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Multidimensional parametric quantum solitons

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We consider the parametric quantum field theory involving cubic and quartic couplings of two bosonic fields. This is exactly soluble for the two-particle energy eigenstates (or quantum solitons) in one, two, and three space dimensions. We estimate the binding energies and corresponding radii in the case of photonic fields in nonlinear optical materials, and Bose-Einstein condensates. [S1050-2947(98)51110-9]

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The search for three-dimensional quantum bound states of interacting Bose fields has resulted in surprisingly few exact solutions. However, a large variety of these quantum solitons are known theoretically in one spatial dimension, and experiments are now possible to test these theories. This means that, complementary to high-energy physics, particlelike structures may be investigated in a larger variety of physical systems.

While theory and experiments are possible for quantum solitons in one spatial dimension [1,2], this still leaves the multidimensional problem open. The difficulty is that field theories with localized attractive interactions usually demonstrate a collapsing behavior in higher dimensions. For example, the Bose gas with an attractive δ-function potential (nonlinear Schrödinger model)—while stable in one dimension [1]—has no lower bound to its Hamiltonian in higher dimensions. At the classical level, this instability causes a self-focusing singularity.

Promising candidates for higher-dimensional quantum solitons therefore include quantum field theories whose classical analogs are stable. Early approximate solutions of this type were investigated by Christ and Lee [1], and recently some exact quantum results were obtained for the classical Davey-Stewartson model [3]. Despite the exact two- and three-dimensional solutions that exist in the quantum Davey-Stewartson model, there are no known physical systems that are described by this quantum field theory. There has also been research into possible two-photon solutions to variations of the attractive Bose-gas problem [4], but without experimental verification.

We report here an investigation into a parametric quantum field theory corresponding to a two-component Bose gas interacting via a three-wave mixing process. This is known to have stable classical solitary-wave solutions in higher dimensions [5]. It is a traveling-wave analog of the quantum theory used to describe quantum squeezing [6], and more recently molecular dissociation in atom optics [7]. We find solutions in two and three spatial dimensions, and estimate the binding energies.

Our results have a number of entirely unexpected features. The most surprising is that we show that the simplest parametric quantum field theory, like the attractive Bose-gas model, is unstable in two and three dimensions. However, unlike the Bose-gas model, this instability shows no trace at the classical level, where there is stable behavior with a lower bound to the Hamiltonian energy [5]. For stable parametric quantum field theories the Hamiltonian must be modified.

We investigate the effects of modifying the nonlinear interactions by adding a quartic term to the Hamiltonian, and by imposing a momentum cutoff on the coupling constants. A quartic term corresponds to four-wave mixing, or a nonlinear refractive index in the corresponding optical medium. It is also found, for example, in atom-atom interactions. With a positive quartic interaction, a rigorous lower bound to the energy does exist, and we demonstrate the existence of exact two-particle bound-state solutions in higher dimensions. These types of quantum solitons have a unique character: the solution has a finite binding energy, but the corresponding two-particle wave function has a zero radius, unless a momentum cutoff is imposed on the couplings. Solutions in one dimension have a finite radius in all cases.

To demonstrate these solutions, consider a quantum interaction Hamiltonian given by [2,6]:

$$H=\int d^{(D)}\mathbf{x}\left[\sum_{i=1,2}\frac{\hbar^2}{2m_i}|\nabla\Phi_i|^2+\hbar\rho\Phi_2^\dagger\Phi_2+\frac{\hbar\chi_D}{2}(\Phi_1^2\Phi_2^\dagger+\Phi_1^{\dagger 2}\Phi_2)+\frac{\hbar\kappa_D}{2}\Phi_1^{\dagger 2}\Phi_1^2\right]. \quad (1)$$

Here $\Phi_i\,(i=1,2)$ are two complex Bose fields with commutation relations of $[\Phi_i(\mathbf{x}),\Phi_j^\dagger(\mathbf{x}')]=\delta_{ij}\delta(\mathbf{x}-\mathbf{x}')$, m_i are the effective masses, ρ is the phase mismatch, and χ_D and κ_D are the coupling constants responsible for the parametric three-wave mixing and four-wave mixing processes, respectively, in D space dimensions ($D=1,2,3$).

