eq. (2.7) $p\to-i\hbar\partial/\partial x$ and $\lambda\to-i\hbar\partial/\partial f$, and

$$\Psi(x,t-T|f)=G(f)\Psi_I(x). \tag{4.3}$$

Now

$$\rho(x,f,t)\equiv|\Psi(x,t|f)|^2 \tag{4.4}$$

yields the joint probability for finding the particle in x and knowing the value of f to accuracy Δ. We note that since the operator $\hat{\mathcal{H}}$ on the right-hand side of Eq. (4.2) is Hermitian, the total probability is conserved,

$$N(t)\equiv\int|\Psi(x,t|f)|^2dx\,df=\text{const}. \tag{4.5}$$

Also, for any $G(f)$, we have

$$\Psi(x,t)=\int_{-\infty}^{\infty}\Psi(x,t|f)df, \tag{4.6}$$

where $\Psi(x,t)$ is the conventional Schrödinger wave function which satisfies the initial condition $\Psi(x,t-T)=C\Psi_I(x)$, $C\equiv\int_{-\infty}^{\infty}G(f-f')df'=\text{const}$. Equation (4.6) follows directly from Eqs. (4.1) and (3.5).

Recalling again that Eq. (4.2) describes the interaction of the particle with a meter, we see that the meter acts to destroy interference between subamplitudes $\Psi(x,t|f)$ corresponding to different values of f. The meter is switched on at $t-T$ when the particle is in the state $\Psi_I(x)$, and operates until the reading (i.e., the pointer position) is taken at the time t. The accuracy of the measurement is determined by initial uncertainty in the pointer position f, which is contained in the initial state of the meter, $G(f)$ in Eq. (4.3). Finally, we note that the von Neumann vN meter [4] is a particular case of Eq. (4.2) obtained when only linear terms are retained in Eq. (2.7),

$$i\hbar\frac{\partial\Psi^{\mathrm{vN}}(x,t|f)}{\partial t}=\left\{\hat{H}_0-\frac{i\hbar\partial}{T\partial f}\hat{\mathcal{F}}\right\}\Psi^{\mathrm{vN}}(x,t|f), \tag{4.7}$$

where, again, $\hat{H}_0$ and $\hat{\mathcal{F}}$ are the operators obtained from the classical quantities by replacing $p\to-i\hbar\partial/\partial x$. In the short-time limit $T\to0$ the second term on the right-hand side of Eq. (4.6) dominates, and we obtain the quantum analog of Eq. (2.8) [4],

$$\Psi^{\mathrm{vN}}(x,t|f)=\sum_i c_i\phi_i(x)G(f-\mathcal{F}_i), \tag{4.8}$$

where c_i and $\mathcal{F}_i$ are given by Eqs. (1.1).

V. QUANTUM MOMENTUM. FEYNMAN VERSUS VON NEUMANN APPROACH

Next consider the particle's momentum,

$$\mathcal{F}(p,x)=p\equiv m\dot{x}. \tag{5.1}$$

Classically, we have a choice of equivalent meters described by different Hamiltonians in Eqs. (2.7). Now we want to decide which one should be used quantally. We can do so either by choosing a particular form of the Hamiltonian in Eq. (2.7) or, equivalently, by specifying the functional $A[x(\cdot),f]$ in Eq. (3.3). In the spirit of Feynman's approach [2,14], we shall define the amplitude $\Psi^F(x,t|p)$ for the particle in x to have time average of the momentum p as the *net Feynman amplitude* $\exp\{iS_0/\hbar\}$ *on those paths, ending in x, for which* $\langle m\dot{x}\rangle_T=p$. Then, from Eqs. (3.1), (3.2), and (4.2), we have

$$A^F[x(\cdot),p]=\delta(p-\langle m\dot{x}\rangle_T), \tag{5.2}$$

$$S_\lambda=S_0-\lambda\frac{m}{T}\int_0^T\dot{x}dt, \tag{5.3}$$

and

$$i\hbar\frac{\partial\Psi^F(x,t|p)}{\partial t}=\left\{-\frac{\hbar^2}{2m}\left(\frac{\partial}{\partial x}-\frac{m}{T}\frac{\partial}{\partial p}\right)^2+V(x)\right\}\Psi^F(x,t|p), \tag{5.4}$$

