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Trapped one-dimensional Bose gas as a Luttinger liquid

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The low-energy fluctuations of a trapped, interacting quasi-one-dimensional Bose gas are studied. Our considerations apply to experiments with highly anisotropic traps. We show that under suitable experimental conditions the system can be described as a Luttinger liquid. This implies that the correlation function of the bosons decays algebraically, preventing Bose-Einstein condensation. At significantly lower temperatures a finite-size gap destroys the Luttinger liquid picture and Bose-Einstein condensation is again possible. [S1050-2947(98)50711-1]

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The experimental realization of Bose-Einstein condensation (BEC) in atomic vapors of ^{87}Rb [1] and ^{23}Na [2,3] has attracted a lot of interest [4]. Recently, a highly anisotropic, quasi-one-dimensional trap has been designed [5]. Up to now, the possibility of BEC in one dimension has mainly been discussed for the noninteracting Bose gas [6,7]. The role of dimensionality has been carefully examined for the *ideal* bose gas by van Druten and Ketterle [8]. In one dimension the interaction between bosons plays an essential role due to the strong constraint in phase space [9]. The question of BEC in a quasi-one-dimensional system is therefore more complicated. The purpose of this paper is to demonstrate that under suitable experimental conditions the low-energy excitations of this system are described by a Luttinger-liquid (LL) [10] model. The superfluid correlations of a LL decay algebraically and the system is not Bose condensed. At much lower temperatures, which are determined by the extension of the trap in the longitudinal direction, the spectrum of the phase fluctuations is again cut off by finite-size effects and the bosons could condense again.

The realization of a Luttinger liquid in a one-dimensional Bose gas would be a highly nontrivial example of an interacting quantum liquid. Fermionic systems that are believed to be described by a Luttinger liquid include quasi-one-dimensional organic metals [11], magnetic chain compounds, quantum wires, and edge states in the quantum Hall effect. While these systems are always embedded in a three-dimensional matrix and thus show a crossover to three-dimensional behavior at low temperatures, the trapped one-dimensional Bose gas would provide a clean testing ground for the concept of a Luttinger liquid.

The paper is organized as follows. First we discuss the circumstances under which a trapped Bose gas can be considered as a one-dimensional quantum system. Next we demonstrate in an explicit calculation that there is a gapless mode with a linear dispersion. We show that the Hamiltonian of the low-lying excitations can be identified as that of a Luttinger liquid and therefore the density-density correlation function decays algebraically. In the remainder of the paper we discuss the implication of the algebraic decay of the particle-particle correlation function for BEC and review the properties of a Luttinger liquid.

We consider the Bose gas in a cylindrical symmetric trap confined to the z axis by a tight trapping potential in the x-y plane. If the extension L of the trap in the z direction is much larger than its radius R, it is justified to approximate the potential in the longitudinal direction by zero. One-dimensional physics will be dominant, if the temperature is much lower than the energy of the lowest radial excitation. The energy scale is set by $\hbar\omega_\perp$, with $\omega_\perp$ being the trap frequency [12,13]. Thus the condition for one dimensionality is

$$\hbar\omega_\perp \gg k_B T, \tag{1}$$

where T the temperature of the Bose gas. A typical value for $\omega_\perp$, which has been realized in the experiments performed at MIT by the Ketterle group [5], is $2\pi\times 240$ Hz. In order to realize a one-dimensional Bose gas for this value of $\omega_\perp$ the temperature has to be lower than 1.8 nK. Another possibility is to increase the value $\omega_\perp$, which might be more feasible experimentally. For instance, permanent magnets can be used to increase trap frequencies by more than an order of magnitude [14].

Assuming that this condition for $\omega_\perp$ can be met experimentally, we can model the system by the following Hamiltonian:

$$H=\int d^3r\,\psi^\dagger(\vec r)\left(-\frac{\hbar^2}{2m}\Delta+U(\vec r)-\mu\right)\psi(\vec r)+\frac{1}{2}\int d^3r\,d^3r'\,\psi^\dagger(\vec r)\psi^\dagger(\vec r\,')g\,\delta(\vec r-\vec r\,')\psi(\vec r\,')\psi(\vec r), \tag{2}$$

where m is the atomic mass, μ is the chemical potential fixed by the particle number $N=\int d^3r|\psi(\vec r)|^2$, and $g=4\pi\hbar^2 a/m$ is the coupling constant, with a being the s-wave scattering length. We only consider repulsive interactions. $U=\frac{1}{2}m\omega_\perp^2(x^2+y^2)$ is the trapping potential. The field operators $\psi^\dagger(\vec r)$ and $\psi(\vec r)$ are bosonic creation and destruction operators.

