Weyl functions and their use in the study of quantum interference

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Weyl functions are shown to be an important tool in quantum phase-space studies. Their properties are studied and relations with other quantities are derived. The use of Weyl functions for the understanding of quantum interference phenomena is discussed. The general theory is applied to superpositions of m coherent states uniformly distributed on a circle (generalized Schrödinger cats). The properties of these states are explored and their interference behavior is discussed, using Weyl functions. [S1050-2947(98)04508-9]

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I. INTRODUCTION

Quantum phase-space methods [1–3] play an important role in fundamental problems in quantum mechanics where they describe the location of a particle in the x-p phase space in a way consistent with quantum mechanics and the uncertainty principle $\Delta x \Delta p \geqslant 0.5$. They also play an important role in more applied problems in quantum optics and quantum electronics where they describe the electromagnetic field in the E-B plane in a way consistent with quantum noise and the uncertainty principle $\Delta E \Delta B \geqslant 0.5$.

The Wigner function and also the P and Q functions play a central role in these techniques and their properties have been studied extensively [1–3]. Numerical calculations of these functions have been used as a practical tool and provided a valuable insight in many quantum optics problems. More recently the Wigner tomography [4] provided a method of constructing the Wigner function from optical measurements.

The Weyl function which can be defined in many ways and which is the Fourier transform of the Wigner function has been used as an auxiliary quantity in theoretical studies (e.g., [5]); but it has not been studied in its own right and it has not been exploited as a practical tool in quantum optics problems. In this paper we discuss the properties of the Weyl function with particular emphasis on its use for the study of quantum interference phenomena. There has been a lot of work on quantum interference phenomena [6,7] (for a review see [8]) and in this paper we show that the Weyl function can play an important role in these studies.

Our general ideas are applied to the important example of superpositions of m coherent states uniformly distributed on a circle (generalized Schrödinger cats). The study of these states is an interesting problem in its own right. Superpositions of two coherent states have been studied extensively [9] both theoretically and experimentally. More generally highly nonclassical states produced as a superposition of many coherent states have been studied in [10]. Superpositions of m coherent states similar to the one considered here have been studied in Refs. [11,12]. We study several properties of these states and especially their interference behavior using Weyl functions.

In Sec. II A we introduce the Weyl function and study its properties. In Sec. II B we consider a quantum state $|s\rangle$ which is a superposition of m other quantum states $|s_i\rangle$, and explain that both its Wigner function and its Weyl function can be decomposed into auto terms and cross terms. The cross terms describe the interference between the various states $|s_i\rangle$, and their properties are discussed. In Sec. III A we consider superpositions of m coherent states uniformly distributed on a circle and study their properties. Numerical results for their quantum statistical properties are presented in Sec. III B, and for their Wigner and Weyl functions in Sec. III C. We conclude in Sec. IV with a discussion of our results.

II. WEYL FUNCTIONS

A. Basic formalism

We consider the harmonic oscillator Hilbert space H spanned by the number eigenstates $\{|N\rangle; N=0,1,2\dots\}$. We also consider the coherent states

$$|A\rangle = \exp\left(-\frac{|A|^2}{2}\right)\sum_{N=0}^{\infty}\frac{A^N}{(N!)^{1/2}}|N\rangle = D(A)|0\rangle, \qquad (1)$$

$$D(A) = \exp[Aa^\dagger - A^* a], \qquad (2)$$

where $a^\dagger, a$ are the usual creation and annihilation operators. $D(A)$ is the displacement operator which can also be expressed in terms of the position and momentum operators $\hat{x}, \hat{p}$ as

$$D(x,p) \equiv D\left(A = \frac{x+ip}{\sqrt{2}}\right) = \exp(ip\hat{x} - ix\hat{p}). \qquad (3)$$

For later purposes we also define the following moments associated with a state described by a density matrix ρ:

$$\langle x^M\rangle = \mathrm{Tr}[\hat{x}^M\rho] = \int x^M \sigma(x)\,dx, \quad \sigma(x) \equiv \langle x|\rho|x\rangle \qquad (4)$$

$$\langle p^M\rangle = \mathrm{Tr}[\hat{p}^M\rho] = \int p^M \tau(p)\,dp, \quad \tau(p) \equiv \langle p|\rho|p\rangle \qquad (5)$$

$$\langle xp + px\rangle = \mathrm{Tr}[(\hat{x}\hat{p} + \hat{p}\hat{x})\rho]. \qquad (6)$$

From them we can find the uncertainties

$$\Delta x = [\langle x^2\rangle - \langle x\rangle^2]^{1/2}, \quad \Delta p = [\langle p^2\rangle - \langle p\rangle^2]^{1/2}, \qquad (7)$$

$$K^2=\frac{1}{2}\langle xp+px\rangle-\langle x\rangle\langle p\rangle. \tag{8}$$

The parity operator is

$$U_0=\exp(i\pi a^\dagger a)=\sum_{N=0}^{\infty}(-1)^N|N\rangle\langle N| \tag{9}$$

and obeys the relations

$$\begin{aligned}U_0=U_0^\dagger,\quad U_0^2=1,\\ U_0D(x,p)U_0^\dagger=D(-x,-p).\end{aligned} \tag{10}$$

The displaced parity operator [5,13] is written as

$$\begin{aligned}U(x,p)&=D(x,p)U_0D^\dagger(x,p)=D(2x,2p)U_0\\ &=U_0D(-2x,-2p).\end{aligned} \tag{11}$$

