

Connection between the closeness of classical orbits and the factorization of the radial Schrödinger equation

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It is shown that the Runge-Lenz vector of a hydrogen atom is equivalent to the raising and lowering operators derived from the factorization of the radial Schrödinger equation. A similar situation exists for an isotropic harmonic oscillator. It is seen that there exists an intimate relation between the conserved quantities responsible for the closeness of classical orbits on the one side and the quantum-mechanical raising and lowering operators on the other side, which is physically connected with the dynamical symmetry of the system considered. Some discussions are made about the factorization of a one-dimensional system.
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I. INTRODUCTION

The factorization method of Schrödinger [1,2] was extended to address the radial Schrödinger equation of a particle in a central potential $V(r)$ [2–7]. It was shown [2,3,7] that only for two kinds of potential, the Coulomb potential and isotropic harmonic oscillator, can the radial Schrödinger equation be factorized. Particularly, it is noted [7] that for a hydrogen atom, directly from the factorization, we can derive only one kind of raising and lowering operator $A_{\pm}(l)$ (with selection rules $\Delta l = \pm 1$, $\Delta n_r = \mp 1$, $\Delta n = 0$, where l and n_r are the angular momentum and radial quantum number, and $n = l + n_r + 1$), but for an isotropic harmonic oscillator, we can derive two kinds of angular momentum and energy raising and lowering operators $A_{\pm}$ ($\Delta l = \pm 1$, $\Delta n_r = \mp 1$, $\Delta n = \mp 1$) and $B_{\pm}$ ($\Delta l = \pm 1$, $\Delta n_r = 0$, $\Delta n = \pm 1$, where $n = l + 2n_r$). This reminds us of the famous Bertrand theorem [8] in classical mechanics, which says: “the only central forces that result in close orbits for all bound particles are the inverse square law and Hooke’s law.” It is expected that the factorizability of radial Schrödinger equation may be intimately connected with the closeness of classical orbits.

In classical mechanics, the maximum number of functional independent conserved quantities of a closed Hamiltonian system with N degrees of freedom is $2N - 1$ [9]. A system with independent conserved quantities no fewer than N is called integrable [10]. An integrable classical system with $N + \Lambda$ independent conserved quantities ($0 \leq \Lambda \leq N - 1$) is called Λ -fold degenerate, and there exist Λ linear relations with integer coefficients between the N frequencies ω_i ($i = 1, 2, \dots, N$) of the system [11]. A classical system for $\Lambda = N - 1$ is called a completely degenerate system, and there remains only one independent frequency. For example, it is well known that for a particle in a central potential $V(r)$, apart from the Hamiltonian, the angular momentum $\mathbf{l}$ is also conserved, and the particle in a general central potential $V(r)$ is onefold degenerate and moves in a plane, but the planar orbits are in general not closed. However, for a classical particle in an attractive Coulomb potential [$V(r) = -\kappa/r$], the orbit is always closed for *any continuous* negative energy ($E < 0$) and positive angular momentum L , i.e., an ellipse, of

which the length of the semimajor axis is ($m = \kappa = 1$) $a = 1/2|E|$ and the eccentricity is $e = \sqrt{1 - 2|E|/L^2}$. The period of motion is $T = 1/\nu = \pi|E|^{-3/2}/\sqrt{2} = 2\pi a^{3/2}$ (Kepler’s law), where ν is the frequency. This is guaranteed by the existence of an additional conserved quantity—the Runge-Lenz vector, $\mathbf{a} = \mathbf{p} \times \mathbf{l} - \mathbf{r}/r$ [12,13]. In fact, the direction of $\mathbf{a}$ is just that of the major axis of the elliptic orbit and the magnitude of $\mathbf{a}$ is the eccentricity ($|\mathbf{a}| = e$). The existence of Runge-Lenz vector implies that the Coulomb potential has a higher dynamical symmetry O_4 than its geometric symmetry O_3 [12]. However, it is seen that $\mathbf{a} \cdot \mathbf{l} = 0$, and $\mathbf{a}^2 = 2Hl^2 + 1$, so the number of independent conserved quantities is 5, and the hydrogen atom is a completely degenerate system. A similar situation exists for an isotropic harmonic oscillator.

In Sec. II it will be shown that for a two-dimensional (2D) hydrogen atom, the Runge-Lenz vector itself is equivalent to the raising and lowering operators derived from the factorization of radial Schrödinger equation. In Sec. III, it will be shown that, for a 3D hydrogen atom, from the Runge-Lenz vector $\mathbf{a}$ and angular momentum $\mathbf{l}$, one can construct three kinds of raising and lowering operators ($\Delta n = 0$, $\Delta l = \pm 1$, $\Delta m = 0, \pm 1$), which are equivalent to the raising and lowering operators derived from the factorization of radial Schrödinger equation. In Sec. IV we will address isotropic harmonic oscillators, and in terms of the two kinds of raising and lowering operators one can construct the conserved quantities characterizing elliptic orbits. In Sec. V we discuss the factorizability of the Schrödinger equation for a 1D system, which has been investigated extensively in supersymmetric quantum mechanics. Finally, conclusions and summary are given in Sec. VI.

