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Bures distance between two displaced thermal states

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The Bures distance between two displaced thermal states and the corresponding geometric quantities (statistical metric, volume element, scalar curvature) are computed. Under nonunitary (dissipative) dynamics, the statistical distance shows the same general features previously reported in the literature by Braunstein and Milburn for two-state systems. The scalar curvature turns out to have new interesting properties when compared to the curvature associated with squeezed thermal states. [S1050-2947(98)08408-X]

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In elementary quantum mechanics, there is a natural Riemannian structure defined on the projective Hilbert space (the space of ''rays''): a distance between two rays and a corresponding metric (the Fubini-Study metric) can be obtained as the smallest transition probability between any two vector states belonging to each of the two rays. The geometrical structure of mixed (impure) states can be revealed by applying the same idea this time for the purifications of two such states (a purification of a mixed state ρ is a pure state in an extended Hilbert space having ρ as the reduced density matrix). The result is the so-called Bures (or statistical) distance and metric.

However, due to the mathematical difficulties in computing this metric, few concrete results have been found in the geometry of mixed (impure) quantum states. The Bures distance and the metric have been computed for the spin-$\frac{1}{2}$ system [1] and for the spin-1 system [2]. Recently the Bures distance between two undisplaced thermal squeezed states was obtained [3]. This is a remarkable result because it is the first example of this type in an infinite dimensional Hilbert space. A general formula for the transition probability between any impure state and a pure squeezed state was obtained in [4]. A class of thermal states that is not treated in [3] is that of displaced thermal states, also called coherent thermal states [5]. The main results of this paper are the formulas for the transition probability and the Bures distance between two displaced thermal states.

The transition probability between two quantum states described by the density matrices ρ_1 and ρ_2 in the Hilbert space $\mathcal{H}$ is given by [6]

$$P(\rho_1,\rho_2)=(\mathrm{Tr}\sqrt{\sqrt{\rho_1}\rho_2\sqrt{\rho_1}})^2, \tag{1}$$

and the Bures distance is given by

$$D_B^2(\rho_1,\rho_2)=2[1-\sqrt{P(\rho_1,\rho_2)}]. \tag{2}$$

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The density matrix of a displaced thermal state is parametrized in the form

$$\rho(\beta,(p,q))=D(p,q)\rho(\beta)D(p,q)^\dagger, \tag{3}$$

where $\rho(\beta)=[1/Z(\beta)]\exp(-\beta H)$ with $H=\frac{1}{2}(P^2+Q^2)$, $[Q,P]=iI$, $Z(\beta)=\mathrm{Tr}\exp(-\beta H)=(2\sinh\beta/2)^{-1}$, and $D(p,q)=\exp i(pQ-qP)$. Here p and q are the displacements in momentum and coordinate and β is the inverse temperature. Then, as follows from [3], one has $\sqrt{\rho(\beta,(p,q))}=D(p,q)\sqrt{\rho(\beta)}D(p,q)^\dagger$. It is well known [6] that the transition probability is invariant under the unitary transformations in $\mathcal{H}$,

$$P(U\rho_1U^\dagger,U\rho_2U^\dagger)=P(\rho_1,\rho_2), \tag{4}$$

and is symmetric

$$P(\rho_1,\rho_2)=P(\rho_2,\rho_1). \tag{5}$$

Hence it suffices to compute $\mathrm{Tr}\sqrt{A^\dagger A}$ where $A=\sqrt{\rho(\beta_2,(p,q))}\sqrt{\rho(\beta_1)}$ and $p=p_2-p_1$, $q=q_2-q_1$. The following equation, proven in [7], allows the computation of $\sqrt{A^\dagger A}$:

$$\begin{aligned}&\exp\{\gamma_1[(P-\nu_1)^2+(Q-\omega_1)^2]\}\\&\quad\times\exp\{\gamma_2[(P-\nu_2)^2+(Q-\omega_2)^2]\}\\&\quad=\exp\{(\gamma_1+\gamma_2)[(P-\nu)^2+(Q-\omega)^2]+\theta I\},\end{aligned} \tag{6}$$

with

$$\theta=[(\nu_1-\nu_2)^2+(\omega_1-\omega_2)^2]f(\gamma_1,\gamma_2),$$

$$\nu=\nu_1g(\gamma_1,\gamma_2)+\nu_2g(\gamma_2,\gamma_1)-i(\omega_2-\omega_1)f(\gamma_1,\gamma_2),$$

and

$$\omega=\omega_1g(\gamma_1,\gamma_2)+\omega_2g(\gamma_2,\gamma_1)-i(\nu_1-\nu_2)f(\gamma_1,\gamma_2).$$