The Wigner function of a state described by a density matrix ρ is defined in terms of the displaced parity operator as

$$\begin{aligned}W(x,p)&=\frac{1}{2\pi}\int dX\left\langle x+\frac{1}{2}X|\rho|x-\frac{1}{2}X\right\rangle\exp(-iXp)\\ &=\frac{1}{2\pi}\int dP\left\langle p+\frac{1}{2}P|\rho|p-\frac{1}{2}P\right\rangle\exp(iPx)\\ &=\frac{1}{\pi}\mathrm{Tr}[\rho U(x,p)].\end{aligned} \tag{12}$$

The equivalence between these expressions is known [5,13,14]. Another function which is useful in phase-space methods is the Weyl function

$$\begin{aligned}\tilde{W}(X,P)&=\int dx\langle x+\tfrac{1}{2}X|\rho|x-\tfrac{1}{2}X\rangle\exp(-iPx)\\ &=\int dp\langle p+\tfrac{1}{2}P|\rho|p-\tfrac{1}{2}P\rangle\exp(ipX)\\ &=\mathrm{Tr}[\rho D(X,P)].\end{aligned} \tag{13}$$

The equivalence between these two expressions is known [1,5,13,14]. For a pure state $|s\rangle$ Eq. (13) becomes

$$\tilde{W}(X,P)=\langle s|D(X,P)|s\rangle. \tag{14}$$

It is seen that the Weyl function of a state is equal to the overlap of the displaced state with the original state. In this sense, the X,P are position and momentum *increments*. The Weyl function can be understood as a *generalized correlation function*. If we have a function $s(x)$ (e.g., the wave function in the x representation), in order to find the correlation we displace it into $s(x+X)$ and take the integral of $s(x)s(x+X)$. In the Weyl function we perform a more general displacement in phase space, i.e., a displacement in both position and momentum. Therefore the correlation is a special case of the Weyl function with $P=0$ (or $X=0$).

The Wigner function is related to the Weyl function through the two-dimensional Fourier transform [1,5,13].

$$\tilde{W}(X,P)=\int\int dx\ dp\ W(x,p)\exp[-i(Px-pX)]. \tag{15}$$

The properties of the Wigner function are well known and are not discussed here. We discuss various properties that elucidate further the physical interpretation of the Weyl function. From Eq. (13) it is easily seen that

$$\tilde{W}(0,0)=1, \tag{16}$$

$$\tilde{W}(X,P)=\tilde{W}^*(-X,-P), \tag{17}$$

$$|\tilde{W}(X,P)|\leqslant 1. \tag{18}$$

We next show that for a *pure* state $|s\rangle$

$$\begin{aligned}&\int\int dX'dP'|\tilde{W}(X',P')|^2\exp[i(P'X-X'P)]\\ &=2\pi|\tilde{W}(X,P)|^2.\end{aligned} \tag{19}$$

In order to prove this we use Eq. (13) to get

$$\tilde{W}(X',P')=\int dx\ s(x+X')s^*(x)\exp(-iP'x), \tag{20}$$

$$[\tilde{W}(X',P')]^*=\int dp\ s^*(p)s(p-P')\exp(-ipX'). \tag{21}$$

Inserting these equations into the left hand side of Eq. (19) and using the Fourier transforms

$$\begin{aligned}&\int\frac{dX'}{\sqrt{2\pi}}s(x+X')\exp[-i(p+P)X']\\ &=s(p+P)\exp[ix(p+P)],\end{aligned} \tag{22}$$

$$\begin{aligned}&\int\frac{dP'}{\sqrt{2\pi}}s(p-P')\exp[-i(x-X)P']\\ &=s(x-X)\exp[-ip(x-X)]\end{aligned} \tag{23}$$

we get the right hand side of Eq. (19). In the special case $X=P=0$ Eq. (19) becomes

$$\frac{1}{2\pi}\int\int dX\ dP|\tilde{W}(X,P)|^2=1. \tag{24}$$

We also prove that

$$\frac{1}{2\pi}\int dX|\tilde{W}(X,P)|^2=\int dp|\langle p|\rho|p+P\rangle|^2, \tag{25}$$

$$\frac{1}{2\pi}\int dP|\tilde{W}(X,P)|^2=\int dx|\langle x|\rho|x+X\rangle|^2. \tag{26}$$

In order to prove this we use Eq. (13) to get

$$\int dX|\tilde{W}(X,P)|^2 = \int dX\, dx\, dx' \langle x+\tfrac{1}{2}X|\rho|x-\tfrac{1}{2}X\rangle \times \langle x'-\tfrac{1}{2}X|\rho|x'+\tfrac{1}{2}X\rangle \exp[iP(x'-x)]. \tag{27}$$

We then use the relations

$$\int dP \exp[iP(x-x')] = 2\pi\delta(x-x'), \tag{28}$$

$$\int dx \langle x+\tfrac{1}{2}X|\rho|x-\tfrac{1}{2}X\rangle\langle x-\tfrac{1}{2}X|\rho|x+\tfrac{1}{2}X\rangle = \int dx |\langle x|\rho|x+X\rangle|^2 \tag{29}$$

to prove Eq. (26). In a similar way we prove Eq. (25). Equations (25), (26) reinforce our interpretation of the Weyl function as a generalized correlation function. If we consider for simplicity a pure state $|s\rangle$ then the right hand side of Eq. (26) is the correlation of the probability distribution $|\langle x|s\rangle|^2$. Therefore the $|\tilde{W}(X,P)|^2$ can be interpreted as a type of density for the correlation function whose integral with respect to P gives the correlation of the probability distribution $|\langle x|s\rangle|^2$; and whose integral with respect to X gives the correlation of the probability distribution $|\langle p|s\rangle|^2$.