II. 2D HYDROGEN ATOM

A. Runge-Lenz vector and a simple algebraic approach to the eigenvalue problem

For a 2D hydrogen atom, the quantum version of the Runge-Lenz vector reads ($\hbar = m = e = 1$)

$$\mathbf{a} = \frac{1}{2}(\mathbf{p} \times \mathbf{l} - \mathbf{l} \times \mathbf{p}) - \mathbf{e}_\rho = \mathbf{p} \times \mathbf{l} - i\mathbf{p} - \mathbf{e}_\rho, \quad (1)$$

where $\mathbf{p} = p_x \mathbf{i} + p_y \mathbf{j}$, $\mathbf{l} = l_z \mathbf{k} = (xp_y - yp_x)\mathbf{k}$, $\rho = \sqrt{x^2 + y^2}$. It is easily shown that

$$[l_z, a_x] = ia_y, [l_z, a_y] = -ia_x, [a_x, a_y] = -i2\hbar l_z. \quad (2)$$

In the subspace spanned by the bound states with eigenenergy E ($E < 0$), a_x and a_y may be replaced by $A_x = \sqrt{-1/2E} a_x$, $A_y = \sqrt{-1/2E} a_y$. Let $A_z = l_z$, then

$$[A_\alpha, A_\beta] = i\varepsilon_{\alpha\beta\gamma} A_\gamma, \quad (3)$$

i.e., A_x , A_y , and A_z constitute a SO_3 Lie algebra, and the eigenvalue of $\mathbf{A}^2 = -\frac{1}{4} - 1/2E$ is $m_0(m_0+1)$, $m_0 = 0, 1, 2, \dots$. Therefore, the energy eigenvalue of a 2D hydrogen atom is

$$E = E_n = -1/2n^2, \quad n = (m_0 + 1/2) = 1/2, 3/2, 5/2, \dots \quad (4)$$

Alternatively, defining $a_\pm = a_x \pm ia_y$, it is easily shown that

$$[l_z, a_\pm] = \pm a_\pm, \quad (5)$$

$$a_- a_+ = \frac{H}{2} (2l_z + 1)^2 + 1, \quad (6)$$

i.e., $a_\pm$ are just the raising and lowering operators of the magnetic quantum number m . Let $|Em\rangle$ denote the eigenstate of (H, l_z) with eigenvalues (E, m) , then $a_\pm |Em\rangle$ are the eigenstates of H with energy E and the eigenstates of l_z with eigenvalues $(m \pm 1)$. For a given energy eigenvalue E , the allowed $|m|$ must have an upper limit m_0 , and $a_+ |Em_0\rangle = 0$. Hence $a_- a_+ |Em_0\rangle = 0$. Using Eq. (6), we get $(E/2)(2m_0+1)^2 = 1$, which is just Eq. (4). The degenerate states belonging to E_n may be expressed as $a_-^k |E_n m_0\rangle$, $k = 0, 1, \dots, 2m_0$, and the degeneracy is $2n = (2m_0+1) = 1, 3, 5, \dots$.

B. Raising and lowering operators derived from factorization

The energy eigenstate of a 2D hydrogen atom may be chosen as the simultaneous eigenstate of (H, l_z) , i.e., $\Psi(\rho, \phi) = e^{im\phi} \chi_m(\rho) / \sqrt{\rho}$, $m = 0, \pm 1, \pm 2, \dots$, and $\chi_m(\rho)$ satisfies $H(m)\chi_m(\rho) = E\chi_m(\rho)$, or

$$D(m)\chi_m(\rho) = \lambda_m \chi_m(\rho), \quad \lambda_m = -2E, \quad (7)$$

$$D(m) = -2H(m) = \frac{d^2}{d\rho^2} - \frac{m^2 - 1/4}{\rho^2} + \frac{2}{\rho}.$$

Directly from the factorization one may derive the raising and lowering operator [7],

$$A_+(m) = \frac{d}{d\rho} - \frac{m+1/2}{\rho} + \frac{1}{m+1/2}, \quad (8)$$

$$A_-(m) = \frac{d}{d\rho} + \frac{m-1/2}{\rho} - \frac{1}{m-1/2},$$

whose selection rules are $\Delta E = 0$ ($\Delta n = 0$) and $\Delta m = \pm 1$. Using $A_\pm(m)$, the energy eigenvalues and eigenstates can be easily obtained. From Eq. (7) it is seen that E depends only on the absolute value of m . Using the Hellmann-Feynman theorem

$$\frac{\partial E}{\partial |m|} = \left\langle \frac{\partial H(m)}{\partial |m|} \right\rangle = \frac{|m|}{\rho^2} > 0. \quad (9)$$

