Infinity-free semiclassical evaluation of Casimir effects

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Electromagnetic Casimir energies are a quantum effect proportional to $\hbar$. We show that in certain cases one can obtain an exact semiclassical expression for them that depends only on periodic orbits of the associated classical problem. A great merit of the approach is that infinities never appear if one considers only periodic orbits that make contact with the boundary surface. This notion is made more precise by classifying the closed orbits in a phase space with boundaries and identifying the classes that contribute to Casimir effects. A semiclassical evaluation of the path integral gives a systematic expansion of the Casimir energy in terms of the lengths of classical periodic orbits. For some simple geometries the semiclassical expansion can be summed and explicitly shown to reproduce known results. This is the case, for example, for the force per unit area between parallel plates at a separation small compared to their linear dimensions. A more interesting example for our purposes is the closely related problem of the force on a conducting sphere arbitrarily close to a conducting wall. We provide a rigorous proof of Derjaguin's result for the leading contribution to the force. The semiclassical approach, which has never been truly exploited in Casimir studies, is relatively simple and transparent, and should have a wide range of applications. The methods presented, however, do not apply to cases where diffraction is important; diffraction can, in principle, also be described within this semiclassical approach, but its implementation presents some technical problems. In cases where diffraction is important, conventional methods of calculating the Casimir energy may often be simpler. [S1050-2947(98)10305-0]

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I. INTRODUCTION

An exact expression for a Casimir effect in terms of an integral or infinite sum can normally be given only under very restricted conditions and simple geometries. Since the effects originate, in one picture, in fluctuations of the electromagnetic field, it can be difficult to devise a perturbative scheme when the exact form cannot be found. The Casimir energies to leading order in $\hbar$ do not depend on $e^2/\hbar c$, and often not on any other ''smallness parameter.'' They are in this case of a geometrical nature and depend on the imposed boundary conditions and the topology. The semiclassical treatment we will present here, however, for certain cases leads to a systematic expansion of the Casimir energy in terms of the lengths of primitive periodic classical orbits. This allows one to compute the Casimir energy exactly in some cases, and often to high accuracy when the geometry is complicated and the exact form cannot be obtained.

It will be useful and illustrative to begin by considering a classic Casimir problem, the determination of the energy $\mathcal{E}_{\rm Cas}(\varepsilon,l)$ between two uncharged ideal parallel plates of area $\mathcal{A}=l_1l_2$ that are separated by a distance $l\ll l_1,l_2$, with a dielectric of uniform permittivity $\varepsilon(\omega^2)$ between them. It is known [1,2] that

$$\mathcal{E}_{\rm Cas}(\varepsilon,l)=\frac{\mathcal{A}\hbar}{2\pi^2}\int_0^\infty d\xi\frac{\xi^2}{v^2(-\xi^2)}\int_1^\infty dq \times q\ln[1-e^{-2l\xi q/v(-\xi^2)}], \tag{1.1}$$

where $v(\omega^2)=c/\sqrt{\varepsilon(\omega^2)}$ is the phase velocity in the dielectric. Via a contour integration, an integration over the frequency ω has been replaced by an integration over $\xi=-i\omega$ in Eq. (1.1). For a vacuum between the plates, that is, $\varepsilon=1$, the variable q in Eq. (1.1) can be identified as the secant of the angle between the direction of a virtual photon incident on a wall and the normal to the wall. (The more general Casimir energy for three uniform dielectric slabs can also be determined in integral form [1]. Letting the permittivities of the outer slabs tend to infinity gives the case under consideration.)

The physical picture is greatly simplified if one recasts the two-dimensional integral in Eq. (1.1) as a one-dimensional integral, as was only very recently recognized as being possible [3]. The result is

$$\mathcal{E}_{\rm Cas}(\varepsilon,l)=\frac{\mathcal{A}\hbar}{4\pi^2}\int_0^\infty d\xi\;\xi\left(\frac{d}{d\xi}\frac{\xi^2}{v^2(-\xi^2)}\right)\times\ln[1-e^{-2l\xi/v(-\xi^2)}]. \tag{1.2}$$

If we represent the logarithm in Eq. (1.2) by the infinite sum

$$\ln[1-e^{-2l\xi/v}]=-\sum_{n=1}^\infty (1/n)e^{-2nl\xi/v}, \tag{1.3}$$

the expanded form of Eq. (1.2) virtually begs for an interpretation as virtual photons traveling perpendicular to the walls in periodic orbits; the length of a path with $2n$ reflections is $2nl$, and, roughly speaking, the wave number k for given ξ is $k=i\xi/v(-\xi^2)$ and the frequency-dependent period is $2nl/v(-\xi^2)$. We also see from Eq. (1.3) that the contribution to the Casimir energy from periodic paths decreases

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with their length. [In the final result for the Casimir energy, the contribution of a particular path is inversely proportional to the third power of its length. The interpretation is unambiguous for $\varepsilon=1$, but the path-length picture will turn out to be valid also for frequency-dependent $\varepsilon=\varepsilon(-\xi^2)$.]

The semiclassical approximation will be seen to reproduce the term of order $\hbar$ of the Casimir energy for a large class of geometries. For metallic objects in a vacuum, this is often the whole effect. Corrections from interactions of the order of the electromagnetic coupling $\alpha\sim 1/137$ of higher order in $\hbar$, which can in principle be computed perturbatively, will not be considered. We wish to stress, however, that the semiclassical approach in many cases gives an expansion of the leading contribution, of order $\hbar$, in terms of the lengths of periodic classical orbits that contribute. This is a purely geometrical expansion of the leading term of the Casimir energy, which does not require the existence of an intrinsically small parameter in the problem and that allows us to compute formally exact (to order $\hbar$) answers also in geometries for which an expansion in terms of eigenmodes of the electromagnetic field appears unmanagable (see Sec. III). Other simple applications of this approach for which exact solutions are not easily derived may also be envisioned, such as for the case of a dielectric with a permittivity $\varepsilon(\omega,\vec{x})$ that depends on the location as well as the frequency—the case of slabs of different permittivity being a special case [4].

For $\varepsilon=1$, the Casimir energy density U has been obtained for a rectangular parallelopiped with ideal walls of arbitrary dimensions l_1, l_2, and l_3 [5]. It is given by

$$U(l_1,l_2,l_3)=-\frac{\hbar c}{16\pi^2}\left[\sum{}' \frac{1}{(n_1^2l_1^2+n_2^2l_2^2+n_3^2l_3^2)^2} - \frac{\pi^3}{3l_1l_2l_3}\left(\frac{1}{l_1}+\frac{1}{l_2}+\frac{1}{l_3}\right)\right], \tag{1.4}$$

where Σ' denotes the sum over all triplets of integers, positive, negative, and zero, other than $(n_1,n_2,n_3)=(0,0,0)$. Note the presence in Eq. (1.4) of the lengths

$$L(n_1,n_2,n_3)=2(n_1^2l_1^2+n_2^2l_2^2+n_3^2l_3^2)^{1/2} \tag{1.5}$$

of classical periodic paths in the box with $2n_1$, $2n_2$, and $2n_3$ reflections off the three pairs of parallel walls. Note too that, as in all cases where there is only a free electromagnetic field, the Casimir energy density (1.4) is strictly proportional to $\hbar$. We will see that Eq. (1.4) is *not* reproduced by the semiclassical approximation. If one of the dimensions of the box is much larger than the others, the second term in Eq. (1.4) is negligible, and we will see that the whole effect *can* be described semiclassically. If, for example, $l_3 \gg l_1$ and l_2, Eq. (1.4) reduces to [6]

$$U(l_3\gg l_1,l_2)=-\frac{\hbar c}{\pi^2}\sum{}' [L(n_1,n_2)]^{-4}, \tag{1.6}$$

where Σ' in this case is the sum over all pairs of integers (n_1,n_2) other than (0,0) and

$$L(n_1,n_2)=2(n_1^2l_1^2+n_2^2l_2^2)^{1/2}. \tag{1.7}$$

Since the only lengths that appear in Eq. (1.6) are those of classical periodic paths (in the plane perpendicular to the axis associated with l_3), one strongly suspects that these classical paths will be of importance in a semiclassical calculation. If, further, $l_2 \gg l_1$, the square bracket in Eq. (1.4) is reduced to the sum over all $n_1>0$ of $2/(l_1^4n_1^4)$, and one readily obtains the standard Casimir energy per unit area for two nearby plates in a vacuum [7],

$$\mathcal{E}_{\text{Cas}}/(l_2l_3)=-\frac{\pi^2}{720}\frac{\hbar c}{l_1^3}. \tag{1.8}$$

A more interesting example from the semiclassical point of view is that of a metallic sphere of radius R and a metallic wall a distance $l \ll R$ from the nearest point on the sphere embedded in a vacuum. Making very reasonable assumptions, the leading contribution to the force F between the sphere and the wall for $l \ll R$ has been determined theoretically [8] using the fact that the force per unit area between walls, the derivative with respect to the separation of $\mathcal{E}_{\text{Cas}}$, is known. (A very recent measurement [9] of F was in good agreement with the theoretical result.) A determination of this force from the electromagnetic eigenmodes has, however, never been accomplished. For $R \gg l$, the only classical periodic orbits of finite length are traversals, arbitrary in number, between the two nearest points of the wall and sphere. Diffraction effects in the semiclassical approach are represented by periodic orbits that pass around the sphere; these have lengths that are of the order of $2l$ plus multiples of R and can be neglected for $R/l\sim\infty$. We will show in Sec. III that the contribution from periodic orbits in the semiclassical approach reproduce the leading expression for the force. F has also been calculated for $l \gg R$; the dipole approximation for the interaction of the sphere with the fluctuating electric field is then valid. We will see that semiclassically the contribution from classical periodic paths that "creep" [10] around the sphere can no longer be neglected in this limit. Although we do not obtain the solution in this limit, there are reasons to believe that the semiclassical approach could describe the force F for arbitrary values of l/R. The exact semiclassical description of the case $R \ll l$ is, however, easily seen to be far more cumbersome than the the conventional one. Thus, the semiclassical method of calculating Casimir energies we propose below, while formally exact in certain cases, is generally superior to conventional approaches in applications only when relatively few and sufficiently simple classical periodic orbits are relevant. However, this includes the important application to complicated geometries for which a systematic *approximation* to the Casimir energy is sought, and where the shortest classical periodic orbits can be found (at least numerically).

The possibility of evaluating Casimir energies by a semiclassical approach (though not in terms of periodic orbits) has often been pointed out in particular cases. Thus, for example, the Casimir-Polder interaction between atoms at a separation arbitrarily large compared to either of their dimensions [11] can be obtained *exactly* by proceeding classically and, in the very last step, replacing the volume integral of the square of a component of the electric field of frequency ω by $(1)/8\pi(\hbar\omega/2)$ [12].

In one of the methods we will now consider, one finds that contributions to $\mathcal{E}_{\rm Cas}$ come only from fluctuations around periodic classical orbits that make contact with the boundary surface; infinities never appear in the semiclassical evaluation of the Casimir energy in this case. This is appealing, since in most Casimir studies one must evaluate differences between infinities, or throw away infinite energy contributions that do not depend on the boundary surfaces *after* having calculated formal expressions. In the semiclassical approach described below it will become clear that these infinities arise due to classical paths of arbitrarily short length that do not depend on relative variations of the boundary surfaces and therefore do not contribute to any forces. These contributions can therefore be isolated and ignored from the outset and the semiclassical evaluation of the Casimir energy is then finite at every stage.

Although the expression (1.2) for the Casimir energy in the case of two parallel conductors can be directly compared to the result of the semiclassical determination of this energy in the next section, we will derive an expression for the Casimir energy associated with two spheres in Sec. III that does not depend on the derivative $\partial/\partial\xi(\xi/v)$. It is therefore perhaps illustrative that Eq. (1.2) can also be cast in such a form after a few elementary manipulations, a point missed in [3]. To this end we combine Eqs. (1.2) and (1.3), use $\partial/\partial\xi(\xi/v)^2=2(\xi/v)\partial/\partial\xi(\xi/v)$ and

$$2e^{-2nl\xi/v}\frac{\xi}{v}\frac{\partial}{\partial\xi}\left(\frac{\xi}{v}\right)=\frac{\partial}{\partial l}\frac{\partial}{\partial\xi}\frac{e^{-2nl\xi/v}}{2n^2l}, \tag{1.9}$$

and integrate by parts. The Casimir energy (1.2) for two parallel conducting plates can then also be expressed as

$$\mathcal{E}_{\rm Cas}(\varepsilon,l)=\frac{\mathcal{A}\hbar}{8\pi^2}\frac{\partial}{\partial l}\sum_{n=1}^{\infty}\frac{1}{n^3l}\int_0^\infty d\xi\; e^{-2nl\xi/v(-\xi^2)}=-\frac{\mathcal{A}\hbar}{4\pi^2l}\sum_{n=1}^{\infty}\frac{1}{n^2}\int_0^\infty d\xi\left[\frac{\xi}{v(-\xi^2)}+\frac{1}{2nl}\right]e^{-2nl\xi/v(-\xi^2)}, \tag{1.10}$$

a derivative-free form. For numerical estimates of the Casimir energy (1.10) is eminently better suited than Eq. (1.2), since it suffices in Eq. (1.10) to approximate the monotonically increasing function $\xi/v(-\xi^2)$. The sum converges rapidly and poses no numerical complication.

II. GENERAL FORMALISM AND THE CASE OF INFINITE PARALLEL PLATES

Mainly for pedagogical reasons, we will first recalculate in an unconventional fashion the Casimir energy per unit area for two ideal parallel plates, one at $z=0$ and one at $z=l$, with a uniform medium between them and l very much smaller than the linear dimensions of the plates. We begin with a discussion of the relevant formalism in a broader context.