We next expand the displacement operator $D(x,p)$ around the point $x=p=0$ as

$$D(x,p) = 1 + (ip\hat{x} - ix\hat{p}) - \frac{(p\hat{x}-x\hat{p})^2}{2} + \cdots . \tag{30}$$

Substitution of Eq. (30) into Eq. (13) gives an expansion of the Weyl function around the origin:

$$|\tilde{W}(X,P)|^2 = 1 - \frac{1}{2}(\Delta x)^2 X^2 - \frac{1}{2}(\Delta p)^2 P^2 + K^2(XP) + \cdots , \tag{31}$$

where the uncertainties $\Delta x, \Delta p, K$ have been defined in Eqs. (7), (8).

There are interesting relations between the Weyl function and the Fourier transforms of the P and Q functions, which we denote as $\tilde{P}$ and $\tilde{Q}$, correspondingly. They have been proved in Ref. [15] and here we simply quote the result in our notation:

$$\tilde{W}(X,P) = \exp\left[\frac{1}{16}(P^2+Q^2)\right]\tilde{Q}(X,P) = \exp\left[-\frac{1}{16}(P^2+Q^2)\right]\tilde{P}(X,P). \tag{32}$$

We also point out that the Weyl function for several quantum states of interest to quantum optics (e.g., coherent states, squeezed states, number eigenstates, etc.) have been calculated in the Appendix of Ref. [16].

B. Interference: Auto terms and cross terms

We consider a state which is the superposition of m other quantum states

$$|s\rangle = \mathcal{N}\sum_{i=1}^{m} |s_i\rangle. \tag{33}$$

The states $|s_i\rangle$ are normalized; and $\mathcal{N}$ is the normalization factor of the state $|s\rangle$:

$$\mathcal{N} = \left(m + \sum_{i\neq j} \langle s_i|s_j\rangle\right)^{-1/2}. \tag{34}$$

We call $\langle x_i\rangle$ and $\langle p_i\rangle$ the average position and momentum of the state $|s_i\rangle$,

$$\langle x_i\rangle = \langle s_i|\hat{x}|s_i\rangle, \tag{35}$$

$$\langle p_i\rangle = \langle s_i|\hat{p}|s_i\rangle. \tag{36}$$

Substitution of Eq. (33) into Eq. (12) gives the Wigner function as

$$W(x,p) = \mathcal{N}^2\sum_{i=1}^{m} W_i(x,p) + \mathcal{N}^2\sum_{i\neq j} W'_{ij}(x,p) = \mathcal{N}^2\sum_{i=1}^{m} W_i(x,p) + \mathcal{N}^2 2\sum_{i<j} \mathrm{Re}[W'_{ij}(x,p)], \tag{37}$$

where

$$W_i(x,p) = \langle s_i|U(x,p)|s_i\rangle \tag{38}$$

are the Wigner functions of the $|s_i\rangle$ states (auto terms); and

$$W'_{ij}(x,p) = \langle s_i|U(x,p)|s_j\rangle = \int dX \langle x+\tfrac{1}{2}X|s_i\rangle\langle s_j|x-\tfrac{1}{2}X\rangle \exp(-iXp) = W'^{*}_{ji}(x,p) \tag{39}$$

are cross terms. It is clear that we have m auto terms, and $m(m-1)$ cross terms. The auto terms are real. The cross terms W'_{ij} are not real; but we can define the cross terms

$$W_{ij} = W'_{ij} + W'_{ji} = 2\ \mathrm{Re}[W'_{ij}], \tag{40}$$

which are real. Note that the Wigner function for the mixed state

$$\rho = \frac{1}{m}\sum_{i=1}^{m} |s_i\rangle\langle s_i| \tag{41}$$

contains only the auto terms

$$W(x,p) = \frac{1}{m}\sum_{i=1}^{m} W_i(x,p) \tag{42}$$

and therefore the Wigner cross terms describe the interference between the states $|s_i\rangle$ and distinguish clearly the pure state (33) from the corresponding mixed state (41). The auto term W_i is located around the point $(\langle x_i\rangle,\langle p_i\rangle)$ (this is seen from the fact that W_i is the Wigner function for the state $|s_i\rangle$). The absolute value of W'_{ij} can be expressed in terms of the two Wigner functions W_i and W_j as

$$|W'_{ij}(x,p)|^2=\int\int W_i(x+\tfrac{1}{2}X,p+\tfrac{1}{2}P)\times W_j(x-\tfrac{1}{2}X,p-\tfrac{1}{2}P)dX\,dP. \tag{43}$$

This relation shows clearly that the cross term W'_{ij} is located around the point $[\frac{1}{2}(\langle x_i\rangle+\langle x_j\rangle),\frac{1}{2}(\langle p_i\rangle+\langle p_j\rangle)]$.