Thus, for states with a given radial quantum number, the energy E increases monotonically with $|m|$. Thus, $m=0$ for the ground state. On the other hand, for a given E , $|m|$ must have an upper limit, say, m_0 ($m_0 > 0$). Then

$$A_+(m_0)\chi_{m_0}(\rho) = \left(\frac{d}{d\rho} - \frac{m_0+1/2}{\rho} + \frac{1}{m_0+1/2} \right) \chi_{m_0}(\rho) = 0. \quad (10)$$

So $\chi_{m_0}(\rho) \sim \rho^{m_0+1/2} e^{-\rho/(m_0+1/2)}$. From Eq. (10) we have $A_-(m_0+1)A_+(m_0)\chi_{m_0}(\rho) = 0$, and using Eqs. (7) and (8), we get

$$[D(m_0) - 1/(m_0+1/2)^2] \chi_{m_0}(\rho) = [-2E_{m_0} - 1/(m_0+1/2)^2] \chi_{m_0}(\rho) = 0. \quad (11)$$

Thus we get $E_{m_0} = -1/2(m_0+1/2)^2$. Using $A_-(m_0), A_-(m_0-1), \dots$, successively operating on $\chi_{m_0}(\rho)$, one may get all the degenerate eigenstates belonging to E_{m_0} , $\chi_{m_0,m}(\rho)$, $m = m_0, m_0-1, \dots, -m_0$. Alternatively, the degenerate eigenstates belonging to $E_n = -1/2n^2$ ($n = 1/2, 3/2, 5/2, \dots$) may be denoted by $\chi_{n,m}(\rho)$, $|m| = n - 1/2, n - 3/2, \dots, 1, 0$, and it can be shown that $\chi_{n,m}(\rho) \sim \rho^{|m|+1/2} e^{-\rho/n} F(-n_\rho, 2|m|+1, 2\rho/n)$, where F is the confluent hypergeometric function, and $n_\rho = (n-1/2) - |m| = 0, 1, 2, \dots, (n-1/2)$.

C. Equivalence of the Runge-Lenz vector and the raising and lowering operators

For a 2D hydrogen atom, the Runge-Lenz vector (1) has two components, a_x and a_y , or $a_\pm = a_x \pm ia_y$. We will show that $a_\pm$ are equivalent to the angular momentum raising and lowering operators $A_\pm(m)$. In polar coordinates $a_\pm$ can be expressed as

$$a_\pm = e^{\pm i\phi} \left[\left(\mp \frac{\partial}{\partial \rho} + \frac{1}{\rho} l_z \right) l_z - \frac{1}{2} \left(\frac{\partial}{\partial \rho} \mp \frac{1}{\rho} l_z \right) - 1 \right]. \quad (12)$$

Operating on the eigenfunction $e^{im\phi} R_m(\rho)$, the raising and lowering of m in the angular function are accomplished by $e^{\pm i\phi}$, and $a_\pm$ are equivalent to the operators $a_\pm(m)$ operating on $R_m(\rho)$,

$$a_\pm(m) = \left[\left(\mp m \frac{\partial}{\partial \rho} + \frac{m^2}{\rho} \right) - \frac{1}{2} \frac{\partial}{\partial \rho} \pm \frac{m}{2\rho} - 1 \right] = - \left(\frac{1}{2} \pm m \right) \frac{\partial}{\partial \rho} + \frac{m(m \pm 1/2)}{\rho} - 1. \quad (13)$$

Apart from a trivial constant factor, $a_\pm(m)$ may be expressed as

$$a_+(m) = \frac{\partial}{\partial \rho} - \frac{m}{\rho} + \frac{1}{m+1/2}, \quad (14)$$

$$a_{-}(m) = \frac{\partial}{\partial \rho} + \frac{m}{\rho} - \frac{1}{m-1/2}.$$

The operators $a_{\pm}(m)$ operating on $R_m(\rho)$ may be replaced by $A_{\pm}(m)$ operating on $\chi_m(\rho) = \sqrt{\rho}R_m(\rho)$

$$A_{+}(m) = \frac{\partial}{\partial \rho} - \frac{m+1/2}{\rho} + \frac{1}{m+1/2}, \quad (15)$$

$$A_{-}(m) = \frac{\partial}{\partial \rho} + \frac{m-1/2}{\rho} - \frac{1}{m-1/2},$$

which are just the angular momentum raising and lowering operators (8) derived directly from the factorization of the Schrödinger equation.