In one space dimension, with $\kappa_1=0$, there are known two-particle bound-state solutions [8]. We now ask what solutions can exist in higher dimensions. In any number of dimensions, the Hamiltonian (1) has a momentum conservation law, and a boson number conservation law in which $N=\int d^{(D)}\mathbf{x}(|\Phi_1|^2+2|\Phi_2|^2)$ is conserved. We therefore search for states that are eigenstates of the momentum operator P, the operator N, and H. In the two-particle case ($N=2$), these have the structure

$$|\Psi\rangle=\left[\int\int d^{(D)}\mathbf{x}d^{(D)}\mathbf{y}g(\mathbf{x}-\mathbf{y})e^{i(\mathbf{K}/2)\cdot(\mathbf{x}+\mathbf{y})}\Phi_1^\dagger(\mathbf{x})\Phi_1^\dagger(\mathbf{y})+\int d^{(D)}\mathbf{x}e^{i\mathbf{K}\cdot\mathbf{x}}\Phi_2^\dagger(\mathbf{x})\right]|0\rangle. \quad (2)$$

In general, the Hamiltonian (1) could also contain a quartic interaction term for the second-harmonic field Φ_2. However, this has no effect on the two-particle eigenstate under consideration. Therefore, the corresponding term in the Hamiltonian is omitted for simplicity.

To prove a lower bound on the Hamiltonian energy, we apply Eq. (1) to $|\Psi\rangle$ and use the symmetry property of the two-particle wave function: $g(\mathbf{x})=g(-\mathbf{x})$. This leads to a general expression for the Hamiltonian energy, in which we can omit the contribution of the positive term $\sim(2\hbar^2/m_1)\int d^{(D)}\mathbf{x}|\nabla g(\mathbf{x})|^2$ to arrive at a lower bound. Applying a chain of algebraic inequalities to this reduced energy, we finally obtain that, if $\kappa_D>0$ and

$$\hbar[\chi_D]^2>2\Delta\kappa_D, \tag{3}$$

where $\Delta\equiv-\hbar^2K^2/(4m_1)+\hbar^2K^2/(2m_2)+\hbar\rho$ and $K=|\mathbf{K}|$, then the lower bound E_l can be defined by

$$E_l=\frac{\hbar^2K^2}{2m_2}+\hbar\rho-\frac{\hbar[\chi_D]^2}{2\kappa_D}\leqslant\frac{\langle\Psi|H|\Psi\rangle}{\langle\Psi|\Psi\rangle}. \tag{4}$$

To evaluate an upper bound to the lowest-energy eigenvalue of our Hamiltonian we use a variational approach. In the one-dimensional case ($D=1$) we choose a trial function $g(r)$ in the form $g(r)=g_0\exp(-|r|/r_0)$, where $r=x-y$, following the structure of the known exact solution for the pure parametric interaction [8]. We calculate then the variational energy $\tilde{E}=\langle\Psi|H|\Psi\rangle/\langle\Psi|\Psi\rangle$ and minimize it with respect to the parameters g_0 and r_0. As a result of this optimization procedure, subject to a localized bound-state formation ($r_0>0$), we obtain that there always exists one positive solution with a finite optimum r_0 value, if the condition (3) is met. Here r_0 is obtained by solving a cubic equation similar to that found in the purely parametric case [8]. The final result for the minimal value of $\tilde{E}$, which corresponds to the exact eigenvalue E in this case, is

$$E=\hbar^2K^2/(4m_1)-\hbar^2/(m_1r_0^2). \tag{5}$$

Thus, a finite-size diphoton quantum soliton is shown to exist in our model in one dimension.

In higher dimensions ($D=2,3$) we use the following trial function: $g(\mathbf{r})=g_0\exp[-(|\mathbf{r}|/r_0)^s]$, where $\mathbf{r}=\mathbf{x}-\mathbf{y}$. The variational energy $\tilde{E}$ approaches its minimal value in the limits $r_0^s\to0, s\to0$, and at $g_0=-\chi_D/(2\kappa_D)$. Again the condition (3) is assumed to be fulfilled to provide localized bound states. The final result for the minimal value of $\tilde{E}$ takes the form of the expression for E_l [see Eq. (4)]. This implies that the exact lowest-energy eigenvalue is given by $E=E_l$. Returning to the form of the trial function $g(\mathbf{r})$ at the optimum values of parameters r_0, s, and g_0, we conclude that

$$g(0)=-\chi_D/(2\kappa_D),$$

$$g(\mathbf{r})=0 \quad \text{if} \quad \mathbf{r}\neq\mathbf{0}, \tag{6}$$

i.e., the quantum solitons in two and three dimensions have a pointlike (zero-radius) structure. This is different from the usual δ-function singularity, and leads to a vanishing integral $\int d^{(D)}\mathbf{x}|\nabla g(\mathbf{x})|^2$ (in the case of the δ-function singularity this would give infinity). The normalization integral for the total two-particle wave function $\langle\Psi|\Psi\rangle\sim1+\int d^{(D)}\mathbf{x}g^2(\mathbf{x})=1$ does not vanish due to the contribution of the second-harmonic field [9].