The proof of Eq. (43) is based on the following relation proved by Moyal [17]:

$$\int\frac{dXdP}{2\pi}\langle u_1|D(X,P)|u_2\rangle\langle u_3|D(-X,-P)|u_4\rangle=\langle u_1|u_4\rangle\langle u_3|u_2\rangle \tag{44}$$

for arbitrary states $|u_1\rangle$, $|u_2\rangle$, $|u_3\rangle$, and $|u_4\rangle$. We choose

$$\begin{aligned}\langle u_1|&=\langle s_i|D(x,p),\\ |u_2\rangle&=U_0D^\dagger(x,p)|s_i\rangle,\\ \langle u_3|&=\langle s_j|D(x,p),\\ |u_4\rangle&=U_0D^\dagger(x,p)|s_j\rangle,\end{aligned} \tag{45}$$

and get

$$\int\frac{dXdP}{2\pi}\langle s_i|D(x,p)D(X,P)U_0D^\dagger(x,p)|s_i\rangle\times\langle s_j|D(x,p)D(-X,-P)U_0D^\dagger(x,p)|s_j\rangle=\langle s_i|D(x,p)U_0D(x,p)|s_j\rangle\langle s_j|D(x,p)U_0D^\dagger(x,p)|s_i\rangle. \tag{46}$$

Using Eq. (11) we can now show that

$$\langle s_i|D(x,p)D(X,P)U_0D^\dagger(x,p)|s_i\rangle=W_i(x+\tfrac{1}{2}X,p+\tfrac{1}{2}P), \tag{47}$$

$$\langle s_j|D(x,p)D(-X,-P)U_0D^\dagger(x,p)|s_j\rangle=W_j(x-\tfrac{1}{2}X,p-\tfrac{1}{2}P), \tag{48}$$

$$\langle s_i|D(x,p)U_0D(x,p)|s_j\rangle=W'_{ij}, \tag{49}$$

$$\langle s_j|D(x,p)U_0D^\dagger(x,p)|s_i\rangle=W'_{ji}=W'^{*}_{ij}. \tag{50}$$

This completes the proof of Eq. (43). The physical significance of this equation lies in the fact that it shows that the Wigner auto terms define uniquely the absolute value (but not the phase) of the Wigner cross terms.

A similar decomposition into auto terms and cross terms can be given for the Weyl function. Substitution of Eq. (33) into Eq. (13) gives

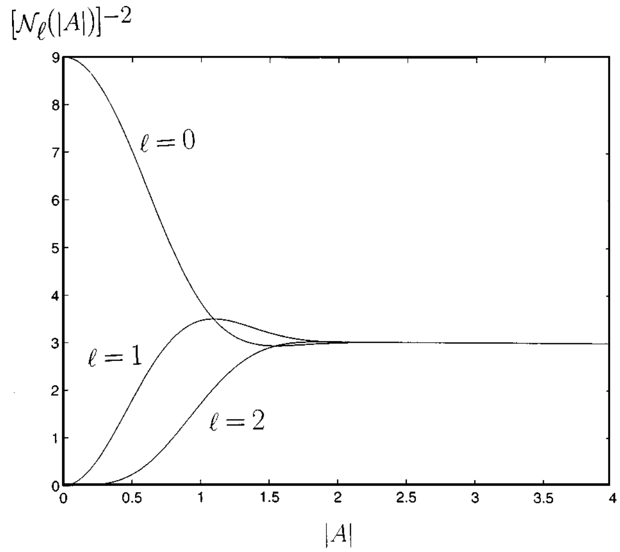

FIG. 1. $[\mathcal{N}_\ell(|A|)]^{-2}$ as a function of $|A|$ for $\ell=0,1,2$.

$$\tilde{W}(X,P)=\mathcal{N}^2\sum_{i=1}^{m}\tilde{W}_i(X,P)+\mathcal{N}^2\sum_{i\neq j}\tilde{W}'_{ij}(X,P)=\mathcal{N}^2\sum_{i=1}^{m}\tilde{W}_i(X,P)+\mathcal{N}^2\sum_{i<j}\tilde{W}_{ij}(X,P), \tag{51}$$

where

$$\tilde{W}(X,P)=\langle s_i|D(X,P)|s_i\rangle, \tag{52}$$

are the Weyl functions of the $|s_i\rangle$ states (auto terms); and

$$\tilde{W}'_{ij}(X,P)=\langle s_i|D(X,P)|s_j\rangle=\int dx\langle x+\tfrac{1}{2}X|s_i\rangle\langle s_j|x-\tfrac{1}{2}X\rangle\exp(-iPx), \tag{53}$$

$$\tilde{W}_{ij}(X,P)=\tilde{W}'_{ij}(X,P)+\tilde{W}'_{ji}(X,P) \tag{54}$$

are the cross terms. The auto terms are located around the origin.

III. EXAMPLE: SUPERPOSITIONS OF *m* COHERENT STATES

A. Basic formalism

We consider the usual harmonic oscillator Hilbert space H which we write as the direct sum

$$H=\sum_{\ell=0}^{m-1}H_\ell, \tag{55}$$

where H_ℓ is spanned by the number eigenstates

$$H_\ell=\{|mN+\ell\rangle;\ \ N=0,1,2,\ldots\}. \tag{56}$$

We call π_ℓ the projection operators into the Hilbert space H_ℓ,

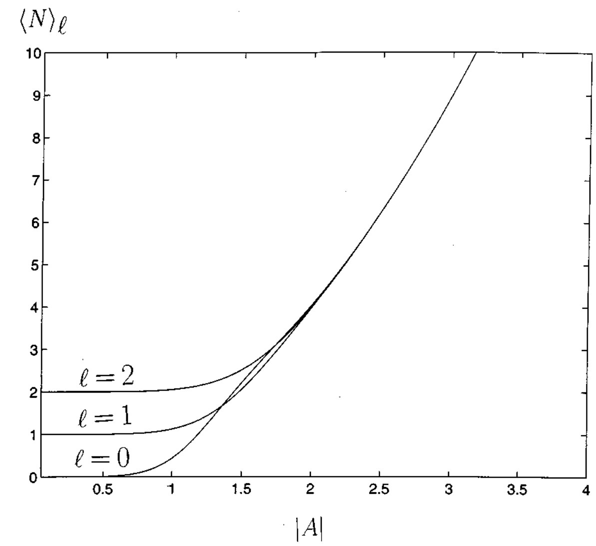

FIG. 2. $\langle N\rangle_\ell$ as a function of $|A|$ for $\ell=0,1,2$.