A. Some general formalism

We determine the Casimir energy in terms of the difference $\rho_{\rm osc}(E)$ between the spectral density $\rho(E,l)$ for a photon in a uniform medium with boundary conditions and the spectral density $\rho_0(E)$ in the uniform medium without boundaries,

$$\rho_{\rm osc}(E,l)=\rho(E,l)-\rho_0(E). \tag{2.1}$$

The parameter l here symbolically represents a collection of shape parameters defining the boundaries. (In the case of two infinite parallel plates, it will simply be the separation l between them if the dimensions of the plates are fixed.) Since the total Casimir energy is just the difference of the zero-point energies with and without boundaries, we can obtain it from the change in the spectral density

$$\mathcal{E}_{\rm Cas}(\varepsilon,l)=\int_0^\infty dE\;\left(\tfrac{1}{2}E\right)\rho_{\rm osc}(E,l). \tag{2.2}$$

In Eq. (2.2) $\frac{1}{2}E$ is the zero-point energy of the oscillator associated with a real photon of energy E and Eq. (2.2) expresses the usual sum over zero-point energies as an integral, since the spectral density $\rho(E)=\sum_n\delta(E-\hbar\omega_n)$ gives the distribution of eigenfrequencies of the harmonic oscillators.

In the following we will exploit the fact that the change $\rho_{\rm osc}(E,l)$ in the spectral density due to a change in the boundaries is related to the change in the imaginary part of the response function $g_{\rm osc}$,

$$\rho_{\rm osc}(E,l)=-\frac{1}{\pi}{\rm Im}\;g_{\rm osc}(E,l), \tag{2.3}$$

and that the response function is the trace of the energy Green function

$$G^{\lambda\lambda'}(\mathbf{x},\mathbf{y};E)=\langle\mathbf{y},\lambda'|(E+i\varepsilon-\hat{H})^{-1}|\mathbf{x},\lambda\rangle, \tag{2.4}$$

where $\hat{H}$ is the Hamiltonian and λ and λ' denote states of polarization. In our case

$$g_{\rm osc}(E;l)=\sum_\lambda\int d\mathbf{x}\lim_{\mathbf{y}\to\mathbf{x}}G^{\lambda\lambda}_{\rm osc}(\mathbf{x},\mathbf{y};E,l), \tag{2.5}$$

where the sum is over the two polarizations of a photon and the spatial integral extends over all the space accessible to it. In the applications we will study, the medium is not optically active and the boundary conditions are such that the Green functions are diagonal and independent of the polarization, that is,

$$G_{\rm osc}^{\lambda\lambda'}(\mathbf{x},\mathbf{y};E,l)=\delta^{\lambda\lambda'}G_{\rm osc}(\mathbf{x},\mathbf{y};E,l). \tag{2.6}$$

The sum over polarizations in the response function (2.5) will thus effectively just give a factor of 2 in this case. The ''oscillatory part'' $G_{\rm osc}$ of the Green function is the difference

$$G_{\rm osc}(\mathbf{x},\mathbf{y};E)=G(\mathbf{x},\mathbf{y};E,l)-G_0(\mathbf{x},\mathbf{y};E), \tag{2.7}$$

between the Green function $G(\mathbf{x},\mathbf{y};E,l)$ in the medium satisfying the boundary conditions and the one in the medium without boundaries. Note that the Green functions G_0 and G are singular in the limit $\mathbf{x}\to\mathbf{y}$, but that this short-range singularity cancels in the difference $G_{\rm osc}$. The limit $\mathbf{x}\to\mathbf{y}$ in the definition (2.5) of $g_{\rm osc}(E,l)$ is therefore generally well defined.

Due to causality, both $G(\mathbf{x},\mathbf{y};E,l)$ and $G_0(\mathbf{x},\mathbf{y};E)$ should be analytic in the first quadrant of the complex E plane (a pole in the first quadrant of the complex E plane would imply the existence of a state whose amplitude *grows* with time). Instead of integrating the imaginary part of $g_{\rm osc}$ along the real axis of E in Eq. (2.2), we may therefore alternatively integrate along the imaginary axis from $E=0$ to $E=i\infty$ and a large quarter circle of radius Ω, $E=\Omega e^{i\phi},\phi\in[\pi/2,0]$. If the integration over the large quarter-circle does not depend on the boundaries, we obtain an alternative expression for the Casimir energy,

$$\mathcal{E}_{\rm Cas}(\varepsilon,l)=-\frac{1}{2\pi}{\rm Im}\int_0^\infty dE\ Eg_{\rm osc}(E,l)$$

$$=\frac{\hbar^2}{2\pi}\int_0^\infty d\xi\ \xi{\rm Im}\,g_{\rm osc}(i\hbar\xi,l), \tag{2.8}$$

in terms of the imaginary part of the response function on the positive imaginary energy axis, where we have expressed the energy as $E=i\hbar\xi$, where $\xi=-i\omega$, with ω the frequency. $g_{\rm osc}$ is given by Eqs. (2.5), (2.6), and (2.7).

The analytic continuation (2.8) of the expression for the Casimir energy will prove useful, since $g_{\rm osc}(E,l)$ is a highly oscillatory function on the real axis near any resonance energies of the medium, whereas it is a smooth function for purely imaginary values of the energy. Of course, the analytic continuation (2.8) of the Casimir energy (2.2) is valid only if the contribution from the quartercircle does not depend on the boundaries, a point that must be checked in each case. We will see that the semiclassical evaluation of the expression (2.8) for the Casimir energy can be directly compared with Eq. (1.2) for the interesting case of a dielectric between two plates.

This way of formulating the Casimir energy exhibits its intimate relation to (the imaginary part of) the Green function. Due to the spatial integration in Eq. (2.3) and the integral over the spectrum in Eq. (2.2), the Casimir energy, however, contains much less information than the Green function and this approach might appear unnecessarily complicated. [Many standard calculations of the Casimir energy involve the determination of the spectrum (and indirectly also the determination of the eigenfunctions), and thus in principle amount to a determination of the (exact) Green function.] We here wish to emphasize that an exact determination of the spectrum (and associated eigenfunctions) is, however, not required. As we show in Appendix A, the Casimir energy of the system can always be related to the coefficient of the term of order $(\hbar c)/(El)$ in the semiclassical expansion of $g_{\rm osc}$, where l is a typical length of the problem. Since this coefficient is not easily found for most situations, the application of this approach is somewhat limited. We here use a different method to extract *exact* Casimir energies from the lowest-order semiclassical approximation to $g_{\rm osc}$ in certain limiting cases.

The method we propose can be used to extract the *leading divergence* of Casimir energies when some of the characteristic lengths on which it depends are taken to be much larger than the other relevant characteristic lengths. It therefore is applicable only in limiting situations where this is actually the case. Important exceptions where this method *does not* give the correct result include the case of a spherical cavity of radius R, where R is the only length (and the energy is proportional to $1/R$). Further, as noted in Sec. I, the semiclassical estimate of $\mathcal{E}_{\rm Cas}$ is not correct if the lengths of a rectangular parallelopiped are all comparable, but does give the *leading* term if the lengths are not comparable. It also gives the leading term for a sphere of radius R at a distance $l\ll R$ from a plane, as will be shown below.

For simplicity let us consider a general cavity in a vacuum. The Casimir energy (2.8) then depends only on a set of lengths $\{\bar{l},l_i\}$, where, without loss of generality, we single out the largest of these and denote it by $\bar{l}$ in the following. The dependence of the Casimir energy (2.8) on dimensionless quantities is

$$\mathcal{E}_{\rm Cas}(\{r_i\};\bar{l})=\frac{\hbar c}{2\pi\bar{l}}\int_0^\infty dx\hat{\rho}(x;\{r_i\}), \tag{2.9}$$

where the constants $r_i=l_i/\bar{l}\leqslant 1$ are dimensionless ratios of the lengths in the problem, and

$$\hat{\rho}(x;\{r_i\})\equiv\hbar\xi\ {\rm Im}\,g_{\rm osc}(i\hbar\xi,\bar{l},\{l_i\})|_{\xi\bar{l}/c=x} \tag{2.10}$$

is a dimensionless function that depends on the energy E only through the ratio $x=(-iE\bar{l})/(\hbar c)=\xi\bar{l}/c$.

If the Casimir energy of the system *diverges* in the limit $\bar{l}\to\infty$ for *any* fixed values of the other lengths in the problem, the integral in Eq. (2.9) diverges as all of the $r_i\to 0$. This can be due to either of two reasons: either the integrand becomes singular within some finite region $x<x_0(\{r_i\})<\infty$ for $r_i\sim 0$, or the integral is divergent in the limit $r_i\to 0$ due to the behavior of the integrand for $x\sim\infty$. We can exclude the first of these possibilities on physical grounds. Thus, let E_0 be defined by $x_0(0)=E_0\bar{l}/\hbar c$. If $x_0(0)$ were finite, the response function $g_{\rm osc}$ would have to become singular for small energies $E\leqslant E_0\sim 0$ as $\bar{l}$ becomes large. The behavior of the oscillating part of the response function of a cavity for small energies can, however, never be more singular than that of the response function without boundaries (which vanishes for small energies). The divergence of the integral in Eq. (2.9) for $\bar{l}\to\infty$ is therefore due to the behavior of the integrand at large values of x. The asymptotic behavior of $\hat{\rho}(x,\{r_i\})$ for large values of $x=(-iE\bar{l})/(\hbar c)$ therefore suffices to extract the *leading* divergence of the integral in the limit $\overline{l}\to\infty$. Since $x=(-iE\overline{l})/(\hbar c)$, this asymptotic behavior of the integrand is given by the leading terms in the semiclassical expansion of $g_{\rm osc}$. The *leading divergence* of the Casimir energy as one of the lengths becomes much larger than all the others is therefore determined by the semiclassical approximation to $g_{\rm osc}$. We will verify in several cases that the semiclassical approximation to $g_{\rm osc}$ does indeed reproduce the *exact* Casimir energy in this limit.

Our basic task now is to apply developments in semiclassical periodic orbit theory to the photons of present interest. An essential element of any semiclassical calculation is the action S of (all) classical paths. A great simplification in the present case and in a number of other semiclassical evaluations is due to the simplicity of the action for classical paths of constant energy. For a massless particle such as a photon, the classical path with constant energy E from a point $\mathbf{x}$ to a point $\mathbf{y}$ extremizes the length $L(\mathbf{x},\mathbf{y})$ of the path (these are *not always* piecewise straight paths, if the motion is constrained, see Sec. III) and the momentum $\mathbf{p}$ is tangent to the path at every point. The wave equation for a medium characterized by an index of refraction $n(\omega^2)$ gives the dispersion relation $\omega^2 n^2(\omega^2)/c^2=\mathbf{k}^2$, which we interpret as $p=n(E)E/c=E/v(E)$, where p is the momentum and $v\equiv v(E)$ is the phase velocity. (For later notational convenience, we have used v rather than say v_p to denote the phase velocity. Since the only other velocity of interest, the group velocity, will be denoted by v_g, there should be no ambiguity.) The classical action S is then

$$S(E,\mathbf{x},\mathbf{y})=\int_{\mathbf{x}}^{\mathbf{y}}\mathbf{p}\cdot d\mathbf{q}=pL(\mathbf{x},\mathbf{y})=EL(\mathbf{x},\mathbf{y})/v. \quad (2.11)$$

The action S of Eq. (2.11) completely determines the semiclassical approximation to the Green function $G(\mathbf{x},\mathbf{y};E,l)$ and thus also the semiclassical expression for $\mathcal{E}_{\rm Cas}$.

The semiclassical approximation to $G(\mathbf{x},\mathbf{y};E,l)$ is [13]

$$G(\mathbf{x},\mathbf{y};E,l)\sim-\frac{1}{2\pi\hbar^2}\sum_\gamma D_\gamma e^{i(S_\gamma/\hbar-\mu_\gamma\pi/2)}, \quad (2.12)$$

where the sum extends over all classical trajectories γ which begin at $\mathbf{x}$ and end at $\mathbf{y}$. The Maslov index $\mu_\gamma=n_t+2n_r$ is given by the number of turning points n_t and the number of reflections n_r along the classical path, and

$$D_\gamma=\mathrm{Det}^{1/2}\left(\begin{array}{c|c}\dfrac{\partial^2 S_\gamma}{\partial\mathbf{x}\partial\mathbf{y}} & \dfrac{\partial^2 S_\gamma}{\partial\mathbf{x}\partial E}\\ \hline \dfrac{\partial^2 S_\gamma}{\partial\mathbf{y}\partial E} & \dfrac{\partial^2 S_\gamma}{\partial E^2}\end{array}\right) \quad (2.13)$$

is the (positive) amplitude resulting from the unconstrained integration over quadratic fluctuations around the classical path γ. The diagonal entries in Eq. (2.13) are 3×3 and 1×1 matrices; the off-diagonal entries are 3×1 and 1×3 matrices.

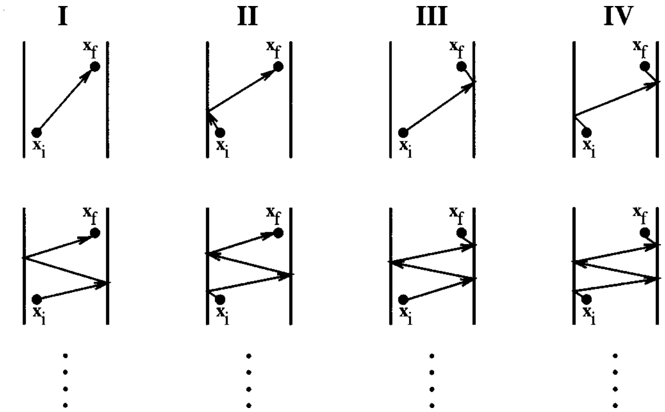

FIG. 1. Classical trajectories from $\mathbf{x}_i$ to $\mathbf{x}_f$ that contribute in the semiclassical approximation to the Green function between two parallel plates. The classical paths fall into four categories: I (even, left), II (odd,left), III (odd,right), and IV (even,right), where even and odd refer to the number of reflections, and right and left refer to the plate from which the last reflection took place. (The upper path in category I involves no reflections, but this path gives no contribution to the Casimir energy.) The shortest and next to shortest paths in each category are shown.