We now illustrate that a gapless mode for a Hamiltonian, such as Eq. (2), exists [10,15]. The dynamics of $\psi(\vec r,t)$ is governed by the equation of motion

$$i\hbar\partial_t\psi=-\frac{\hbar^2}{2m}\Delta\psi+(U-\mu)\psi+g\psi^\dagger\psi\psi, \tag{3}$$

a one-dimensional nonlinear Schrödinger equation, which is well understood [10]. We merely illustrate in the following its application to the problem of trapped bosons. For a macroscopically occupied ground state, the operator ψ can be considered as a classical complex field. Then Eq. (3) becomes the Gross-Pitaevskii equation. We describe the complex field $\psi(\vec{r},t)$ by its density-phase representation: $\psi(\vec{r},t) = \sqrt{\rho(\vec{r},t)}\,\exp[i\theta(\vec{r},t)]$. A saddle-point solution to Eq. (3) is given by a constant phase and static density $\rho(\vec{r},t)=\rho_0(r)$, which only depends on the radius r, due to the axial symmetry of the problem. The solution of the Gross-Pitaevskii equation in a cylindrical trap and its fluctuations in the Thomas-Fermi approximation has been discussed in detail by Zaremba [16]. We only repeat the steps of the calculation that are necessary for our arguments. Expanding in small fluctuations of the phase $\delta\theta$ and density $\delta\rho$ around the saddle-point solution,

$$\psi(\vec{r},t)=\sqrt{\rho_0+\delta\rho(\vec{r},t)}\,e^{i[\theta_0+\delta\theta(\vec{r},t)]},$$

we obtain the linearized equations of motion for $\delta\rho$ and $\delta\theta$,

$$\hbar\,\partial_t\delta\theta=g\,\delta\rho-\frac{\hbar^2}{4m}\frac{1}{\rho_0}\nabla\left(\rho_0\nabla\frac{\delta\rho}{\rho_0}\right), \tag{4}$$

$$\hbar\,\partial_t\delta\rho=\frac{\hbar^2}{m}\nabla(\rho_0\nabla\,\delta\theta). \tag{5}$$

These equations possess a trivial solution ($\delta\rho=0$, $\delta\theta=\mathrm{const}$) whose energy vanishes. This is the Goldstone mode corresponding to global rotations of the condensate's phase. Radial fluctuations can be ignored, because their energy scale is set by $\hbar\omega_\perp$, the trap frequency. Thus it is justified to consider the one-dimensional limit where the equations simplify to

$$\hbar\,\partial_t\delta\theta=g\,\delta\rho-\frac{\hbar^2}{4m}\frac{1}{\rho_0}\partial_z^2\delta\rho, \tag{6}$$

$$\hbar\,\partial_t\delta\rho=\frac{\hbar^2}{m}\rho_0\partial_z^2\delta\theta. \tag{7}$$

The solutions are plane waves ($\delta\rho$, $\delta\theta\propto e^{i(qz-\omega t)}$) with frequencies

$$\hbar^2\omega^2=\hbar^2v_s^2q^2+\left(\frac{\hbar^2}{2m}\right)^2q^4,$$

$$q=\frac{2\pi}{L}n,\quad n=0,1,\ldots,L-1 \tag{8}$$

where the sound velocity is given by $v_s=\sqrt{g\rho_0/m}$, which is the Bogoliubov value for a homogeneous Bose gas (Fig. 1). It has been observed experimentally in anisotropic three-dimensional traps [5].

We draw two important conclusions from this relation. The q^2 term cannot be treated as a small perturbation on the energy of a noninteracting Bose gas; hence the smallest interaction changes the excitation spectrum fundamentally.