$$\pi_\ell=\sum_{N=0}^{\infty}|mN+\ell\rangle\langle mN+\ell|, \tag{57}$$

$$\sum_{\ell=0}^{m-1}\pi_\ell=\mathbf{1}. \tag{58}$$

We now introduce the ''generalized Schrödinger cats'' as superpositions of m coherent states uniformly distributed on a circle with center at the origin:

$$|A;\ell\rangle=\mathcal{N}_\ell(|A|)\sum_{k=0}^{m-1}\omega^{-\ell k}|A\omega^k\rangle, \tag{59}$$

$$\omega=\exp\left(i\frac{2\pi}{m}\right), \tag{60}$$

where $\mathcal{N}_\ell(|A|)$ is the normalization factor:

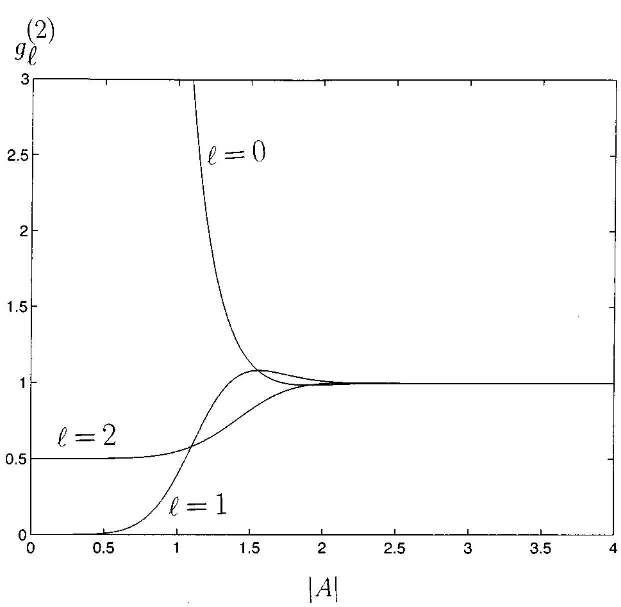

FIG. 3. $g_\ell^{(2)}$ as a function of $|A|$ for $\ell=0,1,2$.

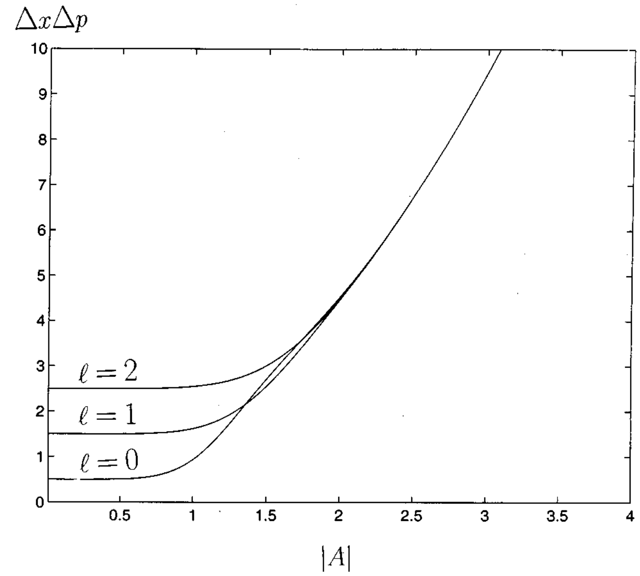

FIG. 4. $\Delta x\Delta p$ as a function of $|A|$ for arg(A)=0.

$$\mathcal{N}_\ell(|A|)=\left(m\sum_{\alpha=0}^{m-1}\omega^{-\ell\alpha}\exp[|A|^2(\omega^\alpha-1)]\right)^{-1/2}. \tag{61}$$

It is easy to see that they are eigenstates at the operator a^m,

$$a^m|A,\ell\rangle=A^m|A;\ell\rangle. \tag{62}$$

A useful relation is that for integers ℓ,k in Z_m (the integers modulo m):

$$\frac{1}{m}\sum_{k=0}^{m-1}\omega^{\ell k}=\delta_{\ell 0}, \tag{63}$$

where $\delta_{\ell 0}$ is equal to 1 if $\ell=0 \pmod m$; and equal to 0 if $\ell\neq 0 \pmod m$. Substituting Eq. (1) into Eq. (59) and using Eq. (63) we find that

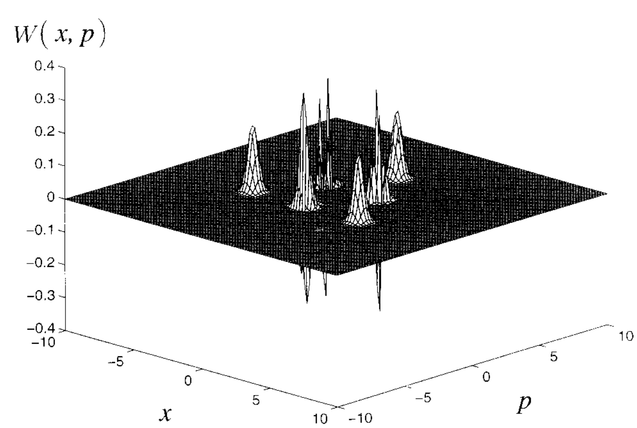

FIG. 5. $W(x,p)$ for the state $|3.2i;0\rangle$ of Eq. (59).

