Path integrals on a flux cone

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This paper considers the Schrödinger propagator on a cone with the conical singularity carrying magnetic flux (''flux cone''). Starting from the operator formalism, and then combining techniques of path integration in polar coordinates and in spaces with constraints, the propagator and its path integral representation are derived. The approach shows that effective Lagrangian contains a quantum correction term and that configuration space presents features of nontrivial connectivity. [S1050-2947(98)02707-3]

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I. INTRODUCTION

Quantum mechanics on cones has been shown to be a fruitful model for studying the interplay between quantum mechanics and geometry. The nearly trivial geometry of the cone (curvature is concentrated at a single point, the conical singularity [1,2], and the result is that the geometry is Euclidean everywhere except on a ray that starts at the singularity [3]) is responsible for Aharonov-Bohm (AB)-like effects that have been discovered throughout the years [3–7]. Such findings can be used in the study of various (real) quantum systems whose backgrounds can be regarded as being conical with good approximation. Quantum matter around cosmic strings and black holes and statistical mechanics of identical particles in two dimensions are examples.

In this paper a path-integral representation for the propagator of the Schrödinger equation is derived from the operator formalism on the cone. A magnetic flux is allowed to run through the cone axis, so that one has an AB setup coupled with the conical geometry. The method contrasts with the one in the literature where path-integral representations in spaces with a singular point are obtained by angular decomposition of the Feynman prescription in Cartesian coordinates, and by assuming a nonsimple connectivity of the configuration space [8–12]. In the present approach, instead, topological features arise naturally.

The paper is organized as follows. In Sec. II the background is briefly discussed (for more detailed accounts, see Ref. [3], and references therein). In Sec. III, path-integral prescription (and propagator) is derived by breaking the evolution operator up into an infinite product of short-time evolution operators, and then inserting completeness relations for configuration-space eigenstates, whose orthonormality relation is expressed in terms of stationary states. (Such a procedure is straightforward in Euclidean space, but rather elaborate in nontrivial backgrounds [12].) Topological features are identified in the resulting expression. The paper closes with final remarks.

II. BACKGROUND

A cone is obtained from the Euclidean plane by removing a wedge of angle $2\pi(1-\alpha)$ (in fact, when $\alpha>1$, a wedge is inserted). Clearly, the line element is given by

$$dl^2=d\rho^2+\rho^2 d\varphi^2, \tag{1}$$

which is the line element of the Euclidean plane written in polar coordinates. The fact that there is a δ function curvature at the origin is encoded in the unusual identification

$$(\rho,\varphi)\sim(\rho,\varphi+2\pi\alpha). \tag{2}$$

The behavior of a free particle with mass M on a cone is determined from the Lagrangian,

$$\mathcal{L}=\tfrac{1}{2}M(dl/dt)^2$$
$$=\tfrac{1}{2}M(\dot{\rho}^2+\rho^2\dot{\varphi}^2). \tag{3}$$

Noting Eq. (2), it follows that orbits of particles (geodesic motion on the cone) are simply broken straight lines with uniform motion. As a constant magnetic flux Φ running through the cone axis does not affect classical motion of a particle (with charge e), then classical motion on a flux cone is nearly trivial. Quantum motion, on the other hand, reveals nontrivial features [13].

III. THE PROPAGATOR AND ITS PATH-INTEGRAL REPRESENTATION

Due to local flatness of the conical geometry the free Hamiltonian operator is just the free Hamiltonian operator on the plane,

$$H=-\frac{\hbar^2}{2M}\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right)+\frac{L^2}{2M\rho^2}, \tag{4}$$

where $L:=-i\hbar\partial/\partial\varphi$. By choosing an appropriate gauge (the one corresponding to a vector potential, which vanishes everywhere, except on a ray) and observing Eq. (2), it follows that solutions of the Schrödinger equation satisfy [3]

$$\psi(\rho,\varphi+2\pi\alpha)=\exp\{i2\pi\sigma\}\psi(\rho,\varphi), \tag{5}$$

with $\sigma:=-e\Phi/ch$. Boundary condition (5) carries all information about the nontrivial geometry and magnetic field.