The following notations have been used

$$f(\gamma_1,\gamma_2)=\frac{\sinh\gamma_1\sinh\gamma_2}{\sinh(\gamma_1+\gamma_2)},\quad g(\gamma_1,\gamma_2)=\frac{\sinh\gamma_1\cosh\gamma_2}{\sinh(\gamma_1+\gamma_2)}. \tag{7}$$

Then

$$A=\frac{1}{\sqrt{Z(\beta_1)Z(\beta_2)}}\exp\left\{-\frac{(\beta_1+\beta_2)}{4}[(P-\xi)^2+(Q-\eta)^2]+\tau I\right\}, \tag{8}$$

with $\tau=-(p^2+q^2)f(\beta_1/4,\beta_2/4)$, $\xi=pg(\beta_2/4,\beta_1/4)-iqf(\beta_1/4,\beta_2/4)$ and $\eta=qg(\beta_2/4,\beta_1/4)+ipf(\beta_1/4,\beta_2/4)$. It follows that

$$A^\dagger=\frac{1}{\sqrt{Z(\beta_1)Z(\beta_2)}}\exp\left\{-\frac{(\beta_1+\beta_2)}{4}[(P-\overline{\xi})^2+(Q-\overline{\eta})^2]+\tau I\right\}, \tag{9}$$

and

$$A^\dagger A=\frac{1}{Z(\beta_1)Z(\beta_2)}\exp\left\{-\frac{(\beta_1+\beta_2)}{2}[(P-\tilde{p})^2+(Q-\tilde{q})^2]+(2\tau+\tilde{\tau})I\right\} \tag{10}$$

where

$$\tilde{\tau}=4(p^2+q^2)\frac{[\sinh(\beta_1/4)\sinh(\beta_2/4)]^2}{\sinh(\beta_1+\beta_2)/2}. \tag{11}$$

Because

$$A^\dagger A=\frac{1}{Z(\beta_1)Z(\beta_2)}\exp(2\tau+\tilde{\tau})D(\tilde{p},\tilde{q})\times\exp\left\{-\frac{(\beta_1+\beta_2)}{2}[P^2+Q^2]\right\}D(\tilde{p},\tilde{q})^\dagger, \tag{12}$$

it follows that

$$\sqrt{A^\dagger A}=\frac{1}{\sqrt{Z(\beta_1)Z(\beta_2)}}\exp\left(\tau+\frac{\tilde{\tau}}{2}\right)D(\tilde{p},\tilde{q})\times\exp\left\{-\frac{(\beta_1+\beta_2)}{4}[P^2+Q^2]\right\}D(\tilde{p},\tilde{q})^\dagger, \tag{13}$$

and

$$\mathrm{Tr}\sqrt{A^\dagger A}=\frac{Z[(\beta_1+\beta_2)/2]}{\sqrt{Z(\beta_1)Z(\beta_2)}}\exp\left(\tau+\frac{\tilde{\tau}}{2}\right). \tag{14}$$

The main result of the paper is

$$P(\rho(\beta_1),\rho(\beta_2,(p,q)))=\frac{Z[(\beta_1+\beta_2)/2]^2}{Z(\beta_1)Z(\beta_2)}\exp\left\{-\frac{Z(\beta_1+\beta_2)}{2Z(\beta_1)Z(\beta_2)}(p^2+q^2)\right\}. \tag{15}$$

Now, using Eq. (2), the statistical distance between any two displaced thermal states can be readily obtained.

When $\beta_2\to\infty$ the state $\rho(\beta_2,(p,q))$ becomes a coherent state and one reobtains the well-known result Eq. (4.4.15) in Ref. [8]. When both $\beta_1,\beta_2\to\infty$ one obtains also the correct result Eq. (4.4.7) in Ref. [8].

The Bures or statistical distance metric is obtained either by considering two states close to each other and making a Taylor expansion with respect to the infinitesimal parameters or, equivalently, as [3]

$$ds_B^2=g_{\mu\nu}dx^\mu dx^\nu \tag{16}$$

$$=\frac{1}{2}\frac{d^2}{dt^2}D_B^2(\rho(\beta),\rho(\beta+t\delta\beta,t\delta p,t\delta q))|_{t=0}, \tag{17}$$

which becomes in our case

$$ds_B^2=\frac{1}{2}\tanh\frac{\beta}{2}(dp^2+dq^2)+\frac{1}{16(\sinh\beta/2)^2}d\beta^2. \tag{18}$$

What this formula shows is that the square of the infinitesimal Bures distance consists of two parts: one given by the difference in the displacements p and q and the other of thermal origin. For $\beta\to\infty$ (pure states) we recover the Fubini-Study (Euclidean) metric $ds_{\mathrm{FS}}^2=\frac{1}{2}(dp^2+dq^2)$ Eq. (2.27) of Ref. [9] for coherent states. The thermal part, $\frac{1}{16}(\sinh\beta/2)^{-2}d\beta^2$, which is obtained from Eq. (18) for $dp=dq=0$, also appears in Twamley's formula [Eq. (29) of [3]] as a squeezing–independent, purely thermal contribution.