It should be mentioned that the pointlike structure, Eq. (6), in the two-particle eigenstate refers to the correlation function of the subharmonic fields. That is, the localization is in the relative position of the two subharmonic quanta, not in their absolute position. The quantum soliton itself has a delocalized center-of-mass, just as one would expect for an energy eigenstate.

To understand our solutions in more detail, we note that parametric couplings of the type found in Eq. (1) are usually restricted to a finite range of relative momenta or wave numbers. Therefore it is more realistic to incorporate the finite range of the couplings into our interaction Hamiltonian. To represent this we can introduce a cutoff at $|\mathbf{k}|=k_m$ in the relative momenta of the interacting fields. The interaction part of the Hamiltonian (1) is then expressed in terms of $a_i(\mathbf{k})$, the Fourier component of Φ_i, so that its cutoff dependence is implemented through the limits of the corresponding integrals.

We can now analyze the energy eigenvalue problem directly, by introducing a Fourier transform of $g(\mathbf{r})$, so that $g(\mathbf{r})=\int d^{(D)}\mathbf{k}G(\mathbf{k})\exp(i\mathbf{k}\cdot\mathbf{r})/(2\pi)^D$. Due to the cutoff in the nonlinearities, we need only investigate eigenstates for which $G(\mathbf{k})=0$ if $|\mathbf{k}|>k_m$. This leads to the following equations (valid for $|\mathbf{k}|<k_m$) for an eigenstate:

$$(k^2+\mu^2)G(\mathbf{k})=-\frac{m_1}{\hbar}\left[\frac{\chi_D}{2}+\kappa_Dg(0)\right],$$

$$E=\frac{\hbar^2K^2}{2m_2}+\hbar\rho+\hbar\chi_Dg(0)=\frac{\hbar^2K^2}{4m_1}-\frac{\hbar^2\mu^2}{m_1}. \tag{7}$$

Here $k=|\mathbf{k}|$, and we have introduced an inverse length scale μ, so that $\mu^2=(K/2)^2-Em_1/\hbar^2$. Clearly, $\hbar^2\mu^2/m_1$ can be interpreted as the binding energy of a solution with momentum $\mathbf{K}$. The solution is bound (against two-particle decay) if μ is real and positive.

In order to evaluate the binding energy and effective radius, we next solve for $g(0)$ and find

$$g(0)=-\frac{\chi_D}{2}\left[\kappa_D+\frac{\hbar}{m_1f_D(\mu,k_m)}\right]^{-1}, \tag{8}$$

where the cutoff structure function f_D ($D=1,2,3$) is

$$f_1(\mu,k_m)=\tan^{-1}(k_m/\mu)/(\pi\mu),$$

$$f_2(\mu,k_m)=\ln(1+k_m^2/\mu^2)/(4\pi),$$

$$f_3(\mu,k_m)=[k_m-\mu\ \tan^{-1}(k_m/\mu)]/(2\pi^2). \tag{9}$$

This result shows the difference caused by the space dimensionality. In one dimension, $f_1(\mu,k_m)$ approaches a constant value at large k_m, while in two and three dimensions $f_D(\mu,k_m)$ has a logarithmic or linear divergence, respectively. The effect of this divergence depends on whether or not the additional quartic interaction term is present. If it is

present (with $\kappa_D>0$), there are exact solutions without cutoff, having the property that $g(\mathbf{r})=0$ if $\mathbf{r}\neq 0$, and $g(0)=-\chi_D/(2\kappa_D)$. In these cases, the energy eigenvalue E takes the form of E_l, Eq. (4). In other words, the solutions in two and three dimensions have a finite energy but zero radius in the limit of $k_m\to\infty$.

In the absence of the quartic term (or if κ_D is negative, as in the case of the nonlinear Schrödinger model), we must impose a cutoff on the couplings to prevent an energy divergence. With a finite cutoff, the general result for the energy eigenvalue E is given by Eqs. (7), (8), and (9). Analysis of Eq. (7) with respect to μ shows that a positive solution is available, provided Eq. (3) is satisfied, in one and two dimensions. In the three-dimensional case it is necessary that $\hbar[\chi_3]^2>2\Delta[\kappa_3+2\pi^2\hbar/(m_1 k_m)]$. This shows that a momentum cutoff can provide both finite energy and finite radius, even without the stabilizing quartic term.