For the action (2.11), one finds that (see Appendix B) D_0 for the straight classical path between the initial and final points is

$$D_0=\frac{1}{2L}\frac{\partial}{\partial E}\left(\frac{E}{v}\right)^2=\frac{E}{Lvv_g}. \quad (2.14)$$

[In the special case where the medium is the vacuum, Eq. (2.14) simplifies to $D_0=E/(c^2L)$.] The semiclassical Green function $G_0(\mathbf{x},\mathbf{y};E)$ in the infinite medium without boundaries is thus

$$G_0(\mathbf{x},\mathbf{y};E)=-\frac{1}{4\pi\hbar^2L}e^{iEL/(\hbar v)}\frac{\partial}{\partial E}\left(\frac{E}{v}\right)^2. \quad (2.15)$$

The semiclassical approximation (2.15) does not reproduce the exact free Green function of a photon [14]; the two differ in their real parts. The semiclassical approximation to the imaginary part of G_0, which enters the calculation of the Casimir energy, is in fact *exact*.

B. Two parallel plates

We now determine in a similar fashion the semiclassical Green function for two parallel plates a distance l apart, and arbitrary initial and final points, $\mathbf{x}_i$ and $\mathbf{x}_f$, between the plates. The classical paths fall into four categories (I–IV); the shortest path for each category is indicated in the upper row of Fig. 1. In each category there are an infinite number of paths, involving more and more traversals of the space between the walls. The lower row in Fig. 1 represents the next shortest path in each category.

Due to angular momentum conservation, the classical paths are planar. With one plate located at $z=0$ and the other at $z=l$ their lengths are

$$L_n^{\rm I}=[(\mathbf{x}_{f\perp}-\mathbf{x}_{i\perp})^2+(2nl+x_{fz}-x_{iz})^2]^{1/2}, \quad n=0,1,2,\ldots,$$

$$L_n^{\mathrm{II}}=[(\mathbf{x}_{f\perp}-\mathbf{x}_{i\perp})^2+(2nl+x_{fz}+x_{iz})^2]^{1/2},\quad n=0,1,2,\ldots,$$

$$L_n^{\mathrm{III}}=[(\mathbf{x}_{f\perp}-\mathbf{x}_{i\perp})^2+(2nl-x_{fz}-x_{iz})^2]^{1/2},\quad n=1,2,3\ldots, \tag{2.16}$$

$$L_n^{\mathrm{IV}}=[(\mathbf{x}_{f\perp}-\mathbf{x}_{i\perp})^2+(2nl-x_{fz}+x_{iz})^2]^{1/2},\quad n=1,2,3\ldots.$$

All classical paths $\gamma=\gamma(J,n)$ between the plates can therefore be classified by their category $J\in\{\mathrm{I,II,III,IV}\}$, and an integer $n\geqslant 0$. L_0^{I} is the length of the direct path $\gamma(I,0)$ from $\mathbf{x}_i$ to $\mathbf{x}_f$. [$\gamma(I,0)$ is the only classical path between the initial and final points in an infinite medium without boundaries; it is of course reflectionless.]

In the semiclassical expression (2.12) for the Green function, we must also evaluate the determinant $D_n^J=D_{\gamma(J,n)}$ for each classical path. D_n^J is given by Eq. (2.13), with $S=EL/v$ replaced by $S_n^J=EL_n^J/v$, where L_n^J differs from $L=L_0^{\mathrm{I}}$ in that $x_{fz}-x_{iz}\to 2nl\pm x_{fz}\pm x_{iz}$. The determinant (2.13) for paths $\gamma(J,n)$ is thus given by D_0 with L replaced by L_n^J, that is,

$$D_n^J=\frac{1}{2L_n^J}\frac{\partial}{\partial E}\left(\frac{E}{v}\right)^2. \tag{2.17}$$

After subtracting the direct contribution, we can set $\mathbf{x}_i=\mathbf{x}_f\equiv\mathbf{x}$ in the semiclassical expression for the oscillatory part of the Green function without encountering any singularities and obtain

$$G_{\mathrm{osc}}(\mathbf{x}_f=\mathbf{x}_i;E,l)\equiv G_{\mathrm{osc}}(\mathbf{x};E,l)$$
$$=-\frac{1}{4\pi\hbar^2}\left(\frac{\partial}{\partial E}\frac{E^2}{v^2}\right)\times\left[2\sum_{n=1}^{\infty}\frac{e^{2inlE/\hbar v}}{2nl}-\sum_{n=-\infty}^{\infty}\frac{e^{2iE|z+nl|/\hbar v}}{2|z+nl|}\right]. \tag{2.18}$$

The first (z-independent) term in Eq. (2.18) is the sum of the (equal) contributions of classes I and IV, while the second (z-dependent) term is the sum of the contributions of classes II and III; we used the fact that for $\mathbf{x}_i=\mathbf{x}_f$, $L_n^{\mathrm{I}}=L_n^{\mathrm{IV}}=2nl$ and $L_n^{\mathrm{III}}=L_{-n}^{\mathrm{II}}$ for $n\neq 0$ [15]. The relative sign of the two terms arises from the Maslov index μ_J because paths in classes I and IV have an even number of reflections, whereas the number of reflections in categories II and III is odd. We now integrate $G_{\mathrm{osc}}(\mathbf{x};E,l)$ over the space between the plates and sum over polarizations to obtain the response function

$$g_{\mathrm{osc}}(E;l)=-\frac{\mathcal{A}}{2\pi\hbar^2}\left(\frac{\partial}{\partial E}\frac{E^2}{v^2}\right)\times\left[\sum_{n=1}^{\infty}\frac{e^{2inlE/\hbar v}}{n}-\int_0^{\infty}\frac{dz}{z}e^{2iEz/\hbar v}\right], \tag{2.19}$$

where $\mathcal{A}$ is the area of the plates. In the second term of Eq. (2.19), the sum over n was accounted for by a change in the range of integration. This term gives an infinite contribution to the response function, but does not depend upon the separation l and therefore does not contribute to any force between the plates. We will therefore drop this infinite term in the following. We note that this divergent but l-independent semiclassical contribution arises from paths of type II and III, which *do not* lead to classical periodic orbits as $\mathbf{x}_i\to\mathbf{x}_f$, the initial and final momenta being equal but opposite. The classical paths of type I and IV, which *are* periodic in the limit when initial and final points are the same, give the l-dependent first term of Eq. (2.19). (We will see in subsection D and also in Appendixes C and D that closed trajectories corresponding to paths of type II and III with initial and final points identical do not arise in a semiclassical evaluation of the path integral for the response function g_{osc}.)

The l-dependent terms of the sum in Eq. (2.19) fall off exponentially with the radius of the large quartercircle and the analytic continuation used to obtain Eq. (2.8) is therefore justified. Dropping the l-independent term in Eq. (2.19) and analytically continuing the remainder to the positive imaginary axis gives

$$\mathrm{Im}\, g_{\mathrm{osc}}(i\hbar\xi,l)=-\frac{\mathcal{A}}{2\pi\hbar}\left(\frac{\partial}{\partial\xi}\left[\frac{\xi}{v}\right]^2\right)\sum_{n=1}^{\infty}\frac{e^{-2nl\xi/v}}{n}$$
$$=\frac{\mathcal{A}}{2\pi\hbar}\left(\frac{\partial}{\partial\xi}\left[\frac{\xi}{v}\right]^2\right)\ln[1-e^{-2l\xi/v}]. \tag{2.20}$$

Inserting Eq. (2.20) in Eq. (2.8) and integrating over the frequencies, one arrives at Eq. (1.2). The Casimir energy of two parallel plates derived semiclassically is thus in fact exact for $l/L\sim 0$. The nature of the above derivation strongly suggests that the semiclassical approach should be widely applicable. As a small point, we note that the semiclassical derivation clearly depends only on the phase velocity v ($-\xi^2$) and therefore apparently applies equally well to homogeneous isotropic media with permeability $\mu\neq 1$.

The semiclassical calculation above, though interesting, is no less—and perhaps even more—complicated than other derivations of the same result. The semiclassical calculation apparently would, however, be considerably simplified if one could (i) legitimately ignore contributions to the Casimir energy from classical paths of type II and III from the outset, since they turned out to be independent of l and therefore are of no physical interest and (ii) avoid calculating the semiclassical Green function between arbitrary points, since the relevant contribution to the oscillatory part of the response function $g_{\mathrm{osc}}(E,l)$ of Eq. (2.5) arose only from periodic classical orbits. We will discuss the alternative approach in Sec. III D and in Appendixes C and D.

We now allow one or both of the walls to be infinitely permeable [16]. The use of periodic orbits renders the analysis transparent and rather trivial. The boundary condition, that the normal derivative of the vector potential, rather than the vector potential itself, vanishes at the surface leads to a reflection coefficient at the surface of a permeable wall of $+1$, rather than -1 as for a conductor. For the periodic path of length $2nl$ the reflection factor is $(\pm 1)^{2n}=1$ for two conductors or two permeable walls, but $(+1)^n(-1)^n=(-1)^n$ for one conductor and one permeable wall. For the latter case, with a dielectric between the walls, the contributions to the Casimir energy from the individual periodic paths therefore alternate and one can effectively insert a factor $(-1)^n$ in the sum (1.3). The minus sign in the argument of the logarithm in Eq. (1.3) and Eq. (1.2) is thereby replaced by a plus sign and the Casimir force between a conductor and a permeable wall is seen to be *repulsive*. In a vacuum, the replacement

$$\sum_1^\infty \frac{1}{n^4}=\frac{\pi^4}{90}\to\sum_1^\infty \frac{(-1)^n}{n^4}=-\frac{7}{8}\frac{\pi^4}{90} \tag{2.21}$$

leads to a repulsive force per unit area with a magnitude 7/8 that for two conducting plates. In the semiclassical picture, the origin for the change of sign and magnitude of the Casimir force between two conducting plates compared to the Casimir force between a conducting plate and a permeable one is due to destructive interference between semiclassical contributions from paths with different numbers of reflections.

C. A long rectangular cavity

The method used to obtain the Casimir energy for two large parallel plates also gives the Casimir energy of a rectangular cavity with dimension $l_1\times l_2\times l_3$ when $l_3\gg l_1,l_2$. Since the Casimir energy diverges as $l_3\to\infty$, the semiclassical approximation is *exact* in this limit. The periodic orbits that reflect $2n_1$ times off one set of parallel walls and $2n_2$ times off the other have lengths $L(n_1,n_2)$ given by Eq. (1.7). Contrary to the case of only two parallel plates, there are now four periodic trajectories of the same length $L(n_1,n_2)$ for any set of integers n_1 and n_2, which, in the limit $\mathbf{x}_i\to\mathbf{x}_f$, arise from classical trajectories that first reflect off any one of the four walls of the cavity. The contribution of the periodic orbits with length $L(n_1,n_2)$ to the semiclassical expression for the oscillating part, $G_{\rm osc}(\mathbf{x}_f=\mathbf{x}_i;E,l_1,l_2)$, of the Green function is otherwise obtained as was $G_{\rm osc}$ of Eq. (2.18) for the case of two parallel plates and found to be

$$-4\frac{1}{4\pi\hbar^2}\left(\frac{\partial}{\partial E}\frac{E^2}{v^2}\right)\frac{e^{iL(n_1,n_2)E/\hbar v}}{L(n_1,n_2)}. \tag{2.22}$$

The factor of 4 in front replaces the factor of 2 in Eq. (2.18) and accounts for the four periodic paths of equal length and the path length $2nl$ in Eq. (2.18) in the present case is replaced by $L(n_1,n_2)$. Note that the semiclassical contribution (2.22) of a periodic orbit to the Green function does not depend on the point $\mathbf{x}$ at which the trajectory starts and ends. The spatial integral in the definition (2.5) of the semiclassical response function $g_{\rm osc}$ therefore just gives the volume $l_1l_2l_3$ of the cavity. To obtain the semiclassical expression for the response function $g_{\rm osc}$ in this case, we, however, have still to determine the possible polarizations for each periodic orbit. There are in general two independent polarizations, but for rays parallel to one of the boundaries there is just one. Since the length $L(n_1,n_2)$ of a trajectory depends only on the squares n_1^2 and n_2^2, we can account for this degeneracy by summing with weight 1/2 over all integer n_1 and n_2, positive and negative and zero, except $n_1=n_2=0$. The oscillating part of the semiclassical response function of a long rectangular cavity thus becomes

$$g_{\rm osc}(l_3\gg l_1,l_2;E)=-l_1l_2l_3\left(\tfrac{1}{2}\sum_{n_1,n_2}{}'\right)\frac{1}{\pi\hbar^2}\left(\frac{\partial}{\partial E}\frac{E^2}{v^2}\right)\times\frac{e^{iL(n_1,n_2)E/\hbar v}}{L(n_1,n_2)}. \tag{2.23}$$

The analytic continuation to imaginary frequencies $E=i\xi\hbar$ in the integral (2.8) for the Casimir energy is possible and we obtain

$$\mathcal{E}_{\rm Cas}(l_3\gg l_1,l_2)/(l_1l_2l_3)=-\frac{\hbar}{4\pi^2}\sum_{n_1,n_2}{}'\int_0^\infty d\xi\,\xi\left(\frac{\partial}{\partial\xi}\frac{\xi^2}{v^2}\right)\times\frac{e^{-L(n_1,n_2)\xi/v}}{L(n_1,n_2)} \tag{2.24}$$

for the Casimir energy per unit volume of a long rectangular cavity. For a vacuum inside the cavity, the phase velocity $v(-\xi^2)=c$ does not depend on the frequency and the integration over ξ in Eq. (2.24) is readily performed. One verifies that the semiclassical evaluation (2.24) gives the exact Casimir energy per unit volume (1.6) of this system.