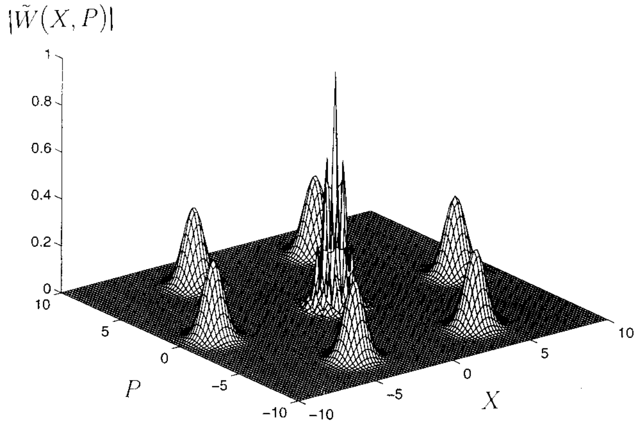

FIG. 6. $|\tilde{W}(X,P)|$ for the state $|3.2i;0\rangle$ of Eq. (59).

$$|A;\ell\rangle=\mathcal{N}_{\ell}(|A|)\exp\left(-\frac{|A|^2}{2}\right)\sum_{N=0}^{\infty}\frac{A^{mN+\ell}}{[(mN+\ell)!]^{1/2}}|mN+\ell\rangle$$

$$=\mathcal{N}_{\ell}(|A|)\,\pi_{\ell}|A\rangle. \tag{64}$$

It is clear that the state $|A;\ell\rangle$ belongs entirely in the Hilbert subspace H_{ℓ}. We can easily prove the following properties for these states:

$$|A\omega;\ell\rangle=\omega^{\ell}|A;\ell\rangle, \tag{65}$$

$$\sum_{\ell=0}^{m-1}[\mathcal{N}_{\ell}(|A|)]^{-1}|A;\ell\rangle=|A\rangle, \tag{66}$$

$$\langle A;\ell|B;k\rangle=\delta_{\ell k}\mathcal{N}_{\ell}(|A|)\mathcal{N}_{\ell}(|B|)\times\left[m\sum_{\alpha=0}^{m-1}\omega^{-\ell\alpha}\times\exp\left(-\frac{|A|^2}{2}-\frac{|B|^2}{2}+A^*B\omega^{\alpha}\right)\right]. \tag{67}$$

Equation (65) shows that it is sufficient to consider the values of A in

$$C_1=\left\{A=r\,\exp(i\phi);\;\;0\leqslant\phi<\frac{2\pi}{m};\quad r>0\right\}. \tag{68}$$

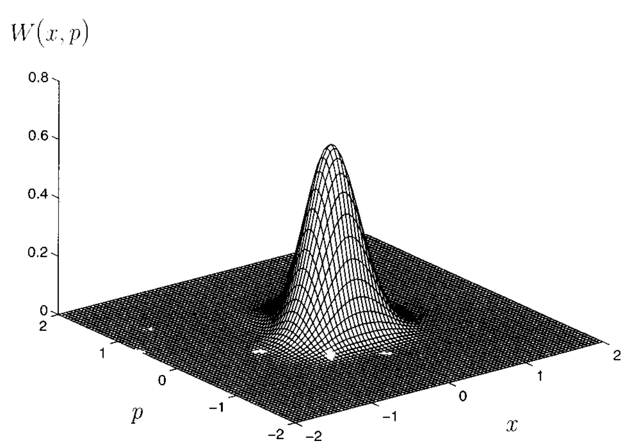

FIG. 7. $W(x,p)$ for the state $|0.2i;0\rangle$ of Eq. (59).

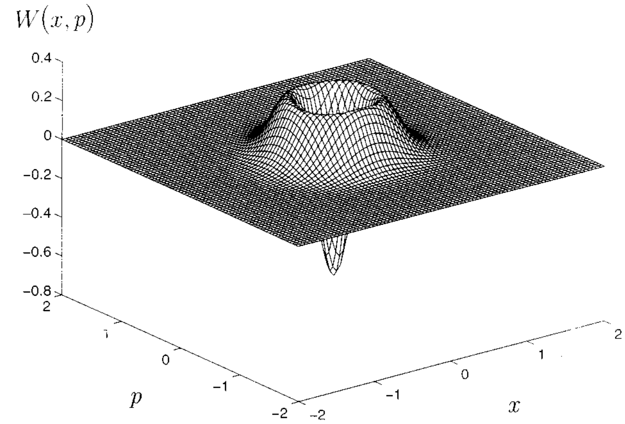

FIG. 8. $W(x,p)$ for the state $|0.2i;1\rangle$ of Eq. (59).

The rest of the values at A give the same coherent states (up to a trivial phase factor). Using the integral

$$\int_C\frac{d^2z}{\pi}\exp(-|z|^2)z^N(z^*)^M=(N!)\,\delta_{NM} \tag{69}$$

we can write a resolution of the identity in the Hilbert space H_{ℓ} using the states $|A;\ell\rangle$,

$$\int_C\frac{d^2A}{\pi}[\mathcal{N}_{\ell}(|A|)]^{-2}|A;\ell\rangle\langle A;\ell|=\pi_{\ell}. \tag{70}$$

It is clear that in Eq. (70) we can integrate over C_1 only, if at the same time multiply the left hand side by m. Summing over l we can have a resolution of the identity in the full Hilbert space H.

$$\sum_{\ell}\int_C\frac{d^2A}{\pi}[\mathcal{N}_{\ell}(|A|)]^{-2}|A;\ell\rangle\langle A;\ell|=\mathbf{1}. \tag{71}$$

Equations (67), (71) show that the $\{|A;\ell\rangle;A\in C_1,\ell\in Z_m\}$ form an overcomplete set of states. Note that states $|A;\ell\rangle$ and $|B,m\rangle$ with $\ell\neq m$ are orthogonal.