Consider the following effective Lagrangian:

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$$\mathcal{L}_{eff}=\frac{M}{2}(\dot{\rho}^2+\rho^2\dot{\varphi}^2)+\frac{\hbar^2}{8M\rho^2}, \tag{6}$$

which is obtained from Eq. (3) by adding a quantum correction. The corresponding Hamiltonian is given by

$$\mathcal{H}_{eff}=\frac{1}{2M}\left(p_\rho^2+\frac{p_\varphi^2}{\rho^2}-\frac{\hbar^2}{4\rho^2}\right). \tag{7}$$

The momentum operators associated with p_ρ and with p_φ are given by

$$p_\rho\to -i\hbar\left(\partial_\rho+\frac{1}{2\rho}\right) \quad p_\varphi\to L, \tag{8}$$

where the presence of the term $-i\hbar/2\rho$ ensures self-adjointness of p_ρ (if the wave functions do not diverge very rapidly at $\rho=0$ [3]), without spoiling the usual canonical commutation relations [12,14]. It turns out that by performing the substitutions (8) in (7), the Hamiltonian operator (4) is reproduced, which obviously would not be the case if the quantum correction was not present in Eq. (7) [15]. The effective Lagrangian (6) will be considered again below.

One seeks stationary states that span a space of wave functions where conservation of probability holds. This implies that the singularity at the origin must not be a source or a sink,

$$\lim_{\rho\to 0}\int_0^{2\pi\alpha} d\varphi\,\rho\,\mathcal{J}_\rho=0, \tag{9}$$

where $\mathcal{J}_\rho$ is the usual expression for the radial component of the probability current on the plane. Condition (9) is automatically guaranteed if the stationary states are finite at the origin. (Mildly divergent boundary conditions can be equally compatible with conservation of probability and square integrability of the wave function [16,17,3]. These possibilities will not be considered here.) Functions

$$\psi_{k,m}(\rho,\varphi)=\langle\rho,\varphi|k,m\rangle=\frac{1}{\sqrt{2\pi\alpha}}J_{|m+\sigma|/\alpha}(k\rho)e^{i(m+\sigma)\varphi/\alpha}, \tag{10}$$

where $0\leqslant k<\infty$, m is an integer and J_ν denotes a Bessel function of the first kind, are simultaneous eigenfunctions of H and L with eigenvalues $\hbar^2k^2/2M$ and $(m+\sigma)\hbar/\alpha$, respectively. Note that since $J_\nu(0)$ is finite for nonnegative ν, these stationary states are finite at the origin.

States $|\rho,\varphi\rangle$ in Eq. (10) are a complete set of configuration-space eigenstates

$$\int_0^\infty d\rho\,\rho\int_0^{2\pi\alpha}d\varphi|\rho,\varphi\rangle\langle\rho,\varphi|=\mathbf{1}. \tag{11}$$

Since their orthonormality relation is

$$\langle\rho,\varphi|\rho',\varphi'\rangle=\frac{1}{\rho}\delta(\rho-\rho')\delta(\varphi-\varphi'),$$

it follows that

$$\langle\rho,\varphi|\rho',\varphi'\rangle=\sum_{m=-\infty}^{\infty}\int_0^\infty dk\,k\,\psi_{k,m}(\rho,\varphi)\psi^*_{k,m}(\rho',\varphi'). \tag{12}$$

This expression is the completeness relation of the eigenfunctions $\psi_{k,m}$,

$$\sum_{m=-\infty}^{\infty}\int_0^\infty dk\,k|k,m\rangle\langle k,m|=\mathbf{1},$$

which may be derived by using the completeness relation of the Bessel functions

$$\int_0^\infty dk\,kJ_\nu(k\rho)J_\nu(k\rho')=\frac{1}{\rho}\delta(\rho-\rho'), \tag{13}$$

with Poisson's formula

$$\sum_{m=-\infty}^{\infty}\delta(\phi+2\pi m)=\frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\exp\{im\phi\}. \tag{14}$$

Expression (12) corresponds to the usual one in Cartesian coordinates where $\langle x|x'\rangle$ is expressed in terms of plane waves $\langle x|x'\rangle=\int(dk/2\pi)\exp\{ik(x-x')\}$. Recall that plane waves are simultaneous eigenfunctions of the free Hamiltonian and linear momentum operators, whereas $\psi_{k,m}(\rho,\varphi)$ are simultaneous eigenfunctions of the free Hamiltonian and angular momentum operators.