On the other hand, proceeding along similar semiclassical lines for a rectangular cavity of *finite* volume does not reproduce the correct expression (1.4). The discrepancy can be traced to the fact that the spectrum in a cavity of *finite* volume is discrete and that certain low-lying states contribute significantly to the Casimir energy individually. As noted above, the semiclassical approximation gives only the *leading* divergent behavior of the Casimir energy as one of the lengths in the problem becomes large compared to the others. This limit corresponds to considering the long rectangular cavity discussed above.

D. Gutzwiller's trace formula

Gutzwiller [17,18] first observed, in a much broader context than that of two plates, that performing the spatial integration in Eq. (2.5) by the saddle-point method to obtain the oscillating response function $g_{\rm osc}(E,l)$ is completely consistent with the semiclassical approximation for the Green function $G_{\rm osc}$. This observation generally leads to the desired simplification of the semiclassical calculation, since one can show [13,18] that the *periodic* classical orbits are the saddle points of this integral. In the case of unconstrained classical paths, the integration in Eq. (2.5) can then be performed explicitly by the method of stationary phase, giving the rather concise expression

$$g_{\rm osc}(E)=\frac{-i}{\hbar}\sum_\lambda\sum_{\gamma\in\{\rm periodic\}}A_\gamma^{\lambda\lambda}\exp\left[\frac{i}{\hbar}S_\gamma^{\lambda\lambda}-\frac{i\pi}{2}\mu_\gamma\right] \tag{2.25}$$

for the semiclassical response function [13,18]. The sum in Eq. (2.25) extends over smooth classical periodic orbits γ only. (If there are two possible *directions* of the classical motion, these are counted as *separate* classical periodic orbits—in our example in Sec. II B, these are classical orbits of type I and IV in the limit when $\mathbf{x}_i \to \mathbf{x}_f$.) In the following examples the polarization λ of the photon does not change upon reflection (because the angle of incidence is always 90°) on the classical periodic paths and we furthermore assume that the medium is not optically active. The action S_γ and the semiclassical amplitude A_γ of a periodic classical orbit in this case are independent of the polarization λ and the sum over polarizations in Eq. (2.25) just gives a factor of 2. To simplify the notation, we again drop polarization indices in the following. The semiclassical amplitude A_γ of a classical periodic orbit is [13]

$$A_\gamma = \int_0^{\tau_\gamma} dt |\mathrm{Det}[M_\gamma(t) - \mathbf{1}]|^{-1/2} / N_\gamma, \qquad (2.26)$$

where the integral over time extends over one period τ_γ of the classical motion on the energy surface. $M_\gamma(t)$ is the 4×4 monodromy matrix, which, for a given periodic path of period $\tau_\gamma = \tau$, relates the infinitesimal deviations perpendicular to the path at a time $t+\tau$, namely, $\delta x_\perp(t+\tau)$ and $\delta p_\perp(t+\tau)$, to the perpendicular deviations $\delta x_\perp(t)$ and $\delta p_\perp(t)$ at time t. Symbolically we thus have

$$M_\gamma(t) = \begin{pmatrix} \dfrac{\partial x_\perp(t+\tau)}{\partial x_\perp(t)} & \dfrac{\partial x_\perp(t+\tau)}{\partial p_\perp(t)} \\ \dfrac{\partial p_\perp(t+\tau)}{\partial x_\perp(t)} & \dfrac{\partial p_\perp(t+\tau)}{\partial p_\perp(t)} \end{pmatrix}_\gamma, \qquad (2.27)$$

where the entries in Eq. (2.27) are 2×2 matrices. The integer N_γ in Eq. (2.26) counts the number of times the classical path γ traverses the *same* generic volume element $d\mathbf{x}$ during one period. It is the ''degree of the map'' for the change of coordinates in the spatial integration to coordinates parallel and transverse to the classical path γ. (The *local* Jacobian for this change of variables is 1, i.e., $d\mathbf{x} = d^2x_\perp dx_{\|}$.)

The expression (2.26) for the amplitude A_γ obtained by a saddle-point approximation for the spatial integration in Eq. (2.5) diverges if the matrix $M_\gamma(t) - \mathbf{1}$ is singular for some classical periodic path γ. This is the case whenever an infinitesimal deviation $(\delta x_\perp, \delta p_\perp)$ in a particular (transverse) direction from the periodic classical path γ is reproduced after one period, i.e., if there is *another* classical periodic path γ' arbitrarily close to γ in phase space with the same action and energy. An infinitesimal time-independent canonical transformation thus relates the two paths. As shown in Appendix C, one can formally extract the ''volume'' of the group generated by this canonical transformation using a procedure analogous to gauge fixing—or, equivalently, introducing collective coordinates to select a representative path. It may be another matter to actually compute the group volume; this is usually possible only for relatively simple groups.

The semiclassical computation using Gutzwiller's trace formula of the Casimir energy between parallel plates is of this kind: Although only classical periodic orbits γ contribute to the final expressions (1.2) and (2.24), A_γ in Eq. (2.26) would diverge due to the invariance of the problem with respect to translations parallel to the plates. A deviation of the initial position transverse to any given classical periodic path γ reproduces itself after one period, since it corresponds to the initial condition for a new classical periodic path that is just a translation of the old one. In the case of two parallel plates of finite area, the ''volume'' of this translation group is the area $\mathcal{A}$ of the plates and we show in Appendix D that one may obtain the result (1.2) by considering periodic classical paths only.

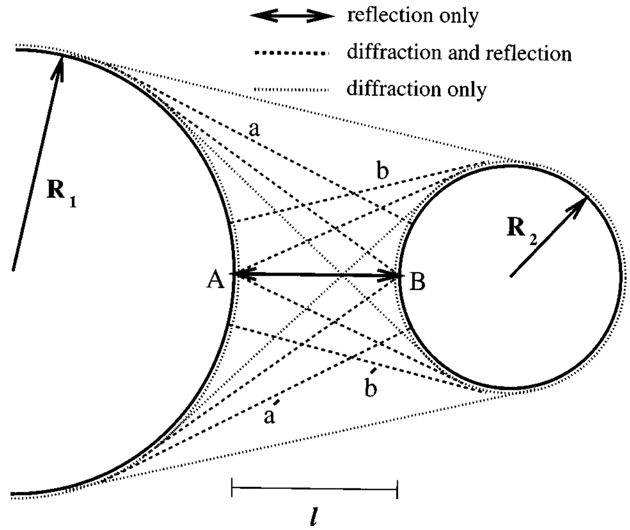

FIG. 2. Periodic classical trajectories for two spheres of radii R_1 and R_2 separated by a distance l. Only paths that do not wrap around either sphere, those which go from A to B to A or from B to A to B once or a number of times, contribute to the Casimir energy when $l \ll R_1$ and R_2. Paths that reflect off one sphere and wrap around the other and paths that wrap around both spheres are also shown. (The path segments denoted by a,a' and b,b' are perpendicular to the surfaces of spheres 2 and 1, respectively, and are retraced after reflection.) The contribution to the Casimir energy from such paths becomes important when the radius of either or both spheres is comparable to or smaller than their separation l, i.e., when diffraction is no longer negligible.

III. THE FORCE BETWEEN TWO PERFECTLY CONDUCTING SPHERES

The degeneracy of periodic classical paths mentioned above does not occur in the calculation of the force on two uncharged conducting spheres of radii R_1 and R_2, a distance $l \ll R_1, R_2$ apart, which are embedded in a uniform medium. This problem was first considered theoretically by Derjaguin [8], who determined the behavior of the force on the spheres for $l \ll R_1, R_2$ from the energy density per unit area in the case of parallel plates. Since the Casimir energy diverges in this limit, a semiclassical evaluation ought to be exact. The semiclassical calculation below provides a more rigorous proof of Derjaguin's result.

The problem Derjaguin considered is axially symmetric with respect to the axis connecting the centers of the two spheres. If the distance l between them is very much less than either radius, the only classical periodic paths of interest are those between the two points of the spheres that are closest. We will have a bit more to say about classical paths that wrap around one or both spheres at the end of this section. These additional paths, shown in Fig. 2, are, however, arbitrarily long compared to l in the limit R_1/l and $R_2/l \to \infty$ and

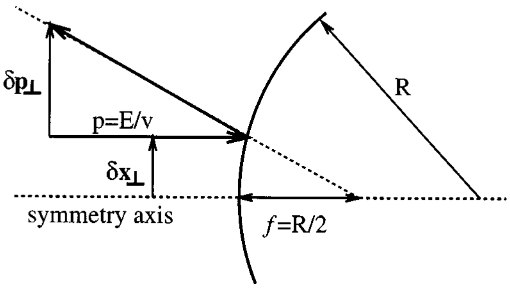

FIG. 3. Reflection by a convex mirror of focal length $f=R/2$ of a monochromatic ray of momentum $p=E/v$ incident parallel to, but displaced an infinitesimal distance $\delta\mathbf{x}_\perp$ from, the symmetry axis of the mirror.

therefore do not contribute to the force between the spheres in the limit that interests us.

A periodic classical path between the closest points of the two spheres is obviously transformed into itself by axial rotations and is therefore isolated in phase space, i.e., it is a fixed point of the symmetry. Such a path $\gamma(J,n)$ is characterized by its ''direction'' (i.e., whether it can be considered as the limit $\mathbf{x}_i\rightarrow\mathbf{x}_f$ of a classical path of type $J=\mathrm{I}$ or of $J=\mathrm{IV}$) and the number n of reflections on either sphere . Using the same notation as for the Casimir force between two plates, the length of a *periodic* classical path with $2n$ reflections off the spheres is $L_n^{\mathrm{I}}=L_n^{\mathrm{IV}}=2nl$, and the associated classical action is therefore

$$S_{\gamma(\mathrm{I},n)}=S_{\gamma(\mathrm{IV},n)}=2nlE/v. \tag{3.1}$$

To obtain the semiclassical expression for the response function, we have to compute the 4×4 monodromy matrices $M_\gamma(t)$ for these periodic paths. This is essentially a problem in geometrical optics. For simplicity, we take the point at time $t=0$ to be midway between the two spheres. The monodromy matrices $M_{\gamma(J,n)}(0)$ are then related to the monodromy matrix for the simplest periodic trajectory, $M\equiv M_{\gamma(I,1)}(0)$, by

$$M_{\gamma(\mathrm{I},n)}(0)=M^T_{\gamma(\mathrm{IV},n)}(0)=(M_{\gamma(\mathrm{I},1)}(0))^n\equiv M^n, \tag{3.2}$$

where T denotes the transpose. To obtain the determinant in Eq. (2.26) for any periodic closed orbit, it therefore suffices to find the eigenvalues of the monodromy matrix M of the simplest periodic orbit of type I. M can be decomposed as

$$M=\mathcal{T}(l/2)\mathcal{R}_2\mathcal{T}(l)\mathcal{R}_1\mathcal{T}(l/2), \tag{3.3}$$

where the matrices $\mathcal{T}$ and $\mathcal{R}_i$ are defined as follows.

The transport matrix $\mathcal{T}(l)$ in Eq. (3.3) relates the infinitesimal transverse deviations $\delta x_\perp(l)$ and $\delta p_\perp(l)$ at the end of a straight path of length l to those at the beginning. Since deviations in the two directions perpendicular to the straight path are independent of one another, the 4×4 matrix $\mathcal{T}(l)$ can be written as the tensor product

$$\mathcal{T}(l)=\begin{pmatrix}1 & (vl/E)\\ 0 & 1\end{pmatrix}\otimes\mathbf{1} \tag{3.4}$$

of a 2×2 matrix and a 2×2 unit matrix; the matrix $\mathcal{T}(l)$ can be read off from the geometrical relations

$$\begin{aligned}\delta x_\perp(l)&=\delta x_\perp(0)+\frac{l}{E/v}\,\delta p_\perp(0),\\ \delta p_\perp(l)&=\delta p_\perp(0)\end{aligned} \tag{3.5}$$

for free motion of a massless particle along a straight path of length l with momentum $p=E/v$. The reflection matrices $\mathcal{R}_i$ in Eq. (3.3) similarly relate the transverse deviations just before a reflection on sphere i to those just after. Since the two radii of curvature of a sphere are the same, the 4×4 matrix $\mathcal{R}_i$ can also be written as a tensor product:

$$\mathcal{R}_i=\begin{pmatrix}1 & 0\\ (2E/R_iv) & 1\end{pmatrix}\otimes\mathbf{1}. \tag{3.6}$$

The matrix $\mathcal{R}_i$ is obtained using geometrical optics for paraxial rays. As shown in Fig. 3 ray optics for reflection on a mirror of focal length $f=R/2$ gives

$$\begin{aligned}\delta x_\perp(\mathrm{after})&=\delta x_\perp(\mathrm{before}),\\ \delta p_\perp(\mathrm{after})&=\frac{E}{fv}\,\delta x_\perp(\mathrm{before})+\delta p_\perp(\mathrm{before}).\end{aligned} \tag{3.7}$$

The matrices $\mathcal{R}_i$ and $\mathcal{T}$ do not depend on the detailed geometry of the problem; our results will hold for any two ideal cylindrically symmetric convex mirrors with focal lengths $f_1=R_1/2$ and $f_2=R_2/2$ whose axes of symmetry coincide and which are a distance $l\ll f_1,f_2$ apart. The approach is valid irrespective of the precise shape of these mirrors, as long as paths between their closest points are the only classical periodic paths with a length comparable to their separation l.