so that the Hamiltonian $\hat{\mathcal{H}}$ is quadratic in the meter's momentum. Equation (5.4), therefore, is different from Eq. (4.7) describing a standard von Neumann meter,

$$i\hbar \frac{\partial \Psi^{\mathrm{vN}}(x,t|p)}{\partial t} = \left\{ -\frac{\hbar^2 \partial^2}{2m_{\partial x^2}} + V(x) - \frac{\hbar^2}{T}\frac{\partial\partial}{\partial p \partial x} \right\} \Psi^{\mathrm{vN}}(x,t|p). \quad (5.5)$$

It is easy to see why the two approaches disagree. In general, rearranging a particle's paths according to the value of $\langle \mathcal{F} \rangle_T$, as in Eq. (5.2), gives the same result as the von Neumann approach for $\Psi(x,T|f)$ only when the Legendre transform connecting the action S_λ [or, more precisely, the Lagrangian $L_\lambda(\dot{x},x)$ in Eq. (3.1)] with the Hamiltonian $H(p,x,\lambda)$,

$$H(p,x,\lambda) = p\dot{x} - m\dot{x}^2/2 + V(x) + \frac{\lambda}{T}\mathcal{F}, \quad (5.6)$$

is linear in λ [14]. This is the case only if $\mathcal{F}(p,x) \equiv \mathcal{F}(x)$ does not depend on p, e.g., for the particle's position or the traversal time [4,14]. The simplest counterexample is the kinetic energy $\mathcal{F} = m\dot{x}^2/2$, where the last term in Eq. (5.6) renormalizes the particle's mass so that $H(p,x,\lambda)$ becomes $p^2/([1-(\lambda/T)]m) + V(x)$. The particle's momentum is another such example. To compare both approaches in more detail we study momentum distributions for a free particle $V(x) \equiv 0$,

$$\Psi^F(x,T|p) = m^{-1}T \int dp' G(p-p') g_0(p'T/m,T) \times \Psi_I(x - p'T/m). \quad (5.7)$$

and

$$\Psi^{\mathrm{vN}}(x,T|p) = \int dp' G(p-k) \exp(-ik^2T/2m\hbar) \times C_I(k)\exp(ikx/\hbar), \quad (5.8)$$

where $g_0(x,T) = (2\pi i\hbar T/m)^{-1/2} \exp(imx^2/2T\hbar)$ is the free-particle propagator, and $C_I(k)$ are the coefficients in the plane wave expansion of the initial particle's state $\Psi_I(x)$,

$$C_I(k) \equiv (2\pi\hbar)^{-1} \int \exp(-ikx/\hbar)\Psi_I(x)dx. \quad (5.9)$$

In addition, we have the relation

$$\Psi^{\mathrm{vN}}(x,T|p) = (iT/2\pi\hbar m)^{1/2} \times \int \exp[-i(p-p')^2T/2m\hbar] \times \Psi^F(x,T|p')dp'. \quad (5.10)$$