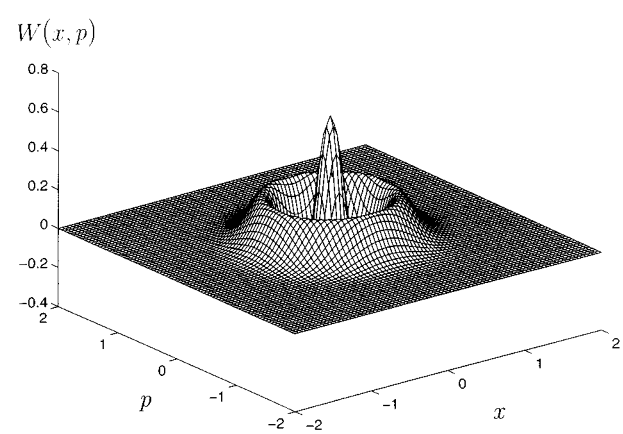

FIG. 9. $W(x,p)$ for the state $|0.2i;2\rangle$ of Eq. (59).

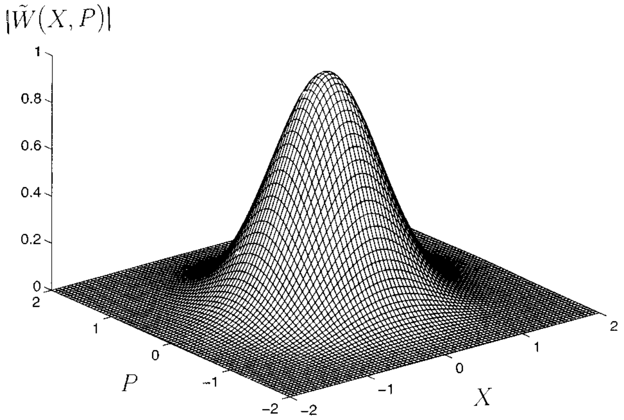

FIG. 10. $|\tilde{W}(X,P)|$ for the state $|0.2i;0\rangle$ of Eq. (59).

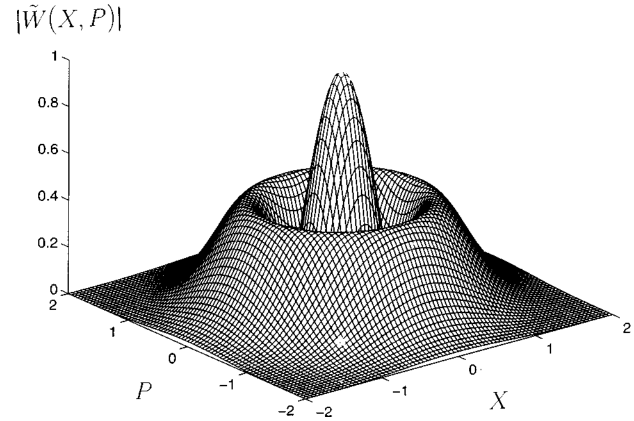

FIG. 11. $|\tilde{W}(X,P)|$ for the state $|0.2i;1\rangle$ of Eq. (59).

B. Quantum statistical properties of these states for the case $m=3$

The case $m=2$ has been studied in the literature in the context of Schrödinger cats [1], so we consider here examples with $m=3$. Using Eq. (59) we calculate the probabilities

$$p_\ell(N)\equiv|\langle N|A,\ell\rangle|^2=[\mathcal{N}_\ell(|A|)]^2\exp(-|A|^2)\times\left|\frac{A^N}{(N!)^{1/2}}+\frac{(A\omega)^N\omega^{-\ell}}{(N!)^{1/2}}+\frac{(A\omega^2)^N\omega^{-2\ell}}{(N!)^{1/2}}\right|^2. \tag{72}$$

It is seen that $p_\ell(N)$ depends only on $|A|$ and is independent of arg(A). Using this we calculate the average number of photons

$$\begin{aligned}\langle N\rangle_\ell\equiv\sum_{N=0}^{\infty}Np_\ell(N)=&[\mathcal{N}_\ell(|A|)]^2|A|^2\\&\times\{3+2\omega^{(1-\ell)}\exp[|A|^2(\omega-1)]\\&+2\omega^{(\ell-1)}\exp[|A|^2(1/\omega-1)]\\&+\omega^{2(1-\ell)}\exp[|A|^2(\omega^2-1)]\\&+\omega^{2(\ell-1)}\exp[|A|^2(1/\omega^2-1)]\}.\end{aligned} \tag{73}$$

We also calculate numerically the quantities

$$\langle N^2\rangle_\ell\equiv\sum_{N=0}^{\infty}N^2p_\ell(N) \tag{74}$$

and the

$$g_\ell^{(2)}=\frac{\langle N^2\rangle_\ell-\langle N\rangle_\ell}{\langle N\rangle_\ell^2}. \tag{75}$$