Inserting Eqs. (3.6) and (3.4) into Eq. (3.3), the monodromy matrix for the path $\gamma(\mathrm{I},1)$ becomes

$$M\equiv M_{\gamma(\mathrm{I},1)}(0)=\begin{pmatrix}1+\dfrac{l}{R_1}+\dfrac{3l}{R_2}+\dfrac{2l^2}{R_1R_2} & \dfrac{vl}{E}\left(2+\dfrac{3l}{2R_1}+\dfrac{3l}{2R_2}+\dfrac{l^2}{R_1R_2}\right)\\ \dfrac{E}{vl}\left(\dfrac{2l}{R_1}+\dfrac{2l}{R_2}+\dfrac{4l^2}{R_1R_2}\right) & 1+\dfrac{3l}{R_1}+\dfrac{l}{R_2}+\dfrac{2l^2}{R_1R_2}\end{pmatrix}\otimes\mathbf{1}. \tag{3.8}$$

Note that the dependence of M on $p=E/v$ appears only in the off-diagonal elements, once as vl/E and once as E/vl. The determinant as well as the trace of M and thus the eigenvalues of M are therefore energy independent; they depend only on geometric properties of the objects, i.e., the radii of the two spheres and l. There is thus a clean separation of the geometric and dynamic aspects of the problem, a result rather simple to understand from the nature of Derjaguin's calculation of the force for $l/R \ll 1$.

Since the determinants of $\mathcal{T}$ and $\mathcal{R}_i$ are each unity, the determinant of M as given by Eq. (3.3) is also unity and the two doubly degenerate eigenvalues $d_\pm$ of M are inverses of each other. The sum of the eigenvalues is the trace of M. We thus have from Eq. (3.8)

$$d_+ + d_- = d_+ + (1/d_+) = 2 + 4a, \tag{3.9}$$

where

$$a = \frac{l}{R_1} + \frac{l}{R_2} + \frac{l^2}{R_1 R_2} \tag{3.10}$$

and one obtains

$$d_\pm = (\sqrt{a+1} \pm \sqrt{a})^2. \tag{3.11}$$

The determinant in Eq. (2.26) can be nicely expressed by proceeding in the usual fashion of casting the eigenvalues (3.11) in exponential form with a single geometrical parameter α

$$d_\pm = e^{\pm 2\alpha}, \quad \text{with} \quad \alpha = \ln(\sqrt{a+1} + \sqrt{a}). \tag{3.12}$$

Using Eq. (3.2) and the fact that each of the eigenvalues (3.12) is doubly degenerate in the matrix (3.8), the determinant in Eq. (2.26) is

$$\begin{aligned} \text{Det}\,[M_{\gamma(J,n)}(0) - \mathbf{1}] &= \text{Det}\,(M^n - \mathbf{1}) = (e^{2n\alpha} - 1)^2 \\ &\quad \times (e^{-2n\alpha} - 1)^2 \\ &= 16 \sinh^4(n\alpha) \end{aligned} \tag{3.13}$$

for the periodic paths characterized by $J = \text{I}$ or IV, with $2n$ reflections. Furthermore, the result (3.13) does not depend on the choice of the initial position on the periodic path at time $t=0$. This is seen by observing that $\mathcal{T}(x)\mathcal{T}(y) = \mathcal{T}(x+y)$. Choosing the initial point at an arbitrary value x rather than at the point $x=0$ midway between the spheres therefore amounts to the replacement

$$M \to \mathcal{T}(x) M \mathcal{T}(-x) = \mathcal{T}(x) M \mathcal{T}^{-1}(x) \tag{3.14}$$

in Eq. (3.13). The determinant (3.13) does not change under the transformation (3.14), and thus is independent of time. The time integration in Eq. (2.26) is thus trivial and just gives the total period $\tau_{\gamma(J,n)}$ for a periodic orbit with $2n$ reflections. The period does not depend on the type J of orbit and is

$$\tau_{\gamma(J,n)} = \tau_n = 2nl/v_g = 2nl \frac{\partial p(E)}{\partial E} = 2nl \frac{\partial}{\partial E}\left(\frac{E}{v}\right) \tag{3.15}$$

since the group velocity in a homogeneous dispersive medium is $v_g(E) = \partial E/\partial p$. [For the special case of a vacuum surrounding the spheres one has $\tau_n^{\text{vac}} = 2nl/c$.] We finally note that a classical periodic path with $2n$ reflections between the closest points of the two spheres (or mirrors) traverses each volume element $d\mathbf{x}$ exactly $2n$ times, so that

$$N_{\gamma(J,n)} = 2n. \tag{3.16}$$

Inserting the expressions (3.16), (3.15), and (3.13) in Eq. (2.26) and performing the (trivial) time integration, we find that the amplitude A_γ for a classical periodic orbit $\gamma(J,n)$,

$$A_{\gamma(J,n)} = \frac{l}{4\sinh^2(n\alpha)} \frac{\partial}{\partial E}\left(\frac{E}{v}\right), \tag{3.17}$$

is the same for $J=\text{I}$ and $J=\text{IV}$ and depends on the geometry of the problem only via the parameter α given in Eqs. (3.12) and (3.10) in terms of the minimal distance l between the spheres and their radii R_i.

Inserting Eq. (3.17) in Eq. (2.25) and using Eq. (3.1) we arrive at the semiclassical expression for the response function

$$g_{\text{osc}}(E; l \ll R_1, R_2) = 4 \frac{-il}{\hbar}\left[\frac{\partial}{\partial E}\left(\frac{E}{v}\right)\right] \sum_{n=1}^{\infty} \frac{e^{2inlE/(\hbar v)}}{4\sinh^2(n\alpha)} \tag{3.18}$$

where the overall factor of 4 in Eq. (3.18) arises from summing over classes I and IV of the periodic orbits and over the two polarizations of the photon. It is quite remarkable that Gutzwiller's extension [18] of the semiclassical method gives such a concise expression for the (semiclassical) response function in the relatively complex situation of two spheres. The derivation of Eq. (3.18) is notably independent of the precise geometry and applies equally well to half spheres, or convex parabolic mirrors, in fact, as commented on above, to any geometry where one may disregard all periodic classical paths apart from the ones we considered. Thus let A and B be the points on the mirrors that are closest. The method is then applicable if the straight line trajectories A to B to A and B to A to B, and their repetitions, are the only relevant classical periodic paths, all other periodic paths having lengths very much greater than l. In general the mirror surfaces at A and B are each characterized by two radii of curvature. The eigenvalues of the monodromy matrix are then no longer doubly degenerate as for axially symmetric coaxial mirrors. This complicates the analysis somewhat but requires no new concepts.

Inserting Eq. (3.18) in Eq. (2.8) and noting that the analytic continuation is valid since the integrand falls off exponentially on the large quarter-circle, the Casimir energy of the system with two spheres semiclassically is

$$\mathcal{E}_{\rm Cas}(l \ll R_1, R_2) = -\sum_{n=1}^{\infty} \frac{l\hbar}{2\pi \sinh^2(n\alpha)} \int_0^\infty d\xi \xi e^{-2nl\xi/v} \left[\frac{\partial}{\partial \xi}\left(\frac{\xi}{v}\right)\right] = -\sum_{n=1}^{\infty} \frac{\hbar}{4\pi n \sinh^2(n\alpha)} \int_0^\infty d\xi e^{-2nl\xi/v}, \quad (3.19)$$

where the last expression is obtained in a fashion similar to that used to arrive at Eq. (1.10).

In a vacuum, the phase velocity $v(-\xi^2)=c$ is independent of the frequency and the integral in Eq. (3.19) is easily evaluated. The Casimir energy $\mathcal{E}^{\rm vac}_{\rm Cas}$ of two spheres in a vacuum is then given semiclassically by the rapidly convergent sum

$$\mathcal{E}^{\rm vac}_{\rm Cas}(l \ll R_1, R_2) = -\sum_{n=1}^{\infty} \frac{\hbar c}{8\pi l n^2 \sinh^2(n\alpha)}. \quad (3.20)$$

The semiclassical expressions (3.19) and (3.20) are strictly valid only in the asymptotic regime $l/R_1 \sim 0$, $l/R_2 \sim 0$ where the semiclassical contribution from paths that wrap around either sphere can be neglected. They furthermore diverge only in the limit $\alpha \to 0$. By our previous argument the Casimir energy of the system is *exactly* reproduced by the semiclassical evaluation in this limit. It is easily seen that the limit $\alpha \to 0$ corresponds to the situation where the radius of the *smaller* of the two spheres is much larger than their separation. Fortunately, this is also the limit in which diffraction effects are negligible and our semiclassical result is valid. From Eqs. (3.19) and (3.20) we thus can obtain the *exact* Casimir energy of the system in the limit $l \ll R_1, R_2$. We retain only the leading (divergent) contribution to the Casimir energy (3.19) in the asymptotic regime $a \sim 0$; with $a \sim 0$, it follows that $\alpha^2 = a + O(a^2) \sim 0$, and keeping only the leading term for $n\alpha \ll 1$,

$$\sinh^2(n\alpha) = n^2 a + O(a^2). \quad (3.21)$$

Inserting Eq. (3.21) in Eq. (3.19), the leading divergent behavior of the Casimir energy for $l/R_1 \sim 0$ and $l/R_2 \sim 0$ is, using Eq. (3.10),

$$\mathcal{E}_{\rm Cas}\left(\frac{l}{R_1} \sim 0 \, \frac{l}{R_2} \sim 0\right) \sim -\sum_{n=1}^{\infty} \frac{\hbar}{4\pi a n^3} \int_0^\infty d\xi e^{-2nl\xi/v} \sim -\sum_{n=1}^{\infty} \frac{\hbar \bar{R}}{4\pi l n^3} \int_0^\infty d\xi e^{-2nl\xi/v}, \quad (3.22)$$

where $\bar{R} = (R_1 R_2)/(R_1 + R_2)$ is the large length scale. The Casimir energy of two spheres embedded in a vacuum a distance $l \ll R_1, R_2$ apart becomes

$$\mathcal{E}^{\rm vac}_{\rm Cas}\left(\frac{l}{R_1} \sim 0 \, \frac{l}{R_2} \sim 0\right) \sim -\frac{\pi^3 \hbar c \bar{R}}{720 l^2}, \quad (3.23)$$

and the attractive force $F^{\rm vac}_{\text{sph-sph}}$ between them for sufficiently small separation l is

$$F^{\rm vac}_{\text{sph-sph}}\left(\frac{l}{R_1} \sim 0 \, \frac{l}{R_2} \sim 0\right) \sim -\frac{\pi^3 \hbar c \bar{R}}{360 l^3}. \quad (3.24)$$

The limiting case of the force, $F^{\rm vac}_{\text{sph-wall}}$, between a conducting sphere of radius R and a conducting wall in a vacuum is obtained by letting one of the radii of curvature become arbitrarily large compared to the other. In this limit Eq. (3.24) implies

$$F^{\rm vac}_{\text{sph-wall}}\left(\frac{l}{R} \sim 0\right) \sim -\frac{\pi^3 \hbar c R}{360 l^3}. \quad (3.25)$$

This leading term of the Casimir forces for small separation, proportional to $\bar{R}/l^3$ and R/l^3, respectively, was first arrived at by Derjaguin [8] using the known dependence of the energy density (1.8) between two parallel plates. His calculation essentially assumes that the vacuum energy density depends upon the separation between opposing infinitesimal surface elements in the same manner as for flat plates and does not depend on their relative orientation, nor on the geometry of the configuration of conductors as a whole. These assumptions can alternatively be formulated as stating that the force between conductors is primarily due to the independent superposition of the retarded interaction between the individual atoms of which they are composed—the additivity principle—if this interaction is normalized to give the Casimir force between two parallel plates—the renormalization principle [19–21]. Both approaches are eminently reasonable from a physical point of view, but the underlying assumptions have not been proven from first principles and higher order corrections cannot be estimated. The semiclassical approach provides the theoretical justification for these physical assumptions.

In the case of spheres, we could ignore diffraction effects only in the limit R_1 and $R_2 \gg l$ we considered. As such, the final result (3.20) gives the true force on the spheres only if $a \ll 1$ and therefore $\alpha \ll 1$, and we could have retained only the leading terms in the expansion of α in powers of a from the outset to derive the semiclassical expression for the force on the spheres in the limit where we can ignore diffraction. As it happens, it is algebraically simpler to proceed as we did, but much more significant is the fact that the contribution to F from the paths we considered, henceforth denoted by $F_{\text{no diff}}$, is the semiclassical result for arbitrary values of l, R_1, and R_2. Our previous discussion concerning the exactness of the semiclassical evaluation of Casimir energies suggests that it is not accurate when the separation l is much larger than the radius of the smaller of the two spheres, because the Casimir energy no longer diverges in this limit and the dominant contribution to the energy integral in Eq. (2.8) does *not* arise from the behavior of the integrand at large energies. We have not looked into the question of whether or not the semiclassical approximation for $g_{\rm osc}$ gives a useful *approximation* to the Casimir energy in this situation.

We return to Eq. (3.20), by itself valid semiclassically also for $l \gg R_1, R_2$, where now $a \gg 1$ and therefore $\alpha = \frac{1}{2}\ln[4l^2/(R_1R_2)]$; we retain only the $n=1$ term, differentiate, and find

$$F_{\text{no diff}}(a \sim \infty) \sim -\frac{3\hbar c R_1 R_2}{8\pi l^4}. \tag{3.26}$$

The Casimir-Polder result [11] for the force between two distant atoms, on the other hand, is

$$F_{\text{At At}} \sim -\frac{161}{4\pi}\frac{\hbar c \alpha_1(0)\alpha_2(0)}{l^8}, \tag{3.27}$$

where $\alpha_i(0)$ is the static (zero-frequency) electric-dipole polarizability of the ith atom and magnetic polarizabilities have been neglected. But $\alpha(0) = R^3$ for a conducting sphere of radius R, and thus for two spheres separated by a distance $l \gg R_1, R_2$, the force is

$$F_{\text{sph-sph}}(l \sim \infty) \sim -\frac{161}{4\pi}\frac{\hbar c R_1^3 R_2^3}{l^8}. \tag{3.28}$$

Since

$$F_{\text{sph-sph}} = F_{\text{no diff}} + F_{\text{diff}}, \tag{3.29}$$

the diffraction contribution F_{diff} would have to cancel the $1/l^4$ term and any other terms falling off less rapidly than $1/l^8$ of $F_{\text{no diff}}$ for the semiclassical approximation to have any justification in the limit $l \sim \infty$. We have not yet studied diffraction effects, but one can see from Fig. 2 that the dominant $1/l^4$ term may well be canceled. Thus, ignoring terms of order R_i/l, the shortest diffractive paths of Fig. 2 have the same length, $2l$, as the shortest nondiffractive path [which is the only path that contributes in the limit $l \sim \infty$ to $F_{\text{no diff}}$ in Eq. (3.26)]. But the shortest diffractive paths undergo one reflection and therefore contribute to the force between the spheres with a sign that is opposite to the contribution of the nondiffractive path of similar length.