The classical version of the purely parametric quantum field theory is well behaved and in wide use as an approximate description of nonlinear optical interactions in parametric nonlinear crystals (sub-second-harmonic generation) [5]. Thus, we have an unusual situation, where the quantum counterpart of a well-defined classical theory requires a momentum cutoff in the nonlinearities before it leads to nonsingular structures. However, an investigation into the origins of the theory shows that paraxial and finite bandwidth approximations are needed to reduce the full nonlinear Maxwell equation Hamiltonian to the simpler form treated here. The paraxial approximation is only valid for $k_\perp\ll 2\pi/\lambda$. Thus a momentum cutoff of at most $k_m\sim 2\pi/\lambda$, where λ is the carrier wavelength of the subharmonic field, is required in order to use these approximations. A similar procedure was employed by Bethe, in using an estimated cutoff of $k_m=m_e c/\hbar$ in the first Lamb-shift calculation [10]. Just as in the Lamb shift, this can be improved by more careful treatment of the theory at large momenta.

After imposing the cutoff at $k_m=2\pi/\lambda$, we can now calculate the radius r_0 (which we define as $r_0=1/\mu$) and the resulting binding energy $E_b=\hbar^2\mu^2/m_1$ of the solution. We note, however, that our results should be slightly modified, before applying them to the case of optical parametric interaction for quantitative estimates. In this case Φ_i in Eq. (1) represent two optical fields with carrier frequencies ω_i, and the $\mathbf{x}$ coordinate is defined in a moving frame with $\mathbf{x}=\mathbf{x}_L-\mathbf{v}t$. Here $\mathbf{x}_L$ is the laboratory-frame coordinate and $\mathbf{v}$ is the group velocity that is assumed equal at both frequencies. The effective Hamiltonian describing this nonlinear optical process in more than one spatial dimension, in the presence of diffraction and dispersion effects, is asymmetric with respect to the longitudinal and transverse coordinates [5,6].

To represent this we should rewrite the kinetic-energy terms in the Hamiltonian (1) as

$$H_K=\int d^{(D)}\mathbf{x}\sum_{i=1,2}\frac{\hbar^2}{2}\left[\frac{|\nabla_\parallel\Phi_i|^2}{m_{i\parallel}}+\frac{|\nabla_\perp\Phi_i|^2}{m_{i\perp}}\right]. \qquad (10)$$

Here $m_{i\parallel}=\hbar/\omega_i''$ represents the longitudinal (dispersive) mass, where $\omega_i''=\partial^2\omega_i/\partial k^2$ is the dispersion in the ith frequency band, while $m_{i\perp}=\hbar\omega_i/|\mathbf{v}|^2$ is the transverse (diffractive) mass. Consequently, the relation K^2/m_i appearing in the subsequent equations must be replaced by $K^2/m_i\to K_\parallel^2/m_{i\parallel}+K_\perp^2/m_{i\perp}$, while the mass m_i appearing in terms that are independent of K^2 must be replaced by $m_{i\perp}$. Also, k^2 and μ^2 in Eq. (7) are now defined as $k^2\equiv\varepsilon^2 k_\parallel^2+k_\perp^2$ and $\mu^2\equiv(\varepsilon^2 K_\parallel^2+K_\perp^2)/4-Em_{1\perp}/\hbar^2$, where $\varepsilon\equiv(m_{1\perp}/m_{1\parallel})^{1/2}$.

As a result of these modifications, the cutoff structure functions $f_{2,3}(\mu,k_m)$ become dependent on ε. In the limits $\varepsilon\ll 1$ and $\varepsilon k_m\gg\mu$ the approximate expressions for $f_{2,3}(\mu,k_m)$ are $f_2(\mu,k_m,\varepsilon)\simeq\ln(2\varepsilon k_m/\mu)/(2\pi\varepsilon)$ and $f_3(\mu,k_m,\varepsilon)\simeq k_m(1-\ln\varepsilon)/(2\pi^2)$. In the case $\kappa_D=0$ and $\Delta=0$ ($\mathbf{K}=\mathbf{0}$, $\rho=\omega_2-2\omega_1=0$) this leads to the following simple result for the soliton binding energy (in the laboratory frame) in two and three dimensions:

$$E_b^{[2]}=\frac{\hbar^2\mu^2}{m_{1\perp}}\simeq\frac{m_{1\perp}[\chi_2]^2}{4\pi\varepsilon}\ln(2\varepsilon k_m/\mu),$$

$$E_b^{[3]}=\frac{\hbar^2\mu^2}{m_{1\perp}}\simeq\frac{m_{1\perp}[\chi_3]^2}{4\pi^2}k_m(1-\ln\varepsilon). \qquad (11)$$

