Time of arrival in quantum and Bohmian mechanics

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In a recent paper Grot, Rovelli, and Tate (GRT) [Phys. Rev. A **54**, 4676 (1996)] derived an expression for the probability distribution $\pi(T;X)$ of intrinsic arrival times $T(X)$ at position $x=X$ for a quantum particle with initial wave function $\psi(x,t=0)$ freely evolving in one dimension. This was done by quantizing the classical expression for the time of arrival of a free particle at X, assuming a particular choice of operator ordering, and then regulating the resulting time of arrival operator. For the special case of a minimum-uncertainty-product wave packet at $t=0$ with average wave number $\langle k\rangle$ and variance Δk they showed that their analytical expression for $\pi(T;X)$ agreed with the probability current density $J(x=X,t=T)$ only to terms of order $\Delta k/\langle k\rangle$. They dismissed the probability current density as a viable candidate for the exact arrival time distribution on the grounds that it can sometimes be negative. This fact is not a problem within Bohmian mechanics where the arrival time distribution for a particle, either free or in the presence of a potential, is rigorously given by $|J(X,T)|$ (suitably normalized) [W. R. McKinnon and C. R. Leavens, Phys. Rev. A **51**, 2748 (1995); C. R. Leavens, Phys. Lett. A **178**, 27 (1993); M. Daumer *et al.*, in *On Three Levels: The Mathematical Physics of Micro-, Meso-, and Macro-Approaches to Physics*, edited by M. Fannes *et al.* (Plenum, New York, 1994); M. Daumer, in *Bohmian Mechanics and Quantum Theory: An Appraisal*, edited by J. T. Cushing *et al.* (Kluwer Academic, Dordrecht, 1996)]. The two theories are compared in this paper and a case presented for which the results could not differ more: According to GRT's theory, every particle in the ensemble reaches a point $x=X$, where $\psi(x,t)$ and $J(x,t)$ are both zero for all t, while no particle ever reaches X according to the theory based on Bohmian mechanics. Some possible implications are discussed. [S1050-2947(98)02008-3]

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I. INTRODUCTION

Grot, Rovelli, and Tate [1] (GRT) recently took up the long-standing challenge [2–6] of formulating within conventional quantum mechanics the concept of the time of arrival $T(X)$ at the spatial location $x=X$ of a quantum particle with given initial wave function and of deriving an expression for the probability distribution $\pi(T;X)$ of such arrival times. They maintained that this is a well-posed problem in simple quantum theory and that there must be a solution. They concentrated on the special case of a freely evolving particle, leaving the general problem with a nonzero potential for a future work.

Their approach begins by quantizing the expression for the arrival time for the corresponding, trivially solved, problem in classical mechanics:

$$T(X)=\frac{m(X-x_0)}{p_0}\Rightarrow \hat{T}(X)=\frac{m(X-\hat{x}_0)}{\hat{p}_0}, \tag{1}$$

where x_0 and p_0 are the initial $(t=0)$ values of the classical particle's position and momentum and $\hat{x}_0$ and $\hat{p}_0$ are the corresponding Heisenberg operators for the quantum particle. GRT tentatively adopted the symmetric ordering

$$\frac{1}{\hat{p}_0^{1/2}}\,\hat{x}_0\,\frac{1}{\hat{p}_0^{1/2}} \tag{2}$$

for the noncommuting operators $\hat{x}_0$ and $\hat{p}_0^{-1}$ and then worked in the Heisenberg momentum $(p=\hbar k)$ basis, writing (for $X=0$)

$$\hat{T}(0)=-i\,\frac{m}{\hbar}\,\frac{1}{k^{1/2}}\,\frac{d}{dk}\,\frac{1}{k^{1/2}}, \tag{3}$$

with $k^{1/2}=i|k|^{1/2}$ for $k<0$. To bypass the difficulty that different eigenfunctions $|T\rangle$ of $\hat{T}(0)$ are not in general orthogonal, a problem that they traced to the singular behavior of Eq. (3) at $k=0$, they replaced Eq. (3) by the regulated time of arrival operator

$$\hat{T}_\epsilon(0)=-i\,\frac{m}{\hbar}\,f_\epsilon(k)^{1/2}\,\frac{d}{dk}\,f_\epsilon(k)^{1/2}, \tag{4}$$

with

$$f_\epsilon(k)=k^{-1}\Theta(|k|-\epsilon)+\epsilon^{-2}k\Theta(\epsilon-|k|), \tag{5}$$

where ϵ is an arbitrary small positive number. They showed that the (doubly degenerate) eigenfunctions $|T,\pm\rangle_\epsilon$ of $\hat{T}_\epsilon(0)$,

$$\langle k|T,\pm\rangle_\epsilon=\Theta(\pm k)\left(\frac{\hbar}{2\pi m f_\epsilon(k)}\right)^{1/2}\exp\left(i\,\frac{\hbar T}{m}\int_{\pm\epsilon}^{k}\frac{dk'}{f_\epsilon(k')}\right), \tag{6}$$

in the k representation, form a complete orthonormal basis. GRT then extended the usual definition of the arrival time problem to include arrival times in the interval $[-\infty,0]$ by imagining that the particle was prepared at $t=-\infty$ in the state that, in the assumed absence of any interaction, would evolve in the Schrödinger position basis to the desired initial state $\psi_0(x)\equiv\psi(x,t=0)$, with Fourier transform $\phi(k)$, at $t=0$. GRT claimed that for the extended arrival time problem, just as in the classical case (with $p_0 \neq 0$), a free quantum particle [in a state with $\phi(0)=0$] is certain to reach $x=X$ at some time $T(X)$ in the interval $[-\infty,+\infty]$ so that complex values of $T(X)$, corresponding to the particle never reaching X, do not occur. $\hat{T}_\epsilon(X)$ is then self-adjoint and the standard probability interpretation applies, leading in a straightforward way [assuming that $\phi(k=0)$ is negligibly small] to