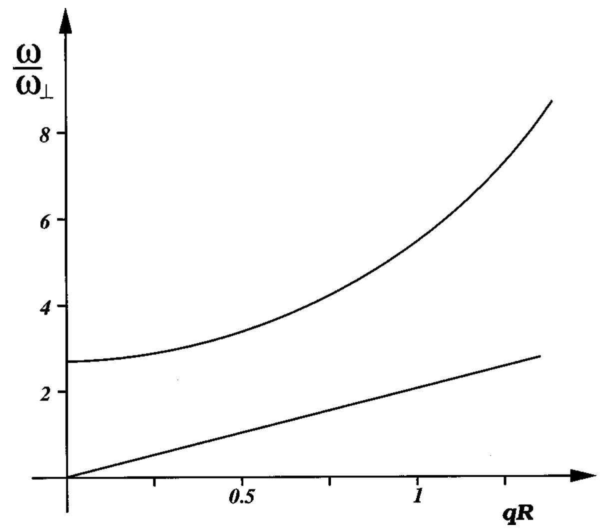

FIG. 1. Qualitative behavior of the low-energy excitation spectrum (lower curve) for a one-dimensional trapped Bose gas. The first radial excitation (upper curve) with energy $\omega\sim\omega_\perp$ is also shown. The discussion in the text focusses on the role of the lower branch.

The existence of a collective mode with linear dispersion for small q is a direct consequence of the interaction between particles. Only for a vanishing coupling constant g, does the spectrum reduce to that of free particles, regardless of the ground-state occupation. In a one-dimensional trap ($L\gg R$) the phase fluctuations of the boson wave function destroy superfluid order due to phase-space constraints [17]. The finite-size gap in three-dimensional traps ($L\approx R$) introduces a cutoff in the phase-space integrals, the phase-space argument does not apply [12,18], and BEC is possible. As all trapped Bose gases are of finite size, in principle the phonon spectrum remains discrete. The level splitting is only relevant in the limit

$$k_BT\ll\hbar v_s\frac{2\pi}{L}. \tag{9}$$

For the MIT trap [5] with the length $L=0.5$ mm, this temperature turns out to be roughly 10^{-13} K (assuming Na atoms). Only in this limit the system can be in a Bose condensed phase. If the length L is not macroscopic the gap in the lowest mode will be appreciable and there is Bose-Einstein condensation for a finite number of particles, as pointed out by Ho and Ma [18]. We stress that this is due to the smallness of L and not a generic feature of the system. A setup with L comparable to R is really three-dimensional. One can check that the gap energy for a one-dimensional trap found by Ho and Ma scales with the inverse axial extension of the system and hence disappears for large systems. We conclude that for L satisfying condition (9) there is a gapless sound mode that inhibits the formation of a condensate at all finite temperatures. Nonetheless the decay of coherence is only weak. This is due to the fact that the system can be described as a Luttinger liquid, as now will be shown.

With the same approximation as for the equations of motion, the Hamiltonian in the long-wavelength limit is

$$H=\int dz\left[\frac{\hbar^2\rho}{2m}(\partial_z\delta\theta)^2+\frac{\kappa}{2\rho^2}\delta\rho^2\right], \tag{10}$$

where ρ is the number of particles per unit length and κ is the compressibility.

The Hamiltonian, Eq. (10), is known as the Luttinger-liquid Hamiltonian [10,19]. This concept has been mostly used to investigate the properties of fermionic systems in one dimension. The Luttinger-liquid Hamiltonian, Eq. (10), can be diagonalized by a Bogoliubov transformation in terms of new bosonic creation and destruction operators $b_q^\dagger, b_q$ for the long-wavelength density-fluctuation modes. This is possible due to the linear dispersion relation. The Bogoliubov transformation is given by

$$\delta\rho=\frac{1}{\sqrt{2}}\sum_{q\neq 0} e^{iqz}f_q(b_q^\dagger+b_{-q}), \tag{11}$$

$$\partial_z\delta\theta=\frac{1}{\sqrt{2}}\sum_{q\neq 0} e^{iqz}g_q(b_q^\dagger-b_{-q}). \tag{12}$$

The $b_q^\dagger, b_q$ satisfy the usual boson commutation relation $[b_q, b_{q'}^\dagger]=\delta_{q,q'}$ and $\delta\theta$ and $\delta\rho$ form a pair of conjugate operators:

$$[\delta\theta(z),\delta\rho(z')]=i\delta(z-z'). \tag{13}$$

This condition fixes the functions f_q and g_q:

$$f_q=\sqrt{|q|}e^{\alpha_q}, \tag{14}$$

$$g_q=sgn(q)\sqrt{|q|}e^{-\alpha_q}, \tag{15}$$

where α_q is the parameter of the Bogoliubov transformation. Inserting the representations (11) and (12) for $\delta\rho$ and $\delta\theta$, respectively, in the Hamiltonian for the fluctuations, the LL in terms of the new bosonic operators is given by

$$H=\sum_{q\neq 0}\hbar\omega_q(b_q^\dagger b_q+\tfrac{1}{2}), \tag{16}$$

with the choice $\exp(2\alpha_q)=\hbar\sqrt{\rho/mg}$. The phonon frequency is given by $\omega_q=v_s|q|$ where v_s is the sound velocity.