For a 2D hydrogen atom, apart from H and l_z , there exists an additional conserved quantity—the Runge-Lenz vector, which guarantees the closeness of classical orbits and is intimately connected with its dynamical symmetry SO_3 . When such a dynamical symmetry is broken (e.g., the screened Coulomb potential term $\propto -1/r^2$ is considered in an alkali-metal atom [9]), the Runge-Lenz vector no longer keeps constant and the closeness of orbits, in general, will be lost [14]. In this case, it can be shown [14] that *no angular momentum* raising and lowering operators can be derived from the factorization of the radial Schrödinger equation, which is consistent with the equivalence of the Runge-Lenz vector and the angular momentum raising and lowering operators. Therefore, it is understandable that the closeness of classical orbits of a 2D hydrogen atom is closely connected with the factorization of the radial Schrödinger equation, and both are physically related to the dynamical symmetry.

III. 3D HYDROGEN ATOM

Now we address the Runge-Lenz vector for a 3D hydrogen atom,

$$\mathbf{a} = \mathbf{p} \times \mathbf{l} - i\mathbf{p} - \mathbf{r}/r, \quad (16)$$

which has three components (a_x, a_y, a_z) , or $(a_{\pm} = a_x \pm ia_y, a_z)$,

$$a_{\pm} = \mp \left(\frac{\partial}{\partial x} \pm i \frac{\partial}{\partial y} \right) (l_z \pm 1) \pm \frac{\partial}{\partial z} l_{\pm} - \frac{x \pm iy}{r}, \quad (17)$$

$$a_z = p_x l_y - p_y l_x - ip_z - z/r.$$

In the spherical coordinate system $a_{\pm}$ and a_z may be expressed as

$$\begin{aligned} a_{\pm} = & \pm \frac{\partial}{\partial r} [\cos \theta l_{\pm} - \sin \theta e^{\pm i\phi} (l_z \pm 1)] \\ & \mp \frac{1}{r} \left[\sin \theta \frac{\partial}{\partial \theta} l_{\pm} \pm \cos \theta l_{\pm} (l_z \pm 1) \mp \sin \theta e^{\pm i\phi} l_z (l_z \pm 1) \right] \\ & - \sin \theta e^{\pm i\phi}, \end{aligned} \quad (18)$$

$$a_z = \frac{\partial}{\partial r} \left[\frac{1}{2} \sin \theta (e^{i\phi} l_- - e^{-i\phi} l_+) - \cos \theta \right]$$

$$\begin{aligned} & + \frac{1}{r} \left\{ \sin \theta \frac{\partial}{\partial \theta} + \frac{1}{2} [\cos \theta (l_+ l_- + l_- l_+) \right. \\ & \left. - \sin \theta (e^{i\phi} l_z l_- + e^{-i\phi} l_z l_+)] \right\} - \cos \theta. \end{aligned} \quad (19)$$

Equations (18) and (19) operating on the simultaneous eigenfunction of $(H, \mathbf{l}^2, l_z)$, $\Psi_{nlm}(r, \theta, \phi) = R_{nl}(r)Y_{lm}(\theta, \phi)$, we get

$$\begin{aligned} a_{\pm} \Psi_{nlm} = & \pm \frac{d}{dr} R_{nl}(r) [(l+1) d_{l, \pm m} Y_{l+1, m \pm 1} \\ & + l d_{l-1, -(\pm m+1)} Y_{l-1, m \pm 1}] \\ & \mp \frac{1}{r} R_{nl}(r) [l(l+1) d_{l, \pm m} Y_{l+1, m \pm 1} \\ & - l(l+1) d_{l-1, -(\pm m+1)} Y_{l-1, m \pm 1}] \pm R_{nl}(r) \\ & \times [d_{l, \pm m} Y_{l+1, m \pm 1} - d_{l-1, -(\pm m+1)} Y_{l-1, m \pm 1}], \end{aligned} \quad (20)$$

$$\begin{aligned} a_z \Psi_{nlm} = & - \frac{d}{dr} R_{nl}(r) [(l+1) c_{l, m} Y_{l+1, m} - l c_{l-1, m} Y_{l-1, m}] \\ & + \frac{1}{r} R_{nl}(r) [l(l+1) c_{l, m} Y_{l+1, m} \\ & + l(l+1) c_{l-1, m} Y_{l-1, m}] \\ & - R_{nl}(r) [c_{l, m} Y_{l+1, m} + c_{l-1, m} Y_{l-1, m}], \end{aligned} \quad (21)$$

where

$$c_{l, m} = \sqrt{\frac{(l+1)^2 - m^2}{(2l+1)(2l+3)}},$$

$$d_{l, m} = \sqrt{\frac{(l+m+1)(l+m+2)}{(2l+1)(2l+3)}}. \quad (22)$$

Using Eqs. (20) and (21), we may get

$$\begin{aligned} l_- a_+ \Psi_{nlm} = & \left[\frac{d}{dr} - \frac{l}{r} + \frac{1}{l+1} \right] R_{nl}(r) (l+1)(l+m+2) \\ & \times c_{l, m} Y_{l+1, m} + \left[\frac{d}{dr} + \frac{l+1}{r} - \frac{1}{l} \right] \\ & \times R_{nl}(r) l(l-m-1) \\ & \times c_{l-1, m} Y_{l-1, m}, \end{aligned} \quad (23)$$