In Fig. 1 we present the normalization constant $[\mathcal{N}_\ell(|A|)]^{-2}$ as a function of $|A|$. It is seen that for large $|A|$, when the three coherent states of Eq. (59) become ''almost orthogonal,'' it takes a value very close to 3. It is also seen that for $\ell=1,2$ it takes very small values for $|A|$ close to zero; and for this reason all our numerical results are for $|A|>0.1$. In Fig. 2 we present the $\langle N\rangle_\ell$ as a function of $|A|$. Equation (64) shows that for small $|A|$ the state $|A;\ell\rangle$ is approximately the number eigenstate $|\ell\rangle$. Therefore, for $|A|$ close to zero, the $\langle N\rangle_\ell$ takes the values 0,1,2 for $\ell=0,1,2$ correspondingly. In Fig. 3 we present $g_\ell^{(2)}$ as a function of $|A|$. It is seen that for $\ell=1,2$ we have antibunching (i.e., $g_\ell^{(2)}<1$) for $|A|<1.5$. For large values of $|A|$ the $g_\ell^{(2)}$ is approximately equal to 1.

With regard to the uncertainties of these states, it is easily seen that $\langle x\rangle=\langle p\rangle=0$. In Fig. 4 we present the uncertainty product $\Delta x\Delta p$ as a function of $|A|$ for arg(A)=0°. Note that in the case arg(A)=0° considered here, $K=0$. As we have explained, for small $|A|$ the state $|A;\ell\rangle$ is approximately the number eigenstate $|\ell\rangle$ and therefore the $\Delta x\Delta p$ is equal to $\ell+\frac{1}{2}$. For large $|A|$ the $\Delta x\Delta p$ takes large values.

C. Wigner and Weyl functions

In this section we apply the formalism developed in Sec. II B to the states $|A;\ell\rangle$ of Eq. (59) (with $m=3$). We consider the state $|3.2i;0\rangle$ and present its Wigner function $W(x,p)$ in Fig. 5. We see clearly the three Gaussian auto terms which are (up to normalization) the Wigner functions of the coherent states $|3.2i\rangle$, $|3.2i\omega\rangle$, and $|3.2i\omega^2\rangle$; and also the three cross terms W_{12}, W_{13}, and W_{23} which describe the interference. We also present the Weyl function for the state $|3.2i;0\rangle$ in Fig. 6. We see clearly the sum of all the auto terms located around the center; and also the six cross terms

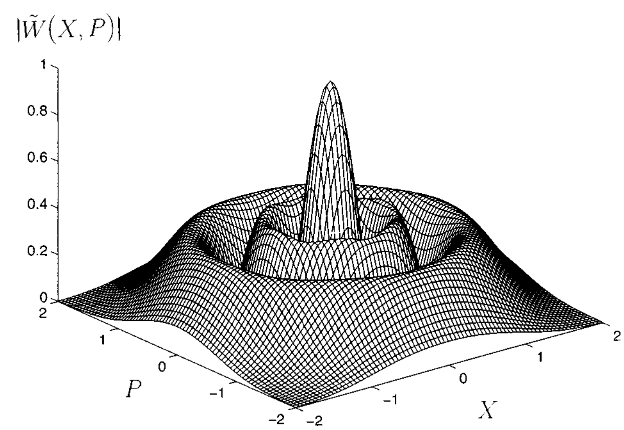

FIG. 12. $|\tilde{W}(X,P)|$ for the state $|0.2i;2\rangle$ of Eq. (59).

$\tilde{W}'_{i,j}$. The cross terms are located at those (X,P) for which displacements of the state $|3.2i;0\rangle$ by (X,P) produce another state which overlaps significantly with the original one. For example, displacement of the $|3.2i;0\rangle$ by $(X=3.9, P=-6.75)$ brings the coherent state $|3.2i\rangle$ to overlap with the coherent state $|3.2i\omega\rangle$; and for this reason we get significant cross terms around the point $(X=3.9,\ P=-6.75)$. We see clearly here the role of the Weyl functions as generalized correlation functions where displacements in both position and momentum are performed.

In the above examples we only present the case $\ell=0$ because the cases $\ell=1$ and $\ell=2$ do not present any differences which are easily visible. In Figs. 7, 8, and 9 we present the Wigner function for the states $|0.2i;0\rangle$, $|0.2i;1\rangle$, and $|0.2i;2\rangle$ correspondingly. It is clear that there are significant differences between these three cases. In Figs. 10, 11, and 12 we present the absolute value at the Weyl function for the states $|0.2i;0\rangle$, $|0.2i;1\rangle$, and $|0.2i;2\rangle$ correspondingly. Here again there are significant differences between the three cases.

IV. DISCUSSION

The Weyl functions are an important tool in the area of quantum phase-space methods. We have studied their properties emphasizing in particular their significance for the description of quantum interference phenomena. The general case of a quantum state which is a superposition of m other quantum states has been considered, and both its Wigner function and its Weyl function have been decomposed into auto terms and cross terms. The cross terms provide a valuable insight into the interference.

The study of superpositions of m coherent states uniformly distributed on a circle is an interesting problem in its own right. Here we have studied the properties of these states and shown that they form an overcomplete basis in the Hilbert space, that there is a resolution of the identity in terms of them, etc. Numerical results (for $m=3$) have demonstrated the quantum statistical properties of these states and shown clearly the cross terms in their Wigner and Weyl functions and the interference effects.

As a final comment we point out that there is a similarity between the phase-space techniques in quantum mechanics and the so-called time-frequency methods in signal processing, as has been explained by Gabor and Ville [18,19]. In this context also the Wigner function and the so-called ambiguity function (which is similar to the Weyl function), play an important role.

It is our belief that the Weyl function should play a central role in quantum mechanics and quantum optics, especially in problems where quantum interference takes place [20].

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