The orthonormality relation for the eigenfunctions $\psi_{k,m}$,

$$\langle k,m|k',m'\rangle=\int_0^\infty d\rho\,\rho\int_0^{2\pi\alpha}d\varphi\,\psi^*_{k,m}(\rho,\varphi)\psi_{k',m'}(\rho,\varphi)$$
$$=\frac{1}{k}\delta(k-k')\delta_{mm'}, \tag{15}$$

follows from the orthonormality relation

$$\int_0^{2\pi\alpha}d\varphi\exp\{i\varphi(m-n)/\alpha\}=2\pi\alpha\delta_{mn}, \tag{16}$$

and Eq. (13).

For a complete set of configuration-space eigenstates $|\rho,\varphi\rangle$ the propagator of the Schrödinger equation is given by $K(\rho,\varphi;\rho',\varphi';\tau)=\langle\rho,\varphi|U(\tau)|\rho',\varphi'\rangle$, where $U(\tau)=\exp\{-iH\tau/\hbar\}$ is the evolution operator and $\tau=t-t'$ is the time interval. Slicing τ into $N+1$ slices of width $\epsilon=\tau_n=t_n-t_{n-1}=\tau/(N+1)$, the propagator reads

$$K(\rho,\varphi;\rho',\varphi';\tau)=\langle\rho,\varphi|\prod_{n=1}^{N+1}U(\tau_n)|\rho',\varphi'\rangle, \tag{17}$$

where the composition law of the evolution operator was used with the identifications $t\equiv t_{N+1}$ and $t'\equiv t_0$. By inserting into Eq. (17) N completeness relations (11) between each pair of evolution operators, one is led to

$$K(\rho,\varphi;\rho',\varphi';\tau)=\prod_{n=1}^{N}\left[\int_0^\infty d\rho_n\,\rho_n\int_0^{2\pi\alpha}d\varphi_n\right]\times\prod_{n=1}^{N+1}\left[\langle\rho_n,\varphi_n|U(\epsilon)|\rho_{n-1},\varphi_{n-1}\rangle\right]. \tag{18}$$

The identifications $|\rho,\varphi\rangle\equiv|\rho_{N+1},\varphi_{N+1}\rangle$ and $|\rho',\varphi'\rangle\equiv|\rho_0,\varphi_0\rangle$ were also used.

The short-time amplitudes in Eq. (18) may be rewritten as

$$\langle\rho_n,\varphi_n|U(\epsilon)|\rho_{n-1},\varphi_{n-1}\rangle=\langle\rho_n,\varphi_n|\rho_{n-1},\varphi_{n-1}\rangle-i\frac{\epsilon}{\hbar}H\langle\rho_n,\varphi_n|\rho_{n-1},\varphi_{n-1}\rangle+\mathcal{O}(\epsilon^2). \tag{19}$$

In order to obtain the action of H on $\langle\rho_n,\varphi_n|\rho_{n-1},\varphi_{n-1}\rangle$ one expresses the latter in terms of eigenfunctions of the former, i.e., Eq. (12) is considered. Then Eq. (19) is recast as

$$\langle\rho_n,\varphi_n|U(\epsilon)|\rho_{n-1},\varphi_{n-1}\rangle=\sum_{m=-\infty}^{\infty}\int_0^\infty dk\; ke^{-i\epsilon E_k/\hbar}\psi_{k,m}(\rho_n,\varphi_n)\psi^*_{k,m}(\rho_{n-1},\varphi_{n-1})+O(\epsilon^2), \tag{20}$$

where E_k denotes the eigenvalue of H, i.e., $\hbar^2k^2/2M$. By using Eq. (20) in Eq. (18) and taking the limit $N\to\infty$ ($\epsilon\to 0$), a partitioned expression for the propagator is obtained,

$$K(\rho,\varphi;\rho',\varphi';\tau)=\lim_{N\to\infty}\prod_{n=1}^{N}\left[\int_0^\infty d\rho_n\,\rho_n\int_0^{2\pi\alpha}d\varphi_n\right]\times\prod_{n=1}^{N+1}\left[\sum_{m=-\infty}^{\infty}\int_0^\infty dk\,ke^{-i\epsilon E_k/\hbar}\psi_{k,m}(\rho_n,\varphi_n)\times\psi^*_{k,m}(\rho_{n-1},\varphi_{n-1})\right]. \tag{21}$$