We see from Eq. (5.7) that to contribute to $\Psi^F(x,T|p)$ at t, the particle must have been approximately (if Δ is small) at $x - pT/m$ at $t-T$, since obviously, for any path $x(t)$, $\langle m\dot{x}\rangle_T = m[x(T)-x(0)]/T$. The von Neumann meter, however, probes the coefficients $C_I(p)$ in the plane-wave expansion of the initial state $\Psi_I(x)$ rather than the particle's position at the time $t-T$. Since for $T\to\infty$ the first term in Eq. (6.9) tends to $\delta(p-p')$, both meters give identical readings if they operate sufficiently long, $\Psi^{\mathrm{vN}}(x,T|p) \approx \Psi^F(x,T|p)$. Results of very fast measurements are, conversely, considerably different. As $T\to 0$, for finite p, the Feynman wave function is nearly independent of p, $\Psi^F(x,T|p) \approx m^{-1}T\Psi_I(x)$. This is what one should expect: Feynman paths are very irregular on a small time scale and, if the very recent past is considered, the particle is seen as arriving at x equally with all possible velocities. For a von Neumann meter, in the same limit, the simple relation between the particle's momentum and its past position is lost. Indeed, as $T\to 0$, the width of the first Gaussian in Eq. (5.10) tends to infinity and $\Psi^{\mathrm{vN}}(x,t|p)$ contains contributions from those paths whose position $x(t=0)$ can be far from $x(T)$, so that the particle's velocity$\approx [x(T)-x(0)]/T$ is very large. Thus, for a very accurate (Δ is small) von Neumann meter, we have $\Psi^{\mathrm{vN}}(x,T|p) \approx C_I(p)\int dp' G(p-p')\exp(ip'x/\hbar)$. Hence for $x \ll \hbar/\Delta$, $\Psi^{\mathrm{vN}}(x,T|p) \approx C_I(p)\exp(ipx/\hbar)$, whereas for $x \gg \hbar/\Delta$ it vanishes. Note that since Δ is small, the coordinate width of $\Psi^{\mathrm{vN}}(x,T|p)$, $\hbar/\Delta$, may exceed the width of the initial wave packet $\Psi_I(x)$. This is a consequence of the Heisenberg uncertainty principle stating that an accurate von Neumann measurement of the particle's momentum destroys information about its position. Finally, from Eq. (5.9) we see that for an initial wave packet $C_I(k)$, with mean values of k, $\overline{k}$, and $\overline{(k-\overline{k})^2} \equiv \sigma^2$, the Feynman and von Neumann approaches agree if

$$T \gg m\hbar/[\max(|p+\Delta|,|\overline{k}+\sigma|)]^2. \quad (5.11)$$

Thus for a typical wavepacket with $E\approx 1$ eV, $\sigma < \overline{k}$, and $\Delta \ll \sigma$, both meters would give the same result if the duration of measurement $T \gg \hbar/E \approx 10^{-15}$ s. Momentum distributions for a particle described by an (unnormalized) Gaussian wave packet

$$\Psi_I(x) = \exp(i\overline{k}x)\exp(-x^2/\delta^2), \quad (5.12)$$

measured by Gaussian,

$$G(p-p') = (\Delta^2\pi/2)^{-1/4} \exp[-(p-p')^2/\Delta^2], \quad (5.13)$$

Feynman (dashed line), and von Neumann (solid line) meters are shown in Fig. 1.

VI. CLASSICAL LIMIT. CLASSICALLY FORBIDDEN EVENTS

Finally, we shall analyze the semiclassical limit of Eq. (4.2). The case of the time spent in the barrier was analyzed in Refs. [14] and [15]. To present a more general argument it is convenient to rewrite Eq. (4.1) introducing a particular solution $\Phi(x,T|f)$ of Eq. (4.2),

$$\Phi(x,t|f) = \int dx' g(x,x',t|f)\Psi_I(x'),$$
$$\Phi(x,0|f) = \delta(f)\Psi_I(x). \quad (6.1)$$