Another, perhaps experimentally more relevant, observation is in order in this context: diffraction effects are negligible, irrespective of the curvature radii at the closest points of the two convex axially symmetric mirrors with a common axis, whenever the linear dimensions of the mirrors are large compared to their separation. One can imagine mirrors with rather small radii of curvature but with mirror dimensions sufficiently large for diffraction to play no role even when the curvature at their closest points is much less than the separation between them. (Consider, for instance, the extreme case of two perfectly aligned ''needles'' pointed at each other, separated by a distance l satisfying $L_i \gg l \gg R_i$, where the R_i are the radii of curvature of their tips and the L_i are the lengths of the needles, or the more realistic case of a single needle pointed at a plane.) In this case, the dependence of the force between these objects on their separation would semiclassically be given by Eq. (3.26), rather than by Eq. (3.27) for $l \gg R_1, R_2$ but $l \ll L_1, L_2$. In other words, Eq. (3.20) would be the semiclassical result for all separations l small compared to the linear dimensions of the mirrors, but not necessarily small compared to their radii of curvature. There is no apparent contradiction with the Casimir-Polder result (3.27), which obviously does not apply when the separation is small compared to the linear dimensions of the objects.

Similar reasoning suggests that the results (3.19) and (3.20) give the semiclassical approximation to the Casimir energy of two (infinite) paraboloids of revolution with radii of curvature R_1 and R_2 and a common symmetry axis. Diffraction effects are negligible. We do not claim that the semiclassical approximation in this case is exact (except for $l \ll R_1, R_2$), but it could be of interest to see whether the l^{-4} dependence of the force for large separations between the paraboloids predicted semiclassically by Eq. (3.26) is at least qualitatively correct.

IV. DISCUSSION

The semiclassical approach to Casimir energies advocated here is clearly applicable to a range of problems we have not considered. The analysis of nearby metallic spheres embedded in a dielectric can obviously be extended to nearby dielectric spheres in a dielectric, since the only significant periodic orbits are the same for the two cases; the reflection matrices $\mathcal{R}_1$ and $\mathcal{R}_2$ would, however, have to be changed to those appropriate for reflection off a dielectric. We also remark that the results deduced simplify greatly for very large and very small separations. In the former case, $\varepsilon(\omega^2)$ of the medium between the metallic objects can be replaced by $\varepsilon(0)$, and the results for the medium in this regime are those for the vacuum with the replacement $c \to c/\sqrt{\varepsilon(0)}$. For sufficiently small separations, $\xi^2\varepsilon(-\xi^2)$ is well approximated by $\xi^2 + \omega_{\text{pl}}^2$, where ω_{pl} is the plasma frequency of the medium, and the integrations can be evaluated [22].

More important for the general usefulness of the semiclassical method we presented for calculating the Casimir energy would be the inclusion of diffraction phenomena. The current status of affairs [10] is somewhat unsatisfactory in this regard, for it requires a decomposition of the scattering amplitude into partial waves, which, after performing the semiclassical approximation for each channel separately, are eventually resummed. The procedure is at best cumbersome and generally leads to approximations (because the summation over partial waves is usually truncated) that are not consistent with an expansion in $\hbar$. Pending a more efficient resolution of this technical problem of the semiclassical approximation, the calculation of Casimir energies using classical periodic paths, although conceptually appealing, is in practice limited to situations in which diffraction effects can be neglected.

When present, as in the examples we discussed, contributions from fluctuations around periodic classical paths completely dominate the semiclassical response function, the contribution from nonperiodic paths being of higher order in $\hbar$. In the absence of classical periodic paths, as, for example, for two large conducting plates that are at an angle β to one another, one must clearly consider nonperiodic paths. The semiclassical Green function can be computed for this case, but it is not clear how the leading approximation to the spatial integral for the response function can be obtained. We conjecture that the spatial integration giving the response function in the $\hbar \to 0$ limit in this case is dominated by endpoint contributions and speculate that the Casimir energy can be found by considering certain classical paths that begin and end at a point *on the boundary*. We are currently investigating this possibility and wish only to remark here that as $\beta\to 0$, the case of two parallel plates, such paths coincide with the periodic classical paths we considered.

Note added. As is apparent from the semiclassical derivation, Eq. (1.6) is the contribution to $\mathcal{E}_{\rm Cas}$ of fluctuations *within* the rectangular parallelepiped. It is the full $\mathcal{E}_{\rm Cas}$ only for a rectangular parallelepiped of vacuum within an infinite metallic region. The total Casimir energy, internal *and* external, has been calculated for an ideal cylinder of arbitrarily small thickness by L. L. DeRaad, Jr. and K. A. Milton, Ann. Phys. (N.Y.) **136**, 229 (1981).

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APPENDIX A: ON THE SEMICLASSICAL EVALUATION OF CASIMIR EFFECTS

We here show that the Casimir energy of a system of conductors in a vacuum can in principle always be found *exactly* from the semiclassical expansion of the response function $g_{\rm osc}(E,l)=g(E,l)-g_0(E,l)$ to *finite* order. In Appendix C we define the oscillating part of the response function without recourse to a semiclassical expansion, as a path integral over periodic orbits which depend on the boundary of the system. For simplicity we here consider only a perfectly conducting cavity of general shape in a vacuum. Even with this restriction, an exact determination of $g_{\rm osc}$ is possible only for very simple geometries of the cavity. The Casimir energy of the system is, however, given by a certain coefficient in the semiclassical expansion of $g_{\rm osc}(E,l)$ for large $|El/\hbar c|$.

The basic idea is to express the energy integral over the imaginary part of $g_{\rm osc}(E,l)$ in Eq. (2.8) as an integral over a contour in the complex E plane using analytic properties of the response function. The contour is chosen so that the integral can be evaluated to arbitrary accuracy using the semiclassical approximation to $g_{\rm osc}(E,l)$. The exact semiclassical evaluation of a moment of the imaginary part of $g_{\rm osc}$ is the basis for many sum rules and the Casimir energy is in this sense just a special case.

We first (re)introduce a cutoff Ω in the energy integral (2.8) and consider $\mathcal{E}_{\rm Cas}$ as the limit $\Omega\to\infty$ of

$$\mathcal{E}_{\rm Cas}(\varepsilon,l;\Omega)=-\frac{1}{2\pi}\lim_{\eta\to 0_+}\int_0^{\Omega}dE\;E\;{\rm Im}\,g_{\rm osc}(E+i\eta,l). \tag{A1}$$

The spectral representation of $g_{\rm osc}(E,l)$ follows from Eq. (2.3); it is

$$g_{\rm osc}(E,l)=\int_0^{\infty}dE'\,\frac{\rho_{\rm osc}(E',l)}{E-E'}, \tag{A2}$$

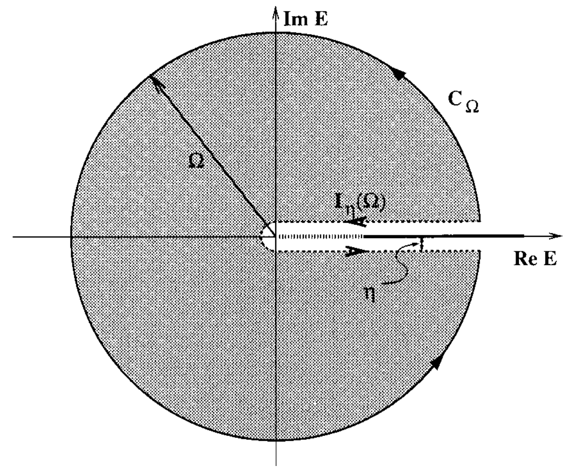

FIG. 4. Contours of integration in the complex energy plane used in the text. Poles and cuts of $g_{\rm osc}(E,l)$ on the positive real E axis are shown schematically. The region where the oscillating part of the response function $g_{\rm osc}$ is an analytic function of E is shaded gray.

where $\rho_{\rm osc}(E,l)$, the oscillatory part of the spectral density, is a real (but not necessarily positive definite) function. $g_{\rm osc}(E,l)$ is thus an analytic function of E in the whole complex plane, apart from a cut (and/or poles) on the positive real E axis. Inserting Eq. (A2) in Eq. (A1) one can express the Casimir energy as

$$\mathcal{E}_{\rm Cas}(\varepsilon,l;\Omega)=\frac{1}{4i\pi}\lim_{\eta\to 0_+}\int_{I_\eta(\Omega)}dE\;Eg_{\rm osc}(E,l), \tag{A3}$$

that is, as an integral over the contour $I_\eta(\Omega)$ in the complex E plane. The contour $I_\eta(\Omega)$, indicated in Fig. 4, starts at the point $\Omega+i\eta$, runs just above the positive real E axis to the origin, encircles the origin, and continues just below the real positive E axis to the point $\Omega-i\eta$. Since the function $g_{\rm osc}(E,l)$ is analytic in the complex E plane apart from singularities on the positive real E axis, the integral in Eq. (A3) can just as well be performed by integrating counterclockwise along a large circle of radius Ω in the complex E plane, the contour C_Ω shown in Fig. 4. We can therefore evaluate the Casimir energy of the system by performing the integral

$$\mathcal{E}_{\rm Cas}(\varepsilon,l)=\frac{1}{4i\pi}\lim_{\Omega\to\infty}\int_{C_\Omega}dE\;Eg_{\rm osc}(E,l)$$
$$=\lim_{\Omega\to\infty}\frac{\Omega^2}{4\pi}\int_0^{2\pi}d\phi e^{2i\phi}g_{\rm osc}(\Omega e^{i\phi},l). \tag{A4}$$

The point is that a semiclassical expansion of the response function $g_{\rm osc}(E,l)$ in Eq. (A4) is possible for sufficiently large radius Ω, and should suffice to determine the Casimir energy of the system *exactly*.

Dimensional analysis shows that the E dependence of the response function is of the form

$$g_{\rm osc}(E,l)=\hat{g}(x,\{r_i\})/E. \tag{A5}$$

$\hat{g}$ is a dimensionless function that depends on the energy only through the dimensionless parameter $x=(El)/(\hbar c)$, with l a typical length of the cavity. The dimensionless parameters $r_i=l_i/l$ are ratios of lengths on which the geometry of the cavity may also depend. To perform the contour integral in Eq. (A4), we need to know $g_{\rm osc}(E,l)$ only for large $|E|=\Omega$ and Eq. (A5) shows that a semiclassical expansion of the response function is valid in this regime. The semiclassical expansion of $\hat{g}(x,\{r_i\})$ for large energy and therefore large x is in general of the form

$$\hat{g}(x\gg 1,\{r_i\})\sim x^q e^{ixc_1(\{r_i\})}\sum_{j=0}^{\infty} x^{-j}c_{-j}(\{r_i\}), \tag{A6}$$

where the coefficients c_j are functions of the dimensionless geometrical ratios r_i only and q is the leading exponent of x in the expansion. Because $g_{\rm osc}(E,l)$ is a single-valued function of the energy E, only integer powers of x appear in the semiclassical expansion (A6). [The semiclassical expansion of other quantities is generally an expansion in half-integer powers of $\hbar$.] In the text we obtained the leading term of this expansion and found that $q=2$ for the parallel plate problem [see Eqs. (2.19) and (2.23)], whereas $q=1$ in the Derjaguin problem [see Eq. (3.18)]. If we use the expansion Eq. (A6) for the oscillating part of the response function (A5) at large energies, the contour integration in Eq. (A4) can be performed term by term. By Cauchy's theorem, only the term proportional to x^{-1}, with coefficient c_{-q-1}, gives a contribution and the Casimir energy of a cavity is

$$\mathcal{E}_{\rm Cas}(\varepsilon,l,\{r_i\})=\frac{\hbar c}{2l}c_{-q-1}(\{r_i\}). \tag{A7}$$

Although interesting, the relation (A7) is generally perhaps of little practical use for a determination of the Casimir energy. It is usually quite difficult to obtain the coefficient c_{-q-1} in the semiclassical expansion of $g_{\rm osc}$ accurately. We therefore did not pursue this approach to evaluate Casimir energies and instead restricted ourselves to special cases for which an asymptotic evaluation of the Casimir energy using only the leading term of Eq. (A6) suffices. The relation (A7) shows that an exact semiclassical evaluation of the Casimir energy is, however, in principle always possible.