The integral over k in Eq. (21) may be evaluated by using the formula [18]

$$\int_0^\infty dx\,xe^{-ax^2}J_\nu(bx)J_\nu(cx)=(\tfrac{1}{2}a)e^{-(b^2+c^2)/4a}I_\nu(bc/2a), \tag{22}$$

where $\mathrm{Re}\,a>0$, $\mathrm{Re}\,\nu>-1$. This integral corresponds, in Cartesian coordinates, to the Gaussian integral. Analytic continuation of Eq. (22) gives

$$K(\rho,\varphi;\rho',\varphi';\tau)=\lim_{N\to\infty}\frac{M}{2\pi\alpha i\epsilon\hbar}\prod_{n=1}^{N}\left[\int_0^\infty d\rho_n\,\rho_n\times\int_0^{2\pi\alpha}\frac{d\varphi_n}{2\pi\alpha i\epsilon\hbar/M}\right]\prod_{n=1}^{N+1}\left[e^{iM(\rho_n^2+\rho_{n-1}^2)/2\hbar\epsilon}\times\sum_{m=-\infty}^{\infty}I_{|m+\sigma|/\alpha}(M\rho_n\rho_{n-1}/i\hbar\epsilon)e^{i(m+\sigma)(\varphi_n-\varphi_{n-1})/\alpha}\right]. \tag{23}$$

When σ is an integer and the space is Euclidean, i.e., $\alpha=1$, Eq. (23) reduces to Feynman's prescription for the propagator of a free particle. Indeed, by considering the Fourier expansion of a plane wave,

$$\exp\{ia\cos\phi\}=\sum_{m=-\infty}^{\infty}I_{|m|}(ia)e^{im\phi}, \tag{24}$$

one sees from Eq. (23) the familiar partitioned expression

$$K_0(\mathbf{x},\mathbf{x}';\tau)=\lim_{N\to\infty}\frac{M}{2\pi i\epsilon\hbar}\prod_{n=1}^{N}\left[\int\frac{d^2x_n}{2\pi i\epsilon\hbar/M}\right]\times\exp\left\{\frac{i}{\hbar}\sum_{n=1}^{N+1}\epsilon\frac{M}{2}\left(\frac{\mathbf{x}_n-\mathbf{x}_{n-1}}{\epsilon}\right)^2\right\}, \tag{25}$$

which is symbolically written as

$$K_0(\mathbf{x},\mathbf{x}';\tau)=\int\mathcal{D}^2x\;\exp\left\{\frac{i}{\hbar}\int_{t'}^{t}dt\;\frac{M}{2}\dot{\mathbf{x}}^2\right\}. \tag{26}$$

Before rewriting the path-integral representation (23) in a symbolic form that is analogous to Eq. (26), the expression for the propagator on the flux cone that was obtained in Ref. [4], using a complex contour method, will be reproduced here from Eq. (21). (References [6,11] have also reproduced this propagator, when $\sigma=0$, using other methods. Reference [11] in particular has used a path-integral approach that is a generalization for the cone of the method used in the AB set up [9]. The propagator, when $\alpha=1$, has been well known in the literature [19].) Observing Eq. (15), it is seen that only one sum over m and one integration over k remain in Eq. (21),

$$K(\rho,\varphi;\rho',\varphi';\tau)=\frac{1}{2\pi\alpha}\int_0^\infty dk\,ke^{-i\tau E_k/\hbar}\times\sum_{m=-\infty}^{\infty}J_{|m+\sigma|/\alpha}(k\rho)J_{|m+\sigma|/\alpha}(k\rho')e^{i(m+\sigma)(\varphi-\varphi')/\alpha}. \tag{27}$$