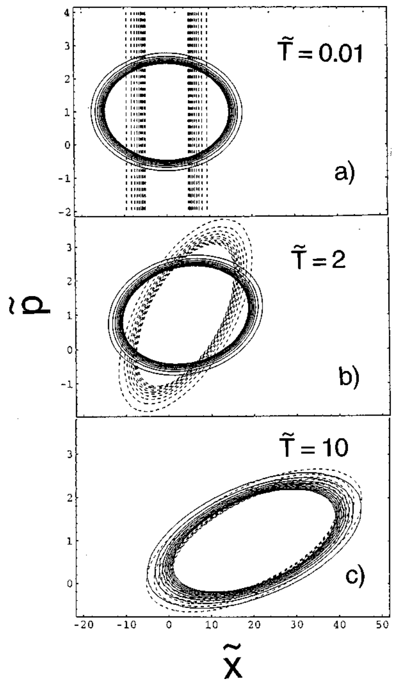

FIG. 1. Contour plots of the probability densities $|\Psi^F(x,T|p)|^2$ (dashed) and $|\Psi^{\mathrm{vN}}(x,T|p)|^2$ (solid) for a free Gaussian wave packet in Eq. (5.11) vs dimensionless $\tilde{p} \equiv p/\overline{k}$ and $\tilde{x} \equiv \overline{k}x/\hbar$, and for $\tilde{T} \equiv T\overline{k}^2/2m\hbar = 0.01$ (a), 2 (b), and 10 (c). The accuracy of the measurement $\tilde{\Delta} \equiv \Delta/\overline{k} = 0.5$, and the coordinate width of the wave packet $\tilde{\delta} \equiv \overline{k}\delta/\hbar = 5$. As in Sec. V, $\overline{k}$ is the centroid of the wave packet in reciprocal space.

Clearly, $\Phi(x,T|f)$ yields the amplitude distribution for the quantity f when measured to an infinite accuracy and is, therefore, unnormalizable, $\int |\Phi(x,t|f)|^2\, dx\, df = \infty$. For a (normalizable) finite accuracy wave function $\Psi(x,t|f)$ in Eq. (4.2), we have [12]

$$\Psi(x,t|f) = \int_{-\infty}^{\infty} df' G(f-f')\Phi(x,t|f'). \tag{6.2}$$

Quantally, f is a distributed quantity. It is instructive to see first how a well-defined value for f is recovered in the (semi)classical limit in the classically allowed region. As $\hbar \to 0$, $\Phi(x,t|f)$ in Eq. (6.2) becomes highly oscillatory everywhere except in the vicinity δf of one (or possible more) critical point where its phase is stationary. The width of the stationary region δf is typically proportional to $\hbar^{1/2}$. We can then measure f with a meter, such that

$$\Delta \gg \delta f. \tag{6.3}$$

Assuming that there is only one critical point f_0, and evaluating the integral in Eq. (6.2) by the stationary phase, for the probability $\rho(x,f)$ to find the value f, we obtain

$$\rho(x,f) = |\Psi(x,t|f)|^2 \approx |G(f-f_0)|^2, \tag{6.4}$$

which is the classical result. Note that a chance to obtain a value significantly different from the classical f_0 is negligible because rapid oscillations of $\Phi(x,t|f')$ over the range of integration Δ make the integral in Eq. (6.2) extremely small.

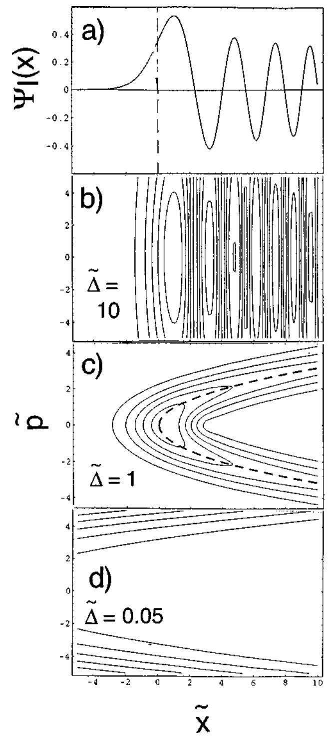

FIG. 2. (a) Wave function $\Psi_I(x) = \mathrm{Ai}(-x/x_0)$ of a particle with zero energy, $E=0$, in a linear potential $V(x) = -e\epsilon x$ vs dimensionless $\tilde{x} \equiv x/x_0$. The vertical dashed line separates the classically forbidden region $x<0$ from the classically allowed region $x>0$. (b) Contour plot of the probability density $|\Psi^{\mathrm{vN}}(x,T\to 0|p)|^2$ vs dimensionless $\tilde{x} \equiv x/x_0$. and $\tilde{p} \equiv px_0/\hbar$ for an inacurate, $\tilde{\Delta} \equiv \Delta x_0/\hbar = 10$, Gaussian meter. (c) Same as (b), but for $\tilde{\Delta} = 1$. Also shown by the dashed line are positions of the critical points $p_{1;2}$ in Eq. (6.9). (d) Same as (b), but for $\tilde{\Delta} = 0.05$ As in Sec. VI, $x_0 \equiv x_0 \equiv (2me\epsilon/\hbar^2)^{-1/3}$