One of the striking features of a Luttinger liquid is that the model has only two microscopic parameters, the sound velocity v_s and the compressibility κ. Another important property is that the correlation functions of the original boson operators decay algebraically in a Luttinger liquid. The asymptotic behavior for large distances, $z\to\infty$, of the boson-boson and density-density correlation function is given by [10]

$$\langle\Psi^\dagger(z)\Psi(0)\rangle\sim 1/z^{1/\eta}, \tag{17}$$

$$\langle\rho(z)\rho(0)\rangle-\langle\rho\rangle^2\sim\eta/z^2, \tag{18}$$

where η is the correlation exponent. A useful naive estimate for η, assuming that the compressibility $\kappa\sim g\rho^2$ is

$$\eta=\pi l_B\sqrt{\frac{2\rho}{a}}, \tag{19}$$

where $l_B=\sqrt{\hbar/\omega_\perp m}$ is the magnetic length of the trap perpendicular to the z axis and a is the scattering length of the trapped atoms. Because the interaction is weak we do not expect the exponent η to be renormalized substantially. For current traps the exponent η is of the order $\eta\sim 1000$, demonstrating that the phase coherence of the bosons decays only very weakly and is experimentally indistinguishable from true BEC [20]. However, for steeper magnetic traps, $\omega_\perp\sim 50$ kHz, particle densities of $\rho\sim 10^4$ particles/cm, and assuming a scattering length of $110a_B$ for Rb [21], the exponent η is $\eta\sim 4$ and it should be possible to observe LL behavior. Below $T\sim 0.4$ nK only the linear mode is excited and the physics is described by LL physics. At still lower temperatures, $T\sim 10^{-12}$ K, the finite-size gap comes into play [18].

Next we compare our results to the ''two-step condensation'' picture put forward by van Druten and Ketterle [8]. The authors consider an *ideal* Bose gas in a highly anisotropic trap. In the noninteracting system there is no fundamental difference between the one and three dimensions, except in the density of states. As soon as interactions have to be considered, the situation changes drastically. Basically we have developed a more precise physical picture of the regime that van Druten and Ketterle call the ''two-step BEC'' [8]. Our claim is that in this regime the ground state is described by a Luttinger liquid and not by an ideal Bose gas.

Since the Luttinger-liquid model has a harmonic Hamiltonian, Eq. (16), for the phase and density fluctuations, any expectation value and dynamical correlation function of the boson operators in the long-wavelength limit can be evaluated. Luttinger liquids are well understood and many results can be carried over to the one-dimensional trapped Bose gas. At larger densities the parameters of the Luttinger-liquid model will be renormalized from the saddle-point values by short-range fluctuations and also by the three-dimensional density profile of the trapped Bose gas. The renormalized parameters can be obtained by considering more realistic interactions in one dimension. For a repulsive δ-function potential the sound velocity and the compressibility have been obtained exactly [22]. We are currently working on models with longer-range interactions that can be treated by the density-matrix–renormalization-group method. Results of this work will be presented elsewhere [23]. In a complementary approach, we calculate the finite-size effects in the experimental setup on the dynamics of the bosons [24]. Another interesting problem that we are currently investigating is the response of the system to an impurity atom. The finite mass leads to an unusual behavior of the mobility [25,26]. Also the transport properties should differ significantly from the conventional Bose condensate if the Bose gas is in the Luttinger-liquid regime.

To summarize, we have shown under which experimental conditions a trapped quasi-one-dimensional system of interacting Bosons is described by a Luttinger-liquid Hamiltonian. An experimental realization of such a system would provide a clean laboratory for testing the properties of a Luttinger liquid. Its behavior deviates significantly from the noninteracting Bose gas. Unlike other systems that are realizations of a Luttinger liquid, three-dimensional effects become less important for lower temperatures. Moreover, it would be possible to tune important parameters like the density and the length, i.e., the trap frequency $\omega_\perp$, which is impossible in a solid.

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