$$\begin{aligned} l_+ a_z \Psi_{nlm} = & - \left[\frac{d}{dr} - \frac{l}{r} + \frac{1}{l+1} \right] R_{nl}(r) (l+1)(l-m+1) \\ & \times d_{l, m} Y_{l+1, m+1} + \left[\frac{d}{dr} + \frac{l+1}{r} - \frac{1}{l} \right] \\ & \times R_{nl}(r) l(l+m) \\ & \times d_{l-1, -(m+1)} Y_{l-1, m+1}, \end{aligned} \quad (24)$$

$$\begin{aligned}
l_- a_z \Psi_{nlm} = & - \left[\frac{d}{dr} - \frac{l}{r} + \frac{1}{l+1} \right] R_{nl}(r) (l+1) (l+m+1) \\
& \times d_{l,-m} Y_{l+1,m-1} + \left[\frac{d}{dr} + \frac{l+1}{r} - \frac{1}{l} \right] \\
& \times R_{nl}(r) l (l-m) \\
& \times d_{l-1,m-1} Y_{l-1,m-1}.
\end{aligned} \quad (25)$$

Now, we may define two operators $S_{\pm}$ in terms of the conserved quantities (the Runge-Lenz vector and angular momentum)

$$\begin{aligned}
S_+ &= l_- a_+ + (l_z - l + 1) a_z, \\
S_- &= l_- a_+ + (l_z + l + 2) a_z.
\end{aligned} \quad (26)$$

It is easily verified that

$$\begin{aligned}
S_+ \Psi_{nlm} &= \left(\frac{d}{dr} - \frac{l}{r} + \frac{1}{l+1} \right) R_{nl}(r) (l+1) (2l+1) \\
& \times c_{l,m} Y_{l+1,m}(\theta, \phi) \\
& \propto a_+(l) R_{nl}(r) Y_{l+1,m}(\theta, \phi), \\
S_- \Psi_{nlm} &= \left(\frac{d}{dr} + \frac{l+1}{r} - \frac{1}{l} \right) R_{nl}(r) l (2l+1) \\
& \times c_{l-1,m} Y_{l-1,m}(\theta, \phi) \\
& \propto a_-(l) R_{nl}(r) Y_{l-1,m}(\theta, \phi),
\end{aligned} \quad (27)$$

where

$$a_+(l) = \left(\frac{d}{dr} - \frac{l}{r} + \frac{1}{l+1} \right), \quad a_-(l) = \left(\frac{d}{dr} + \frac{l+1}{r} - \frac{1}{l} \right), \quad (28)$$

which are equivalent to the angular momentum raising and lowering operators $A_{\pm}(l)$ derived from the factorization of radial Schrödinger equation for a 3D hydrogen atom [7]

$$A_+(l) = \left(\frac{d}{dr} - \frac{l+1}{r} + \frac{1}{l+1} \right), \quad A_-(l) = \left(\frac{d}{dr} + \frac{l}{r} - \frac{1}{l} \right), \quad (29)$$

operating on the radial wave function $\chi_{nl}(r) = r R_{nl}(r)$. It is seen that the effect of S_+ (S_-) is to increase (decrease) the angular momentum l by 1, but keep the energy and magnetic quantum number m unchanged [note that considering $n = l + n_r + 1$, A_+ (A_-) decreases (increases) the radial quantum number n_r by 1]. Therefore, to clearly indicate the selection rules, S_+ and S_- may be relabeled as

$$\begin{aligned}
S_+ &\rightarrow S(n, l \uparrow, m) = l_- a_+ + (l_z - l + 1) a_z, \\
S_- &\rightarrow S(n, l \downarrow, m) = l_- a_+ + (l_z + l + 2) a_z.
\end{aligned} \quad (30)$$

It is noted that in the radial Schrödinger equation the magnetic quantum number m disappears. When the shift of m is considered, using the Runge-Lenz vector ($a_{\pm}$ and a_z) and angular momentum operator ($l_{\pm}, l_z$), one may construct the

other two kinds of angular momentum raising and lowering operators, which also keeps the energy unchanged,

$$S(n, l \uparrow, m \uparrow) = (l_z + l - 1) a_+ - l_+ a_z, \quad (31)$$

$$S(n, l \downarrow, m \uparrow) = (l_z - l - 2) a_+ - l_+ a_z,$$

$$S(n, l \uparrow, m \downarrow) = (l_z - l + 1) a_- - l_- a_z, \quad (32)$$

$$S(n, l \downarrow, m \downarrow) = (l_z + l + 2) a_- - l_- a_z.$$

Thus, similar to the situation of a 2D hydrogen atom, we have shown the equivalence of the conserved quantities ($\mathbf{l}, \mathbf{a}$) responsible for the closeness of classical orbits on the one side and the six angular momentum raising and lowering operators on the other side. Therefore, it is understandable that there exists an intimate connection between the closeness of classical orbits and the factorization of radial Schrödinger equation.