In the classically forbidden region $\Phi(x,t|f)$ has no critical points on the real f axis. Rather, it may have saddle points in the complex f plane, so that in the limit $\hbar \to 0$ the conventional Schrödinger wave function $\Psi(x,t)$ [cf. Eq. (4.6)],

$$\Psi(x,t) = \int_{-\infty}^{\infty} \Phi(x,t|f)\, df, \tag{6.5}$$

is exponentially small. In the absence of a well-defined real stationary region, a meter will produce readings distributed over a wide range of values. The shape of the distribution will depend on the properties of the meter, namely, its apparatus function $G(f)$. In this sense, in the classically forbidden region we cannot assign a unique value to f even in the limit $\hbar \to 0$ [16].

The simplest system demonstrating both classically allowed and classically forbidden behaviors is a quantum particle of mass m and charge e in a constant electric field ε,

$$V(x) = -e\varepsilon x \equiv -Fx. \tag{6.6}$$

We shall choose the particle to have zero energy, $E=0$, so that the region $x>0$ is classically allowed, while $x<0$ is classically forbidden. The initial state of the particle is, therefore, the Airy function [17] shown in Fig. 2(a),

$$\Psi_I(x)=\mathrm{Ai}(-x/x_0),$$
$$x_0\equiv(2me\varepsilon/\hbar^2)^{-1/3}. \tag{6.7}$$

For simplicity we shall consider a very fast, $(T\to 0)$ measurement of the particle's momentum by a von Neumann meter [Eq. (4.8)]. From Eqs. (1.1) and (4.8), we have

$$\lim_{T\to 0}\ \Phi^{\mathrm{vN}}(x,T|p)=x_0(4\pi\hbar^2)^{-1/2}$$
$$\times\exp[-ip^3x_0^3/(3\hbar^3)+ipx/\hbar]. \tag{6.8}$$

The phase of the exponential in Eq. (6.8) has two critical points $p_{1;2}$:

$$p_{1;2}=\pm\hbar x^{1/2}x_0^{-3/2}. \tag{6.9}$$

For $x>0$, the classical values $p_{1;2}$ are real, corresponding to the particle moving in the left and right directions, respectively. The width of the stationary region δp increases as the particle approaches the turning point $x=0$,

$$\delta p\approx\pi^{1/2}\hbar x^{-1/4}x_0^{-3/4}=\pi^{1/2}\hbar^{1/2}x^{-1/4}(2me\varepsilon)^{3/4}. \tag{6.10}$$

At the turning point $x=0$, the critical points of the Airy integral (6.9) coalesce, and then, as x becomes negative, move into the complex plane. The results of measuring the particle's momentum to accuracy Δ by a von Neumann meter with a Gaussian apparatus function (5.13) are shown in Fig. 2. For a very inaccurate meter [Fig. 2(b)] Δ is large and, for finite p, $G(p-p')$ does not restrict integration in Eq. (4.1). As a result, $\rho(x,p,T\to 0)$ in Eq. (4.4) is nearly independent of p, $\rho(x,p,T\to 0)\approx|\Psi_I(x)|^2$. In the classically forbidden region $\rho(x,p,T\to 0)$ is exponentially small, while in the classically allowed region it repeat oscillations of the Airy function. Clearly, such a meter does not distinguish between different momenta, and this limit corresponds to Aharonov's weak measurement regime [8]. We shall not discuss the weak measurement limit any further. As the accuracy of the measurement improves, $\Delta\approx\delta p$ [Fig. 2(c)] in the classically allowed region the meter begins to resolve the two stationary regions corresponding to the classical values of the momentum in Eq. (6.9). At the turning point $x=0$ where the two classical values coalesce, the meter's readings are centered around $p=0$. Further into the classically forbidden region of negative x there is no preferred real value of p, and the readings remain to be distributed around the origin. A further increase in the accuracy of the measurement, $\Delta\ll\delta p$, again destroys the classical picture as the apparatus function begins to vary rapidly across the stationary region. Equivalently, the perturbation produced by an accurate meter is greater than the effect of the original potential $V(x)$. For this reason, there is little difference between formerly classically allowed and classically forbidden regions [Fig. 2(d)].