APPENDIX B: THE DETERMINANT D_0

To prove Eq. (2.14), let Q represent the matrix in Eq. (2.13) for the direct path between $\mathbf{x}$ and $\mathbf{y}$. With S given by Eq. (2.11), Q becomes

$$\boldsymbol{Q}=\left(\begin{array}{c|c} -\frac{E}{Lv}\left[\delta_{ij}-\frac{X_iX_j}{L^2}\right] & \frac{X_i}{v_gL} \\ \hline -\frac{X_j}{v_gL} & -\frac{L}{v_g^2}\frac{\partial v_g}{\partial E}\end{array}\right), \tag{B1}$$

where $\mathbf{X}=\mathbf{x}-\mathbf{y}$ is the displacement vector and

$$v_g(E)=\left(\frac{\partial}{\partial E}\frac{E}{v}\right)^{-1} \tag{B2}$$

denotes the group velocity. The elements of Q are not all of the same dimension. In evaluating $\det Q$, we therefore extract a factor $(-E/Lv)$ from each of the first three rows, $(-1/v_g)$ from the fourth row, and Lv/Ev_g from the resultant fourth column, and obtain

$$\det Q=\left(\frac{E}{Lvv_g}\right)^2\det Q',$$

$$\textbf{with}\ \ \boldsymbol{Q'}=\left(\begin{array}{c|c} \delta_{ij}-\frac{X_iX_j}{L^2} & -\frac{X_i}{L} \\ \hline \frac{X_j}{L} & \frac{E}{v}\frac{\partial v_g}{\partial E}\end{array}\right), \tag{B3}$$

with the elements of Q' dimensionless. The matrix Q' has two eigenvectors of the form $Z_\pm=(\mathbf{X},f_\pm)$ with corresponding eigenvalues

$$\sigma_\pm=\frac{1}{2}\left(\frac{E}{v}\frac{\partial v_g}{\partial E}\pm\sqrt{\left(\frac{E}{v}\frac{\partial v_g}{\partial E}\right)^2-4}\right), \tag{B4}$$

so that $\sigma_+\sigma_-=1$. The other two linearly independent eigenvectors $Z_{1,2}=(\mathbf{Y}_{1,2},0)$ of Q', with $\mathbf{Y}_{1,2}\cdot\mathbf{X}=0$, are orthogonal to $Z_\pm$ and correspond to a doubly degenerate eigenvalue 1. The determinant of Q is therefore

$$\det Q=D_0^2=\left[\frac{E}{Lvv_g}\right]^2, \tag{B5}$$

and Eq. (2.14) follows.

APPENDIX C: PERIODIC PATHS AND SYMMETRIES OF A PHASE SPACE WITH BOUNDARY

We will here consider a bounded (one-particle) phase space $\mathcal{P}$ with a five-dimensional boundary $\partial\mathcal{P}$ that is described by the constraint

$$\Phi(\mathbf{x},\mathbf{p})=0. \tag{C1}$$

A periodic path γ_τ of period τ in the phase space $\mathcal{P}$ can be considered as the map

$$\gamma_\tau: t\in[0,\tau]\rightarrow\{(\mathbf{x}(t),\mathbf{p}(t))\in\mathcal{P};\ \text{with}\ \ \mathbf{x}(0)=\mathbf{x}(\tau)\ \ \text{and}\ \ \mathbf{p}(0)=\mathbf{p}(\tau)\} \tag{C2}$$

of the time interval $t\in[0,\tau]$ onto a closed path in the phase space $\mathcal{P}$. Since the path is periodic, the time interval $[0,\tau]$ can be thought of as a parametrization of the circle S_1. For reflections at an x-space boundary to be possible, we cannot restrict our attention to continuous maps γ_τ, since the momentum of a classical path in general is discontinuous at an x-space boundary $\partial\mathcal{P}$ of $\mathcal{P}$. We will, however, consider only piecewise continuous maps γ_τ, continuous everywhere except for (possibly) a set of points $\{t_i\}$ of vanishing measure, which is mapped onto the boundary $\partial\mathcal{P}$, i.e.,

$$\Phi(\gamma_\tau(t))|_{t\in\{t_i\}}=0. \tag{C3}$$

The response function $g(E)$ is, loosely speaking, the Laplace transform of a path integral over all such periodic paths (C2) of period τ. This path integral generally diverges in a semiclassical evaluation. Some of these divergences, however, are of no physical significance for Casimir effects, and others are due to the (naive) semiclassical evaluation of the path integral. It would be conceptually preferable and computationally advantageous to avoid such divergences in the formulation of the problem *before performing the semiclassical evaluation*. We will here consider two types of divergences that can be circumvented. The first is associated with the short-range singularity arising from periodic paths of arbitrarily short length, while the second is due to the fact that classical periodic paths are not isolated in the presence of continuous phase-space symmetries. Although these two types of divergences encountered in a straightforward semiclassical evaluation of the path integral representation for the response function are of a different kind, they both have their origin in periodic paths that are equivalent in some sense.

Let us first address the issue of arbitrarily short periodic paths. These paths can be understood as fluctuations around a classical path γ_τ^0, which is a single point in phase space. In the restricted set of piecewise continuous maps γ_τ defined above with a Hamiltonian $H=cp$ describing a massless particle, there is no stationary periodic classical path of vanishing length for $\tau>0$, and the problem can in principle be avoided in a semiclassical evaluation of the path integral by a restriction to paths with $\tau>0$. However, the periodic path $\gamma_\tau^0:[0,\tau]\to\{(\mathbf{x}(t)=\mathbf{x}_0,\ \mathbf{p}(t)=0)\notin\partial\mathcal{P}\}$ is a classical (i.e., stationary) periodic trajectory of vanishing length for a *massive* particle and the fluctuations around such paths do give rise to a divergence in the semiclassical evaluation of the response function that cannot be avoided by a restriction to paths of period $\tau>0$. More significant from a physical point of view is that the contribution to the response function from such arbitrarily short paths can at most depend on *local* variations of the boundary, if the (arbitrarily short) periodic path begins and ends on the boundary. Periodic paths that lie wholly in the *interior* of $\mathcal{P}$ are unaffected by a small variation of the boundary. They therefore do not give a boundary-dependent contribution to $g(E)$ nor to the Casimir energy. We are not interested in these generally divergent but boundary-independent terms, since a Casimir energy should depend on the imposed boundary. One usually simply ignores these boundary-independent contributions to the energy but it would be conceptually and computationally preferable to formulate $g_{\rm osc}(E)$ in a way that avoids them from the outset. In particular, we would like to exclude contributions to the path integral from (short) periodic paths that are wholly in $\mathcal{P}\setminus\partial\mathcal{P}$ or at most include *one* point of the boundary.

The restriction of the space of periodic paths can be cast in a more mathematical form by noting that a short periodic path that depends only *locally* (or not at all) on the boundary can be continuously deformed *in the interior* $\mathcal{P}\setminus\partial\mathcal{P}$ to a trajectory that consists of a single point. More rigorously, we require an equivalence relation that relates paths that can be ''smoothly'' deformed into each other—where smoothly essentially means that the action varies continuously as we deform one path into another. The required equivalence relation is rather close to the notion of *homotopy* [23] on $\mathcal{P}\setminus\partial\mathcal{P}$. Two periodic paths γ and γ' are usually said to be homotopically equivalent if there is a continuous deformation ρ of γ into γ',

$$\rho:s\in[0,1]\to\gamma(s),\quad\text{with}\quad \gamma(0)=\gamma\quad\text{and}\quad\gamma(1)=\gamma'. \tag{C4}$$

However, we clearly require a more restrictive equivalence relation, since the definition (C4) would imply that all the periodic paths in say the Derjaguin problem are homotopically trivial. Intuitively we would like to arrange matters such that paths that differ in the number of reflections are inequivalent, especially since the Maslov index generally differs for such curves. In effect, we would like the ''homotopy'' map to preserve the Maslov index of a path. To guarantee this, it appears to be sufficient for the map ρ of Eq. (C4) to preserve the *number* of time intervals (which can be isolated points) during which the paths are on the boundary $\partial\mathcal{P}$ of the phase space $\mathcal{P}$. We can define a corresponding index $i(\gamma_\tau)$, for any periodic path $\gamma_\tau\in\mathcal{P}$, which simply counts the number of time intervals for which the path γ_τ is on the boundary $\partial\mathcal{P}$. Consider the set of disjoint closed time intervals for which the periodic path γ_τ is part of the boundary,

$$m(\gamma_\tau):=\{I_i=[t_i,t_{i+1}],I_i\cap I_j=\varnothing\,\forall i\neq j:\gamma_\tau(t)\notin\partial\mathcal{P}\ \text{for}\ t\in\sim\bigcup_i I_i\}, \tag{C5}$$

where the periodic time interval $[0,\tau]$ is considered a parametrization of a circle. We define the index $i(\gamma_\tau)$ as the number of elements in $m(\gamma_\tau)$, i.e., the order $N(m)$ of the set m,

$$i(\gamma_\tau):=N(m(\gamma_\tau)). \tag{C6}$$

The desired equivalence relation between two periodic paths is then given by a homotopy map $\bar{\rho}$ that does not change the index of a path. The restricted homotopy map $\bar{\rho}$, in particular, does not change the number of reflections of a periodic path. The trivial equivalence class $\{0\}$ will denote those periodic curves that can be continuously deformed to a point in $\mathcal{P}$ without changing the index $i(\gamma_\tau)$. Since a point in $\mathcal{P}$ either is or is not on the boundary, these periodic paths have index $i=1$ or $i=0$. They are precisely the periodic paths whose contribution to the Casimir energy does not depend on global variations of the boundary and we can exclude them in the definition of the oscillating part of the response function $g_{\rm osc}$. Note that the restricted homotopic equivalence relation defined above fits our requirements quite well and is sufficiently refined to distinguish between *many* of the classical periodic paths of the two-plate and Derjaguin problems. (Reflected paths and paths that follow the boundary for a while can, however, belong to the same equivalence class in the above sense, and one may wonder whether an even more restrictive definition of equivalence distinguishes between them). There always, however, seems to be at least one classical periodic path in each (restricted) equivalence class. Our equivalence between periodic paths on the bounded phase space $\mathcal{P}$ thus appears to be in agreement with the expectation that the action is a continuous functional within each equivalence class only.

We can now eliminate from the outset the boundary-independent but generally divergent contribution to the response function from periodic paths in the trivial equivalence class $\{0\}$ and define the oscillating part of the response function, $g_{\rm osc}(E)$, by the Laplace transform of the path integral

$$g_{\rm osc}(E)=\frac{-2i}{\hbar}\int_0^\infty d\tau e^{i(E+i\varepsilon)\tau/\hbar}\int_{\gamma_\tau\notin\{0\}}[d\mathbf{x}d\mathbf{p}]e^{iS[\gamma_\tau]/\hbar}, \tag{C7}$$

where the factor of two accounts for the sum over polarizations. Since classical periodic paths that belong to a nontrivial equivalence class generally have a minimal length (which depends on the boundary of $\mathcal{P}$), the restricted path integral (C7) is free from divergent contributions due to fluctuations around arbitrarily short classical paths. The restriction of Eq. (C7) to periodic paths in nontrivial equivalence classes is equivalent to the previously used definition (2.5) of the oscillating part of the response function semiclassically, but Eq. (C7) specifies the boundary dependent (oscillating) part of the response function without reference to the generally divergent response function in the absence of boundaries. In the absence of a boundary, all periodic paths on a contractable phase space are trivial and $g_{\rm osc}(E)$ given by Eq. (C7) then vanishes by definition.

The measure $[d\mathbf{x}d\mathbf{p}]$ and the corresponding path integral in Eq. (C7) should be understood as the limit of a finite dimensional integral obtained by suitably discretizing the time interval $[0,\tau]$. Note that momenta and coordinates enter symmetrically in the discretized path integral for the response function (C7)

$$\lim_{n\to\infty}\int\left[\prod_{i=1}^{n}\frac{d\mathbf{x}(i\tau/n)d\mathbf{p}[(i-\frac{1}{2})\tau/n]}{(2\pi\hbar)^3}\right]e^{iS[\gamma_\tau]|_{\rm disc}/\hbar} \tag{C8}$$

and the integrals in Eq. (C7) extend over the available phase-space volume. The momentum and coordinate integrations in the path integral representation of a Green function, on the other hand, are not symmetric. (The end points $\mathbf{x}_0$ and $\mathbf{x}_n$ of a Green function are fixed and there is an ''extra'' $\mathbf{p}$-space integral in its path integral representation.)

Let us next consider continuous symmetries of the phase space $\mathcal{P}$. A function $\lambda(\mathbf{x},\mathbf{p})$ on the phase space $\mathcal{P}$ that does not explicitly depend on time and whose Poisson bracket with the Hamiltonian vanishes generates an (infinitesimal) canonical transformation Λ_ε:

$$\Lambda_\varepsilon(\mathbf{x})=\mathbf{x}+\varepsilon[\mathbf{x},\lambda]_P+O(\varepsilon^2)=\mathbf{x}+\varepsilon\nabla_{\mathbf{p}}\lambda+O(\varepsilon^2),$$
$$\Lambda_\varepsilon(\mathbf{p})=\mathbf{p}+\varepsilon[\mathbf{p},\lambda]_P+O(\varepsilon^2)=\mathbf{p}-\varepsilon\nabla_{\mathbf{x}}\lambda+O(\varepsilon^2), \tag{C9}$$
$$\Lambda_\varepsilon(H)=H=E$$

on the energy surface $H(\mathbf{x},\mathbf{p})=E$. Here ε is an infinitesimal parameter and the Poisson bracket $[F,G]_P$ of two functions F and G on the phase space is as usual

$$[F,G]_P:=\frac{\partial F}{\partial\mathbf{x}}\cdot\frac{\partial G}{\partial\mathbf{p}}-\frac{\partial G}{\partial\mathbf{x}}\cdot\frac{\partial F}{\partial\mathbf{p}}. \tag{C10}$$

Such a canonical transformation will, however, generally also move the boundary $\partial\mathcal{P}$ of the phase space $\mathcal{P}$. One therefore requires that a canonical generator λ on the phase space $\mathcal{P}$ with boundary $\partial\mathcal{P}$ also satisfy

$$0=[\Phi,\lambda]_P|_{\Phi(\mathbf{x},\mathbf{p})=0}. \tag{C11}$$

Note that this requirement greatly restricts the canonical transformations we are considering. Of interest to us will be the special case where the bounded phase space and Hamiltonian possess some ''obvious'' continuous symmetries. The generators λ_a of the symmetry group $\mathcal{G}$ in this case form the basis of an r-dimensional Lie algebra:

$$\{\lambda_a,a=1,\dots,r:[\lambda_a,\lambda_b]_P=f_{ab}^c\lambda_c\} \tag{C12}$$

with structure constants f_{ab}^c. In the Casimir problem with two parallel plates of dimension $L\times L$ located at $z=0$ and $z=l$ with $L\gg l$, the ''obvious'' symmetry is that of translations parallel to the plates. It is generated by the momentum components p_x and p_y. In the Derjaguin problem, the boundary of phase space is axially symmetric with respect to the z axis. The time-independent symmetry is generated by the z component of angular momentum,

$$\lambda=(\mathbf{x}\times\mathbf{p})_z=x_xp_y-x_yp_x. \tag{C13}$$

In a rotationally symmetric spherical cavity, the symmetry generators are the components of angular momentum, etc.