After completing this work, it was noted that in Ref. [15] the relation between the raising and lowering operators and the dynamical symmetry of hydrogen atom was addressed using the group theoretical approach.

IV. ISOTROPIC HARMONIC OSCILLATORS

It is well known that an nD isotropic harmonic oscillator has the dynamical symmetry SU_n . For a 3D isotropic harmonic oscillator, apart from the Hamiltonian H and angular momentum $\mathbf{l}$, there exist five additional conserved quantities, which constitute a quadruple tensor

$$\begin{aligned}
Q_{xy} &= xy + p_x p_y, \quad Q_{yz} = yz + p_y p_z, \quad Q_{zx} = zx + p_z p_x, \\
Q_1 &= \frac{1}{2} [(x^2 - y^2) + (p_x^2 - p_y^2)],
\end{aligned} \quad (33)$$

$$Q_0 = \frac{1}{2\sqrt{3}} [(x^2 + y^2 - 2z^2) + (p_x^2 + p_y^2 - 2p_z^2)].$$

It can be shown that there exist four relations among the nine conserved quantities, so a 3D isotropic harmonic oscillator also is a completely degenerate system and moves, in general, along an elliptic orbit and the direction of the semiaxes and eccentricity are characterized by the quadruple tensor. It has been shown [2,3,7] that the radial Schrödinger equation of an isotropic harmonic oscillator as well as a hydrogen atom can be factorized. Nevertheless, it was noted that for an isotropic harmonic oscillator, two (rather than one) kinds of raising and lowering operators can be derived directly from factorization. Moreover, the selection rules for the raising and lowering operators are quite different for a hydrogen atom and an isotropic harmonic oscillator. For a hydrogen atom, the raising and lowering operators as well as the Runge-Lenz vector are conserved quantities, thus the equivalence of the Runge-Lenz vector and the raising and lowering operators are straightforward. For an isotropic harmonic oscillator, the two kinds of raising and lowering operators ($A_{\pm}$ and $B_{\pm}$) themselves are not conserved quantities. Thus the equivalence of the raising and lowering operators ($A_{\pm}$ and $B_{\pm}$) and the conserved quantities (energy, angular momentum, and quadrupole tensors) is not so straightforward. How-

ever, it is noted that, though both $A_{\pm}$ and $B_{\pm}$ are angular momentum raising and lowering operators ($\Delta l = \pm 1$), the other selection rules are different, i.e., while A_{+} (A_{-}) decreases (increases) both the radial quantum number and energy by 1 ($\Delta n_r = \mp 1$, $\Delta E = \mp 1$), B_{+} (B_{-}) keeps the radial quantum number ($\Delta n_r = 0$), but increases (decreases) the energy by 1 ($\Delta E = \pm 1$). Therefore, by using these differences, it can be shown that in terms of the two kinds of raising and lowering operators one can construct the conserved quantities (energy, angular momentum, and quadrupole tensors) characterizing an elliptic orbit.

For simplicity, we take a 2D isotropic harmonic oscillator as an example. A 2D isotropic harmonic oscillator has the dynamical symmetry SU_2 , which is locally isomorphic to SO_3 (the dynamical symmetry of a 2D hydrogen atom). For a classical 2D isotropic harmonic oscillator, the planar orbit is also an ellipse, and the lengths of its semimajor axis a and semiminor axis b are determined by the energy E and angular momentum L ($\hbar = \omega = 1$), $E = \frac{1}{2}(a^2 + b^2)$, and $L = a^2 b^2$. The intersection angle γ of the major axis with the x axis is determined by $\tan 2\gamma = Q_{xy}/Q_1$, and the eccentricity $\propto [Q_{xy}^2 + Q_1^2]^{1/2}$.