To give numerical estimates we choose $n=3$, $\omega_1''=0.1\ \mathrm{m^2/s}$, $\lambda=2\mu\ \mathrm{m}$, and the nonlinearity $\chi_B^{(2)}=10^{-7}\ \mathrm{m/V}$, typical of highly nonlinear parametric media (e.g., GaAs asymmetric quantum wells and related systems [11]). With a characteristic waveguide diameter of 5 μm, this gives $\varepsilon\simeq 0.097$ and $\chi_1\simeq 7.39\times 10^7\sqrt{\mathrm{m}}/\mathrm{s}$, $\chi_2\simeq 1.65\times 10^5\ \mathrm{m/s}$, $\chi_3\simeq 369.5\ \mathrm{m^{3/2}/s}$, in one, two, and three space dimensions, respectively.

With a reasonable choice of the cutoff at $k_m=2\pi/\lambda=\pi(\mu\mathrm{m})^{-1}$, the resulting solutions in two and three dimensions have binding energies ($E_b^{[2]}\simeq 4.43\times 10^{-6}$ eV, $E_b^{[3]}\simeq 2.25\times 10^{-6}$ eV, for $\kappa_{2,3}=0$) and radii ($r_0^{[2]}\simeq 39.7\ \mu$m, $r_0^{[3]}\simeq 55.6\ \mu$m) comparable to the known results [8] for a one-dimensional waveguide ($E_b^{[1]}\simeq 1.75\times 10^{-5}$ eV, $r_0^{[1]}\simeq 1.94\ \mu$m). In fact, we find that $r_0^{[1]}<r_0^{[2]}<r_0^{[3]}$ and $E_b^{[1]}>E_b^{[2]}>E_b^{[3]}$, i.e., the higher-dimensional solitons are less strongly bound and of larger radius than those in one dimension.

In summary, we have presented bound-state solutions or quantum solitons to a parametric quantum field theory in more than one spatial dimension. The results have the remarkable character that in the presence of the positive quartic term the solutions correspond to quantum pointlike structures with finite energy. Imposing a momentum cutoff on the nonlinear couplings leads to finite radii of the solutions, and finite binding energies, even if the stabilizing quartic term is absent. It should not be impossible to resolve these binding energies—either by using cryogenic means or else by means of transient experiments on time and length scales shorter than those of competing thermal Raman processes [12] and absorption processes in nonlinear optical media.

The physical interpretation of these bound states is that they are a superposition of a second-harmonic and two subharmonic photons, which can propagate without either down-conversion of the higher-frequency photon or dispersive spreading of the subharmonic photons. In practical terms, of course, most photon pairs created by down-conversion are in unbound (continuum) states, which are not treated in detail here. The possibility of creating bound states that are immune to further down-conversion does not seem

to have been treated in earlier theories of this process, although nonclassical spatial oscillations were predicted in earlier work [13].

Most significantly, the solitons form in nonlinear crystals and planar waveguides accessible with currently available technology. It is therefore possible that this parametric quantum theory, as well as being theoretically interesting, could result in the first experimental test of multidimensional quantum soliton theory for Bose fields.

Even more promising physical systems that could be employed as a testing ground for our theory are Bose-Einstein condensates of atomic gases. Bose-Einstein condensate (BEC) experiments are progressing very rapidly, and recent achievements include formation of ultracold diatomic molecules through a Feshbach resonance or Raman photoassociation [14]. These hybrid atomic-molecular BEC systems can directly be treated within our theory, where the fields Φ_1 and Φ_2 would represent atomic and molecular species, respectively, and $m_{1,2}$ are the corresponding masses. The coupling constant χ_D would relate now to the molecular formation rate, while κ_D is the effective self-interaction of the atomic field. The simplest nontrivial objects in such systems, which can be described by our two-particle quantum soliton solutions, are ''dressed'' molecules, each of which exists in a linear superposition with a pair of atoms. With a χ_3-value estimate of about $\chi_3 \sim 10^{-6}$ m$^{3/2}$/sec [14], the atomic mass $m_1 \sim 10^{-25}$ kg and a choice of the cutoff at an inverse scattering length, $k_m \sim 1$ nm^{-1}, the corresponding binding energy $E_b^{[3]}$ and the radius $r_0^{[3]}$ would be $E_b^{[3]} \sim 1.6 \times 10^{-11}$ eV and $r_0^{[3]} \sim 21$ μm. Further details on the applications of our results to BEC systems will be presented elsewhere.

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