Under certain classes of Hamiltonian dynamics the temperature β may remain constant so only the first term of the right-hand side of Eq. (18) is responsible for changes in the statistical distance. But the temperature can be altered via nonunitary transformations. For example, a well-known model from quantum optics [10] for the damped quantum oscillator yields the following master equation of the statistical matrix $\rho\equiv\rho(\beta,(p,q))$:

$$\dot{\rho}=-i[\omega a^+a,\rho]+\gamma_\downarrow\{[a,\rho a^+]+[a\rho,a^+]\}+\gamma_\uparrow\{[a^+,\rho a]+[a^+\rho,a]\}, \tag{19}$$

with $a=(1/\sqrt{2})(Q+iP)$, $\gamma_\downarrow>\gamma_\uparrow\geqslant0$, $\omega>0$. With the notations $k\equiv\gamma_\downarrow-\gamma_\uparrow$ and $\beta_\infty\equiv\ln\gamma_\downarrow/\gamma_\uparrow$ it can be shown [10] that the parameters p, q, and β, which characterize the state $\rho(\beta,(p,q))$ at any moment t have the following behavior:

$$q_t=(q_0\cos\omega t+p_0\sin\omega t)e^{-kt}, \tag{20}$$

$$p_t=(-q_0\sin\omega t+p_0\cos\omega t)e^{-kt}, \tag{21}$$

$$\coth\frac{\beta_t}{2}=e^{-2kt}\coth\frac{\beta_0}{2}+(1-e^{-2kt})\coth\frac{\beta_\infty}{2}. \tag{22}$$

The rate of change in the statistical distance is then

$$\left(\frac{ds}{dt}\right)^2=\frac{1}{2}\tanh\frac{\beta_t}{2}(k^2+\omega^2)(q_t^2+p_t^2)+k^2\sinh^2\frac{\beta_t}{2}\left(\coth\frac{\beta_t}{2}-\coth\frac{\beta_\infty}{2}\right)^2. \tag{23}$$

A similar quantity has been previously analyzed by Braunstein and Milburn for a two-state system under nonunitary dynamics [11]. They found that $(ds/dt)\to\infty$ at $t=0$ if the system is initially in a pure state. From Eq. (23) we see that if $\beta_0\to\infty$ (initial pure coherent state), then indeed $(ds/dt)_{t=0}\to\infty$. At $t\to\infty$, the thermal contribution of ds/dt vanishes, a feature that also appeared in [11]. The practical consequence of Braunstein and Milburn's results was an improvement in the accuracy of ''one-tick'' clocks; we can see now that their conclusions can be extended from two-dimensional to infinite-dimensional spaces.

The volume element is $dv=\frac{1}{8}\operatorname{sech}(\beta/2)dp\,dq\,d\beta$. It was noticed in the literature [12] that, since the Bures metric is proportional (up to a factor of four) to the statistical distinguishability metric [13], the volume element has the significance of a quantum analog of the (classical) Jeffrey's prior [12]. A similar quantity has been calculated and studied for finite-dimensional systems (spin $\frac{1}{2}$ and 1) and for squeezed thermal states [12].

The scalar curvature associated with the Riemannian metric (18) is found to be

$$R=-6+14\tanh^2\frac{\beta}{2}. \tag{24}$$

A new feature with respect to the result of [3] is the vanishing of the scalar curvature for $\beta=2\tanh^{-1}\sqrt{3/7}$ and any value of p and q. Also, for $\beta\to\infty$ the curvature is 8, so it does not diverge. This shows that, while ds is indeed a measure of the distinguishability of two states, R has a more complicated significance than that suggested in [3], characterizing locally a relation between not only two, but three (or more) states. Indeed, the connection between Bures distance and statistical distinguishability provides a natural partition of the three-dimensional Riemannian manifold defined by Eq. (18) into cells [3,11,13]: by imposing a maximum number of generalized measurements performed on identically prepared copies of the system, one can fix the dimensions of the cells to arbitrary (small) values. The manifold is therefore discretized into minimal distinguishable states (each such state being contained in a different cell). Now, roughly speaking, the curvature R is a local measure of the number of states (as defined by the above partition) equally distinguishable from a certain state $\beta,(p,q)$—R actually shows how big is the difference between the number of cells intersected by a sphere centered in $\beta,(p,q)$ and the corresponding number for a flat space. Also, one can see that this number depends only of temperature, and not of the other parameters of the state—a property that was also noticed for squeezed thermal states.

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