An element $g\in\mathcal{G}$ of the group of canonical transformations generated by the λ_a of Eq. (C12) maps any periodic path γ_τ onto another path $\gamma_\tau^g=g\circ\gamma_\tau$ of the same period τ via

$$\gamma_\tau^g=g\circ\gamma_\tau:=\{(g\circ\mathbf{x}(t),g\circ\mathbf{p}(t)),t\in[0,\tau]\} \tag{C14}$$

and respects the boundary conditions (C1). Since $g\in\mathcal{G}$ has an inverse g^{-1} the index $i(\gamma^g)=i(\gamma)$ defined in Eq. (C6) does not change and the path γ_τ^g therefore belongs to the same equivalence class as γ_τ. The Lie group $\mathcal{G}$ thus induces an equivalence relation *within* each of the previous equivalence classes. One says that γ_τ^g is on the same *orbit* as γ_τ. The action as well as the measure $[d\mathbf{x}d\mathbf{p}]$ of the path integral are invariant under these canonical transformations, and in particular are invariant under infinitesimal canonical transformations generated by the λ_a's.

A semiclassical evaluation of the path integral (C7) is therefore plagued by zero modes, infinitesimal deformations of the path that do not change the action, which are a manifestation of this symmetry. The remedy is well known: It consists of choosing a particular representative on each orbit and then performing the integration along the orbit exactly. We choose the representative γ_σ of an orbit σ by demanding that it satisfy the subsidiary conditions

$$F_a[\gamma_\sigma;\sigma]=0,\quad a=1,\dots,r, \tag{C15}$$

where the F_a are functionals of the path. As indicated, the conditions (C15) may explicitly depend on the orbit σ. They must have a solution γ_σ on the orbit σ, which preferably is *unique*. In this case the $F_a[\gamma;\sigma]$ can be regarded as collective coordinates that give the position of the path on the orbit σ on which the action $S[\gamma]$ does not depend. The remainder of this appendix is then just the change of integration variables in the path integral from collective coordinates to parameters of the symmetry group.

For a semiclassical evaluation of Eq. (C7), however, it suffices that a *classical* periodic path on the orbit σ satisfy Eq. (C15) and that the Faddeev-Popov determinant (to be defined below) does not vanish at this solution. There are no gauge-equivalent classical paths in the immediate vicinity of γ_σ in this case, and a semiclassical evaluation of the fluctuations around this representative path is possible. We use the fact that the parameter space of the Lie group $\mathcal{G}$ is a metric space to write

$$N_\sigma = \int_{\mathcal{G}} Dg |\det M| \prod_{a=1}^{r} \delta(F_a[g\circ\gamma;\sigma]), \tag{C16}$$

where the integral is over the manifold of the parameter space of the invariance group $\mathcal{G}$ with the appropriate measure and N_σ is a positive *integer* that gives the number of times the subsidiary conditions (C15) are satisfied on the orbit. (Note that we take the *absolute value* of the determinant—N_σ is not the degree of the map, and vanishes only if the subsidiary conditions cannot be satisfied.) The $r\times r$ matrix M in Eq. (C16) has elements

$$M_{ab} = \left.\frac{\partial F_a}{\partial \varepsilon^b}\right|_\gamma \equiv \int_0^\tau \left(\frac{\delta F_a}{\delta \mathbf{x}(t)} \frac{\partial \mathbf{x}(t)}{\partial \varepsilon^b} + \frac{\delta F_a}{\delta \mathbf{p}(t)} \frac{\partial \mathbf{p}(t)}{\partial \varepsilon^b} \right)_\gamma$$
$$= \int_0^\tau \left(\frac{\delta F_a}{\delta \mathbf{x}(t)} \frac{\partial \lambda_b}{\partial \mathbf{p}} - \frac{\delta F_a}{\delta \mathbf{p}(t)} \frac{\partial \lambda_b}{\partial \mathbf{x}} \right)_\gamma \tag{C17}$$

and is the finite-dimensional analog of the Faddeev-Popov [24] matrix in gauge theories. It relates the change of the collective coordinates of γ to an infinitesimal change of the group parameters and the determinant of this matrix is just the Jacobian for the change of variables from the collective coordinates F_a to the group parameters [in the case where the solution of Eq. (C15) is unique and $N_\sigma = 1$]. The determinant of M needs to be known only at points that satisfy Eq. (C15) and we demand that the collective coordinates are chosen so that $\det M$ does not vanish for the classical periodic paths on the orbit σ. We insert Eq. (C16) in Eq. (C7) and use the fact that the action S, the measure $[d\mathbf{x}d\mathbf{p}]$, $\det M$, and N_σ are invariant under the action of the group $\mathcal{G}$. One can thus rewrite the response function as

$$g_{\rm osc}(E) = \frac{-2i}{\hbar}\left(\int_{\mathcal{G}} Dg\right)\int_0^\infty d\tau e^{i(E+i\varepsilon)\tau/\hbar}\int_{\gamma_\tau \notin \{0\}} [d\mathbf{x}d\mathbf{p}] N_\sigma^{-1} |\det M| \prod_{a=1}^{r} \delta[F_a(\gamma_\tau;\sigma)] e^{iS[\gamma_\tau]/\hbar}, \tag{C18}$$

that is, as a path integral over periodic representative paths γ_σ of each orbit σ that satisfy the constraints (C15). The volume $V_{\mathcal{G}} = \int_{\mathcal{G}} Dg$ of the symmetry group from the integration over the orbit has been separated, and one may proceed to a semiclassical evaluation of the path integral in Eq. (C18). As is shown in Appendix D for the Casimir problem with two plates, a judicious choice of the subsidiary conditions (or collective coordinates) F_a often allows for a semiclassical evaluation of Eq. (C18) in terms of a semiclassical Green function. Using the Fourier representation of the δ function, Eq. (C18) can also be written in the form of a path integral over an enlarged phase space supplemented by Lagrange multipliers s^a,

$$g_{\rm osc}(E) = \frac{-2i}{\hbar}\left(\int_{\mathcal{G}} Dg\right)\int_0^\infty d\tau e^{i(E+i\varepsilon)\tau/\hbar}\int_{\gamma_\tau \notin \{0\}} [d\mathbf{x}d\mathbf{p}] N_\sigma^{-1} |\det M| \left(\prod_{a=1}^{r}\int_{-\infty}^{\infty}\frac{ds^a}{2\pi\hbar}\right) e^{i(S[\gamma_\tau]+s^aF_a[\gamma_\tau;\sigma])/\hbar}. \tag{C19}$$

It is consistent with the semiclassical approximation to evaluate all the integrals in Eq. (C19) by the method of stationary phase and consider only quadratic fluctuations around classical periodic paths γ_σ that are representatives of the orbit σ that satisfy the subsidiary conditions (C15).

APPENDIX D: THE TWO-PLATE CASIMIR PROBLEM REVISITED

The boundaries at $z=0$ and $z=l$ of the Casimir problem are described by the constraint

$$\Phi(\mathbf{x},\mathbf{p}) = z(l-z) = 0. \tag{D1}$$

To avoid the edge effects of two finite plates and for conceptual clarity we impose periodic boundary conditions in the x and y directions of period $L \gg l$. The symmetry of the problem in this case is that of the translation group on a two-dimensional torus. The corresponding canonical transformations are generated by the momenta p_x and p_y, whose Poisson brackets with the constraint (D1) vanish. Since

$$\lambda_1 = p_x, \quad \lambda_2 = p_y, \quad \text{and} \quad [\lambda_1,\lambda_2]_P = [p_x,p_y]_P = 0, \tag{D2}$$

there are no additional generators and the symmetry group $\mathcal{G}$ of translations parallel to the plates is two-dimensional, i.e., $r=2$ in this case. Periodic paths that are the same up to a translation parallel to the plates are thus equivalent and belong to the same orbit. We choose the representative of such an orbit to be the path whose initial point $(\mathbf{x}(0),\mathbf{p}(0))$ is on the z axis and select it by imposing the subsidiary conditions

$$F_1[\gamma] = x(0) = 0,$$
$$F_2[\gamma] = y(0) = 0. \tag{D3}$$

Note that the subsidiary conditions (D3) can be chosen independent of the orbit σ, since a representative periodic path satisfying them can be found for any orbit.

A group element $g(d\mathbf{a}_\perp)\in\mathcal{G}$ in the vicinity of the identity depends on two infinitesimal parameters da_1 and da_2, $d\mathbf{a}_\perp=(da_1,da_2,0)$. The corresponding infinitesimal canonical transformation $\Lambda_{d\mathbf{a}_\perp}$ is generated by

$$\lambda_1 da_1+\lambda_2 da_2=\mathbf{p}\cdot d\mathbf{a}_\perp, \tag{D4}$$

and effects an infinitesimal translation $d\mathbf{a}_\perp$ parallel to the plates

$$\Lambda_{d\mathbf{a}_\perp}(\mathbf{x})=\mathbf{x}+[\mathbf{x},d\mathbf{a}_\perp\cdot\mathbf{p}]_P=\mathbf{x}+d\mathbf{a}_\perp, \quad \Lambda_{d\mathbf{a}_\perp}(\mathbf{p})=\mathbf{p}, \quad \Lambda_{d\mathbf{a}_\perp}(H)=H, \tag{D5}$$

on the energy surface $H=E$.

Due to the periodic boundary conditions, a translation in the x or y direction by L is equivalent to no translation in that direction at all. The group element corresponding to such a translation by L is thus the identity. The parameter space of $\mathcal{G}$ is therefore topologically a symmetric two-dimensional torus $\mathcal{T}_2$ of dimension $L\times L$. The metric on this parameter space of the Lie group $\mathcal{G}$ is flat, because $\mathcal{G}$ is Abelian, and the manifold of the parameter space of the group $\mathcal{G}$ is thus isomorphic to $\mathcal{T}_2$. The volume of the translation group $\mathcal{G}$, that of a two-dimensional symmetric torus $\mathcal{T}_2$ of dimension $L\times L$, is just the area of the plates

$$V_{\mathcal{G}}=\int_{\mathcal{G}}Dg=\int_{\mathcal{T}_2}da_1da_2=L^2. \tag{D6}$$

The Faddeev-Popov matrix M of the two-plate Casimir problem with the subsidiary conditions (D3) is the 2 by 2 unit matrix

$$M_{ij}=\frac{\partial F_i}{\partial a_j}=\delta_{ij}\Rightarrow \det M=1 \tag{D7}$$

with unit determinant. Although obvious, one may also prove that the representative γ_σ chosen by the subsidiary conditions (D3) is unique: since the change of the subsidiary conditions by a group element infinitesimally close to the unit element of $\mathcal{G}$ is given by the matrix M and $\det M=1>0$ *for any* value of the subsidiary conditions, there is only one solution $g\in\mathcal{G}$ to $F_1[g\circ\gamma]=F_2[g\circ\gamma]=0$ for any path γ. The multiplicity constant N_σ is thus

$$N_\sigma=1, \tag{D8}$$

for any orbit σ.

Inserting Eqs. (D6), (D8), (D7), and (D3) in Eq. (C18), we obtain for the oscillating response function $g_{\rm osc}^{\text{2-plate}}(E)$ of the two-plate Casimir problem

$$g_{\rm osc}^{\text{2-plate}}(E)=\frac{-2iL^2}{\hbar}\int_0^\infty d\tau e^{i(E+i\varepsilon)\tau/\hbar}\times\int_{\gamma_\tau\notin\{0\}}[d\mathbf{x}d\mathbf{p}]\delta(x(0))\delta(y(0))e^{iS[\gamma_\tau]/\hbar}. \tag{D9}$$

Only periodic paths that begin and end on the z axis and that do not belong to the trivial equivalence class contribute to the path integral in Eq. (D9). To evaluate Eq. (D9) semiclassically, observe that

$$\frac{-i}{\hbar}\int d\mathbf{x}(0)\,\delta(x(0))\delta(y(0))\int_0^\infty d\tau e^{i(E+i\varepsilon)\tau/\hbar}\int_{\gamma_\tau\notin\{0\}}\frac{d\mathbf{p}(0)}{(2\pi\hbar)^3}\times\int[d\mathbf{x}d\mathbf{p}]e^{iS[\gamma_\tau]/\hbar}=\int_0^l dzG(z,z;E)\big|_{\gamma\notin\{0\}} \tag{D10}$$

is an integral over z of the Green function $G(z,z;E)$ from a point on the z axis to the same point, where the contribution from periodic paths in the trivial equivalence class has been excluded. The semiclassical approximation to this Green function was obtained above and is given by the first term in Eq. (2.18).

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[13] See, for example, L.E. Reichl, *The Transition to Chaos* (Springer, New York, 1992), Chap. 8 and references therein.

[14] The semiclassical approximation to a Green function is accurate at least to order $\hbar$. For a photon in free space the semiclassical approximation to $G_0(\mathbf{x},\mathbf{y};\tau)$ and to $\mathrm{Im}\, G_0(\mathbf{x},\mathbf{y};E)$ happens to be exact, but it is accurate only to first order in $\hbar$ for $\mathrm{Re}\, G_0(\mathbf{x},\mathbf{y};E)$. The saddle-point approximation to the Laplace transform of $G_0(\mathbf{x},\mathbf{y};\tau)$ is exact only for $\mathrm{Im}\, G_0(\mathbf{x},\mathbf{y};E)$, since it is proportional to a δ function, i.e., a Gaussian of arbitrarily small width.

[15] Note that, not surprisingly, Eq. (2.18) has a one-dimensional form. On setting $\mathbf{x}_i = \mathbf{x}_f$, the perpendicular components in Eq. (2.16) drop out but, because of reflections (or images), the z components do not. Integration over $\mathbf{x}_\perp$ is therefore trivial.

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