In Ref. [7], it was shown that from the factorization of radial Schrödinger equation for a 2D isotropic oscillator, two kinds of raising and lowering operators, $A_{\pm}$ and $B_{\pm}$, operating on the radial wave function $\chi_m(\rho)$, can be derived

$$A_{\pm}(m) = \frac{d}{d\rho} \mp \frac{m \pm 1/2}{\rho} \pm \rho, \quad (34)$$

$$B_{\pm}(m) = \frac{d}{d\rho} \mp \frac{m \pm 1/2}{\rho} \mp \rho.$$

The selection rules for $A_{\pm}$ are $\Delta m = \pm 1, \Delta n_{\rho} = \mp 1, \Delta E = \mp 1$, and for $B_{\pm}$, $\Delta m = \pm 1, \Delta n_{\rho} = 0, \Delta E = \pm 1$. Now, it will be shown that in terms of $A_{\pm}$ and $B_{\pm}$, the conserved quantities characterizing a closed orbit (the angular momentum l_z , energy E , and quadrupole tensor Q_{xy} and Q_1) can be constructed. The operators $A_{\pm}$ may be replaced by $a_{\pm}(m)$ and $b_{\pm}(m)$ operating on $R_m(\rho) = \chi_m(\rho)/\sqrt{\rho}$,

$$a_{\pm}(m) = \frac{d}{d\rho} \mp \frac{m}{\rho} \pm \rho, \quad (35)$$

$$b_{\pm}(m) = \frac{d}{d\rho} \mp \frac{m}{\rho} \mp \rho.$$

When operating on the whole wave function $e^{im\phi}R_m(\rho)$, $a_{\pm}(m)$ and $b_{\pm}(m)$ may be replaced by $a_{\pm}$ and $b_{\pm}$,

$$\begin{aligned} a_{+} &= e^{i\phi} \left[\frac{\partial}{\partial \rho} - \frac{1}{\rho} l_z + \rho \right], \\ a_{-} &= e^{-i\phi} \left[\frac{\partial}{\partial \rho} + \frac{1}{\rho} l_z - \rho \right], \\ b_{+} &= e^{i\phi} \left[\frac{\partial}{\partial \rho} - \frac{1}{\rho} l_z - \rho \right], \end{aligned} \quad (36)$$

$$b_{-} = e^{-i\phi} \left[\frac{\partial}{\partial \rho} + \frac{1}{\rho} l_z + \rho \right].$$

It is easily shown that, similar to Eq. (5) for a 2D hydrogen atom, for a 2D isotropic oscillator we have

$$[l_z, a_{\pm}] = \pm a_{\pm}, [l_z, b_{\pm}] = \pm b_{\pm}, \quad (37)$$

i.e., both $a_{\pm}$ and $b_{\pm}$ are angular momentum raising and lowering operators, but unlike the hydrogen atom, here $a_{\pm}$ and $b_{\pm}$ are not conserved quantities,

$$[H, a_{\pm}] = \mp a_{\pm}, [H, b_{\pm}] = \pm b_{\pm}. \quad (38)$$

Therefore, the operators $a_{\pm}$ and $b_{\pm}$ themselves cannot be directly equivalent to the conserved quantities characterizing an elliptic orbit. However, it can be shown by straightforward calculation that, as expected, the conserved quantities (l_z , energy E , and quadrupole tensors Q_1 and Q_{xy}) can be constructed in terms of $a_{\pm}$ and $b_{\pm}$ as follows:

$$\begin{aligned} l_z &= \frac{1}{4} (a_{-} a_{+} - b_{+} b_{-}), \\ H &= \frac{1}{4} (a_{-} a_{+} + b_{+} b_{-}) + 1, \end{aligned} \quad (39)$$

$$Q_1 = -\frac{1}{4} (a_{-} b_{-} + b_{+} a_{+}),$$

$$Q_{xy} = -\frac{i}{4} (a_{-} b_{-} - b_{+} a_{+}).$$

V. 1D SYSTEMS

The Schrödinger's factorization method and the concept of raising and lowering operators were extended extensively in supersymmetric quantum mechanics [16–20] to treat the Schrödinger equation for a particle in a general 1D potential $V(x)$. It was shown that for a potential $V(x)$, provided the ground bound state energy E_0 is finite ($E_0 \neq -\infty$) and the ground-state wave function $\Psi_0(x)$ is differentiable, the Schrödinger equation can always be factorized and the corresponding raising and lowering operators, A^{+} and A , can be constructed. It is interesting to note that classical orbits of all bound particles in a regular 1D potential are obviously closed. The supersymmetric partner Hamiltonian, $H_{-} = A^{+}A$ and $H_{+} = AA^{+}$, have the same energy spectra, $E_n^{(+)} = E_{n+1}^{-}$ ($n=0,1,2,\dots$), except the ground-state energy of H_{-} ($E_0^{(-)}=0$), and the eigenstates with the same eigenvalue of H_{-} and H_{+} are connected with each other by A^{+} and A . It was shown that this is due to the shape invariance [20] of $V(x)$, which may also be considered as a special kind of dynamical symmetry.