Then, using Eq. (22) to evaluate the integration over k, results in

$$K(\rho,\varphi;\rho',\varphi';\tau) = \frac{M}{2\pi\alpha i\tau\hbar} e^{iM(\rho^2+\rho'^2)/2\hbar\tau} \times \sum_{m=-\infty}^{\infty} I_{|m+\sigma|/\alpha}(M\rho\rho'/i\hbar\tau) e^{i(m+\sigma)(\varphi-\varphi')/\alpha}, \tag{28}$$

which could have been guessed from Eq. (23). From Eq. (24) it follows that when σ is an integer and $\alpha=1$, Eq. (28) collapses into the free Schrödinger propagator on the Euclidean plane, viz.

$$K_0(\mathbf{x},\mathbf{x}';\tau) = \frac{M}{2\pi i\tau\hbar} e^{iM(\mathbf{x}-\mathbf{x}')^2/2\hbar\tau}.$$

Noting that $\int_{-\infty}^{\infty} d\lambda I_\lambda(z)\,\delta(\lambda-\nu)\exp\{i\lambda\phi\} = I_\nu(z)\exp\{i\nu\phi\}$, and using Eq. (14), Eq. (28) becomes

$$K(\rho,\varphi;\rho',\varphi';\tau) = \sum_{l=-\infty}^{\infty} e^{-i2\pi l\sigma} \tilde{K}(\rho,\varphi+2\pi\alpha l;\rho',\varphi';\tau), \tag{29}$$

with

$$\tilde{K}(\rho,\varphi;\rho',\varphi';\tau) := \frac{M}{2\pi i\tau\hbar} e^{iM(\rho^2+\rho'^2)/2\hbar\tau} \times \int_{-\infty}^{\infty} d\lambda\, I_{|\lambda|}(M\rho\rho'/i\hbar\tau) e^{i\lambda(\varphi-\varphi')}. \tag{30}$$

Likewise, Eq. (23) may be rewritten as

$$K(\rho,\varphi;\rho',\varphi';\tau) = \lim_{N\to\infty} \frac{M}{2\pi i\epsilon\hbar} \times \prod_{n=1}^{N} \left[\int_0^\infty d\rho_n \rho_n \int_0^{2\pi\alpha} \frac{d\varphi_n}{2\pi i\epsilon\hbar/M} \right] \times \prod_{n=1}^{N+1} \left[\sum_{l=-\infty}^{\infty} e^{-i2\pi l\sigma} e^{iM(\rho_n^2+\rho_{n-1}^2)/2\hbar\epsilon} \times \int_{-\infty}^{\infty} d\lambda\, I_{|\lambda|}(M\rho_n\rho_{n-1}/i\hbar\epsilon) \times e^{i\lambda(\varphi_n-\varphi_{n-1}+2\pi\alpha l)} \right]. \tag{31}$$

Now, by using

$$\sum_{k,l=-\infty}^{\infty} e^{(k+l)z} \int_0^c dx\, f(kc+x) g(lc-x) = \sum_{l=-\infty}^{\infty} e^{lz} \int_{-\infty}^{\infty} dx\, f(x) g(lc-x),$$

one may extend the range of integration of φ from $(0,2\pi\alpha)$ to $(-\infty,\infty)$. This leaves only one sum in Eq. (31), leading to Eq. (29), but now $\tilde{K}(\rho,\varphi+2\pi\alpha l;\rho',\varphi';\tau)$ is given as a partitioned expression

$$\tilde{K}(\rho,\varphi+2\pi\alpha l;\rho',\varphi';\tau) = \lim_{N\to\infty} \frac{M}{2\pi i\epsilon\hbar} \prod_{n=1}^{N} \left[\int_0^\infty d\rho_n\, \rho_n \int_{-\infty}^{\infty} \frac{d\varphi_n}{2\pi i\epsilon\hbar/M} \right] \times \prod_{n=1}^{N+1} \left[e^{iM(\rho_n^2+\rho_{n-1}^2)/2\hbar\epsilon} \times \int_{-\infty}^{\infty} d\lambda\, I_{|\lambda|}(M\rho_n\rho_{n-1}/i\hbar\epsilon) e^{i\lambda(\varphi_n-\varphi_{n-1}+2\pi\alpha l\delta_{n,N+1})} \right]. \tag{32}$$