VII. CONCLUSIONS

In general, a quantum act of measurement, complete at time t, can be understood as the destruction of interference between different components (subamplitudes) $\Psi(x,t|f)$ of the particle's wave function $\Psi(x,t)$, related to the particle's history for $t'<t$. A particular choice of the variable f provides different decompositions of $\Psi(x,t)$, where the value of the measured variable f plays the role of a continuous index labeling subamplitudes. Typically, a measurement is characterized by a duration T, showing how far into the past the particle's behavior is probed, and an accuracy Δ which specifies the degree to which the interference between different $\Psi(x,t|f)$ is destroyed.

A particular variable f is measurable in practice (or, rather, at the *gedankenexperiment* level) if the single-particle generalized Schrödinger equation (4.2) for $\Psi(x,t|f)$ can be interpreted as a Schrödinger equation representing both the particle plus a meter. Since a meter acts as a ''slit'' in the f coordinate, i.e., it filters in subamplitudes $\Psi(x,t|f)$ in the vicinity Δ of the measured value f_0 and discards the rest, it would, in general, perturb the particle's motion. Conversely, the perturbation produced by a meter is the one necessary to project $\Psi(x,t|f)$ onto the apparatus function $G(f)$. The time average of a dynamical variable in Eq. (2.1) can always be measured, but more esoteric quantities, such as the quantum first passage time [18,19], may not necessarily be measurable.

Importantly, there is no unique recipe for constructing quantum subamplitudes $\Psi(x,t|f)$ for a given classical dynamical variable $\mathcal{F}$. Measurements, equivalent in the classical limit, may differ in the quantum case, and correspond, therefore, to different decompositions of $\Psi(x,t)$. In general, a measurement does not necessarily correspond to a rearrangement of Feynman's paths into classes according to the value of a particular classical functional. We have analyzed two possibilities. A special choice of the Hamiltonian in Eq. (2.2), linear in the meter's momentum, yields, quantully, the wave function $\Psi^{\mathrm{vN}}(x,T|f)$, which is consistent with the von Neumann procedure and, in the limit $T\to 0$, with the eigenfunction expansion (1.1). Defining $\Psi(x,t|f)$ as the net amplitude on those Feynman paths for which $\langle\mathcal{F}\rangle_T$ in Eq. (1.1) has exactly the value f yields the Feynman wave function $\Psi^F(x,T|f)$ which is, in general, different from $\Psi^{\mathrm{vN}}(x,T|f)$.

One of the simple examples is the particle's momentum p. Classically, p is the variable canonically conjugate to particle's position x. It is also related to the rate of change of x, $p=m\dot{x}$. Quantally, defining in the usual way the momentum to be conjugate to x leads to the operator $-i\hbar(\partial/\partial x)$ and von Neumann subamplitudes $\Psi^{\mathrm{vN}}(x,T|p)$ in Eq. (5.5). In this way one loses, however, the simple classical relation between p and the particle's position in the past, as $\Psi^{\mathrm{vN}}(x,T|p)$ contains contributions from Feynman paths with a wide range of velocities. Rearranging Feynman paths according to the value $\langle m\dot{x}\rangle_T$ yields $\Psi^F(x,T|p)$ in Eq. (5.4), which considerably differs from $\Psi^{\mathrm{vN}}(x,T|p)$ in the short-time limit. As far as measurements are concerned, both wave functions are equally meaningful. We can, in principle, construct an apparatus which would measure either the probability density $|\Psi^{\mathrm{vN}}(x,t|p)|^2$ or $|\Psi^{F}(x,t|p)|^2$. A choice between the two requires additional assumptions, such as postulating Eq. (1.1). In this sense, there is no unique definition of the quantum particle's momentum.

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