For a 1D harmonic oscillator, the energy eigenvalues and eigenstates are well known, which are quite similar to those for a 3D isotropic harmonic oscillator ($l=0$ case). The raising and lowering operators derived from factorization are $a^{+} = (1/\sqrt{2})(x - d/dx)$ and $a = (1/\sqrt{2})(x + d/dx)$, which connect the neighboring eigenstates with opposite parity ($\Delta N = 1$). However, it should be noted that the 1D harmonic oscillator potential formally corresponding to a 3D isotropic oscillator [$V(r) = r^2/2$, $r \geq 0$] is

$$V(x) = \begin{cases} x^2/2, & x \geq 0 \\ \infty, & x < 0, \end{cases} \quad (40)$$

whose energy levels are $E_N = (N+1/2)$, $N=1,3,5,\dots$. The usually adopted 1D harmonic oscillator is $V(x) = x^2/2$ ($-\infty < x < +\infty$) with reflection symmetry, whose levels are $E_n = (N+1/2)$, $N=0,1,2,3,\dots$, and the neighboring eigenstates are of opposite parity. From this, one can understand why there exist two kinds of raising and lowering operators, $A_{\pm}$ and $B_{\pm}$, for an nD ($n \geq 2$) isotropic harmonic oscillator, and $A_{\pm}$ and $B_{\pm}$ are different in form from the operators $a^{\dagger}$ and a for a 1D harmonic oscillator. However, one may use the product operator of A and B [7], i.e., the operator C , as the raising and lowering operators connecting the neighboring eigenstates with the *same parity*. In fact, for a 3D isotropic harmonic oscillator [7]

$$C(l=0, N \uparrow \uparrow) = \frac{d^2}{dr^2} + r^2 - 2r \frac{d}{dr} - 1 \quad (41)$$

is the same form as $2a^{\dagger}a = d^2/dx^2 + x^2 - 2x d/dx - 1$ for a 1D harmonic oscillator, and both have the selection rule $\Delta N = 2$.

As for the 1D hydrogen atom [21] with a quite singular potential,

$$V(x) = V(-x) = -\frac{1}{|x|} \quad (-\infty < x < +\infty), \quad (42)$$

to our knowledge, the corresponding Schrödinger equation cannot be factorized. This is understandable because its ground-state energy $E_0 = -\infty$, and $\Psi_0(x) \sim \sqrt{\delta(x)}$, is not differentiable at the origin. In fact, the 1D hydrogen atom formally corresponding to a 3D hydrogen atom [$V(r) = -1/r$, $r > 0$] is

$$V(x) = \begin{cases} -1/x, & x > 0 \\ \infty, & x \leq 0. \end{cases} \quad (43)$$

For this 1D potential, the energy levels $E_n = -1/2n^2$, $n = 1, 2, \dots$, are nondegenerate, which are the same as the spectra of $l=0$ states for a 3D hydrogen atom. Thus it is understandable why for a 1D hydrogen atom there exists no analogous raising and lowering operators $A_{\pm}$ connecting degenerate states with the same energy of a 3D hydrogen atom.

However, similar to a 3D hydrogen atom [7], one may construct the energy raising and lowering operators for a 1D hydrogen atom

$$B(l=0, N \uparrow) = \left(x \frac{d}{dx} - \frac{x}{n+1} + n \right) M \left(\frac{n}{n+1} \right),$$

$$B(l=0, N \downarrow) = \left(x \frac{d}{dx} + \frac{x}{n-1} - n \right) M \left(\frac{n}{n-1} \right) \quad (n > 1). \quad (44)$$

VI. CONCLUSIONS AND SUMMARY

In the preceding sections we have shown the equivalence of the raising and lowering operators derived from the factorization of the radial Schrödinger equation on the one side and the conserved quantities responsible for the closeness of classical orbits on the other side, and from the physical point of view, both are intimately connected with the dynamical symmetry of hydrogen atoms and isotropic harmonic oscillators. For a classical hydrogen atom, the orbits are always closed for *any continuous* negative energy $E < 0$ and positive angular momentum $L > 0$. In quantum mechanics, both the angular momentum and bound energy eigenvalues are *discrete*. Thus, it seems understandable that for hydrogen atoms there exist four kinds of operators connecting neighboring energy and angular momentum eigenstates. Particularly, these operators can *simultaneously* raise or lower both the energy and angular momentum. If the dynamical symmetry is broken (e.g., the screened Coulomb potential $\propto -1/r^2$ is considered in an alkali atom), the Runge-Lenz vector no longer keeps constant and the closeness of classical orbits, *in general*, is lost [14]. On the other hand, it is shown [14] that in this case, only the *energy* raising and lowering operators can be constructed from the factorization of the radial Schrödinger equation, but *no angular momentum* raising and lowering operators can be derived. The situation for isotropic harmonic oscillators is similar. Therefore, it seems understandable that there exist simple relations between the raising and lowering operators in quantum mechanics on the one side and the conserved quantities characterizing the closeness of classical orbits on the other side, and both are physically connected with the dynamical symmetry of the system considered.

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