Now the asymptotic behavior of $I_\nu(z)$ for large $|z|$ can be used to derive [9]

$$\int_{-\infty}^{\infty} d\lambda\, I_{|\lambda|}(z)\exp\{i\lambda\phi\} \approx \exp\{z+1/8z - z\phi^2/2\},$$

which when used in Eq. (32) finally gives

$$\tilde{K}(\rho,\varphi+2\pi\alpha l;\rho',\varphi';\tau) = \lim_{N\to\infty} \frac{M}{2\pi i\epsilon\hbar} \prod_{n=1}^{N} \left[\int_0^\infty d\rho_n\, \rho_n \int_{-\infty}^{\infty} \frac{d\varphi_n}{2\pi i\epsilon\hbar/M} \right] \times \exp\left\{ \frac{i}{\hbar} \sum_{n=1}^{N+1} \epsilon \left(\frac{M}{2} \left[\left(\frac{\rho_n-\rho_{n-1}}{\epsilon} \right)^2 + \rho_n\rho_{n-1} \left(\frac{\varphi_n+2\pi\alpha l\delta_{n,N+1}-\varphi_{n-1}}{\epsilon} \right)^2 \right] + \frac{\hbar^2}{8M\rho_n\rho_{n-1}} \right) \right\}, \tag{33}$$

or symbolically,

$$\tilde{K}(\rho,\varphi+2\pi\alpha l;\rho',\varphi';\tau) = \int_0^\infty \mathcal{D}\rho\, \rho \int_{-\infty}^{\infty} \mathcal{D}\varphi \exp\left\{ \frac{i}{\hbar} \int_{t'}^{t} dt \left[\frac{M}{2}(\dot{\rho}^2+\rho^2\dot{\varphi}^2) + \frac{\hbar^2}{8M\rho^2} \right] \right\}. \tag{34}$$

IV. FINAL REMARKS

Expressions (29) and (34) are the path-integral prescriptions where the corresponding action is the one made up of the effective Lagrangian $\mathcal{L}_{eff}$, (6). Recall that $\mathcal{L}_{eff}$ is the appropriate Lagrangian for quantization through the ‘‘substitution principle’’ (8). It is important to note that a naïve change from Cartesian to polar coordinates in the Feynman prescription (26) does not lead to Eq. (34), since the ‘‘quantum correction'' $\hbar^2/8M\rho^2$ would be missing. (Quantum corrections such as this one are typical of path integrals in nontrivial backgrounds [14]). This is a simple example showing that coordinate transformations within path-integral representations raise subtle issues.

Examining expressions (29) and (34) leads to the following interpretation of this path-integral representation. Since there is a conical singularity and/or a magnetic flux at the origin, the configuration space is not simply connected. The propagator is given by a sum of modulated propagators, each one of them giving the contribution of all paths belonging to a homotopy class labeled by the winding number l. Then the sum over l in Eq. (29) takes into account all paths circling around the ''hole'' at the origin. The modulated factors are a unitary representation of the fundamental group Z, and the particle travels in the covering space of $R^2-\{0\}$. The particle is not free, but interacts with the ''nontrivial'' topology through the quantum correction in the effective Lagrangian $\mathcal{L}_{eff}$.

Recalling a study of quantum flow in Ref. [3], one sees that this interpretation may be appropriate when $\alpha<1$ and/or σ is a noninteger. But, strictly speaking, it is incorrect when $\alpha\geqslant 1$ and σ is an integer. In particular, when $\alpha=1$ and $\sigma=0$, Eqs. (29) and (34) are just polar coordinate path-integral prescriptions for a free particle moving on the Euclidean plane—the apparent nontrivial topology is imparted by the use of polar coordinates that are singular at the origin.

In principle, the material in this paper may be reconsidered in the context of other possible boundary conditions at the singularity. The result of such an investigation might reveal different features from the ones seen here. Proceeding as in Sec. III, the crucial point would be the use of new stationary states to obtain the new propagators and their corresponding path-integral representations. This procedure seems to answer a question in Ref. [16], namely, how different boundary conditions at the singularity are related to the path-integral approach. The use of the present method in the context of other geometries is also worth investigating.

Using the proper time representation for the Green functions, the extension of the method to second quantization is straightforward. It would be interesting to investigate the connections between this paper and Ref. [20], where path integrals in a black-hole background are considered.

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