$$\pi(T;0)=|_\epsilon\langle T,+|\psi\rangle|^2+|_\epsilon\langle T,-|\psi\rangle|^2 = \pi_+(T;0)+\pi_-(T;0), \tag{7}$$

with[1]

$$\pi_\pm(T;0)=\frac{\hbar}{2\pi m}\left|\int_0^{\pm\infty} dk k^{1/2}\exp\left(-i\frac{\hbar k^2 T}{2m}\right)\phi(k)\right|^2. \tag{8}$$

The corresponding distribution for $X\neq 0$ contains a factor of $\exp(ikX)$ in the integrand of Eq. (8). The components of $\pi(T;X)$ associated with positive and negative values of k respectively were interpreted as the contributions $\pi_\pm(T;X)$ to the arrival time distribution from particles arriving at $x=X$ from the left and from the right. This decomposition follows, according to GRT, from the key point that $\hat{T}_\epsilon$ commutes with the operator giving the sign of k.

Grot, Rovelli, and Tate used their theory to calculate arrival time distributions for the special case in which the "initial" ($t=0$) wave function $\psi_0(x)$ is a minimum-uncertainty-product Gaussian. They ignored the contribution from $|k|<\epsilon$ on the grounds that ϵ can be taken to be arbitrarily small and in the numerical calculations used parameters such that the relative width $\Delta k/\langle k\rangle$ of $|\phi(k)|^2$ was very small. They compared their analytic expression for $\pi(T;X)$ with the probability current density and found by expansion that they agreed only to first order in $\Delta k/\langle k\rangle$. They stressed the long familiar fact [2] that $J(X,T)$ cannot be identified with the correct arrival time distribution because it can be negative.[2] They suggested that whether or not their result for $\pi(T;X)$ based on the particular operator ordering (2) is physically correct might be decided experimentally. An extension of their theory to the relativistic (Klein-Gordon) case by León [7] provided some strong support for Eq. (2).

The purpose of the present paper is to raise the following theoretical points. The arrival time problem is simply and unambiguously solved not only in classical mechanics but also in Bohmian mechanics [8–14] where, for arbitrary scattering potential $V(x)$, one finds [15–17] *for those particles that actually reach* $x=X$

$$\pi(T;X)=\frac{|J(X,T)|}{\int_{-\infty}^{+\infty}dt|J(X,t)|}, \tag{9}$$

with

$$\pi_\pm(T;X)=\frac{\pm J(X,T)\Theta(\pm J(X,T))}{\int_{-\infty}^{+\infty}dt|J(X,t)|}. \tag{10}$$

(For the usual formulation of the problem where only arrival times subsequent to the initial time $t=0$ are considered, the lower limit on the integrals should be 0.) The arrival time distribution of Bohmian mechanics is in general different, and can be qualitatively different, from that derived by GRT. GRT's claim for the extended arrival time problem that every freely evolving particle is certain to reach $x=X$, for arbitrary X and initial state $\psi_0(x)$ with $\phi(0)=0$, is definitely not the case within Bohmian mechanics.[3] It is important to know whether their claim is a rigorous result of conventional quantum mechanics or simply a plausible conjecture.

Section II contains a very brief sketch of the ingredients of Bohmian mechanics needed for the one-dimensional arrival time problem. Section III compares GRT's theory of arrival time distributions with that based on Bohmian mechanics using two simple case studies for illustrative purposes. Emphasis is given to the key question of whether or not every free particle in a state having $\phi(0)=0$ must arrive at an arbitrary point $x=X$ at some (real) time T. Concluding remarks are made in Sec. IV.

II. ESSENTIALS OF BOHMIAN MECHANICS

In Bohmian mechanics [8–14], tailored to the problem of interest here, it is postulated that an electron, say, propagating in a potential $V(x)$ is an actual pointlike particle that is always associated with a field that probes the potential and guides the particle's motion accordingly so that it has a well-defined position $x(t)$ and velocity $v(t)$ at each instant of time t. It is also postulated that the guiding field in the nonrelativistic case is the solution $\psi(x,t)$ of the time-dependent Schrödinger equation and that the particle's equation of motion is $v(t)\equiv dx(t)/dt=v(x,t)_{x=x(t)}$, where the velocity field $v(x,t)$ is given by

$$v(x,t)=\frac{J(x,t)}{|\psi(x,t)|^2}, \tag{11}$$

with $J(x,t)=(\hbar/m)\mathrm{Im}[\psi^*(x,t)\partial\psi(x,t)/\partial x]$. (This is the simplest equation of motion that is Galilean and time-reversal invariant [10].) It follows from these postulates generalized to the N-particle case [8] that $|\psi(x,t)|^2dx$ is, as assumed and partially justified by Bohm, the probability of the particle *being* between x and $x+dx$ at time t [10]. These basic postulates do not mention measurement, which is not regarded as a primary concept in Bohm's theory (hence the use of "*being*" instead of "*being found*" in the previous sentence). An experiment on a quantum particle does not as a rule reveal the intrinsic value of the property supposedly being measured. An important exception is an ideal position mea-

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[1]The author has taken the liberty of correcting a sign error in the argument of the exponential.

[2]It is the fact that $J(X,T)$ can change sign as T is varied, even if $\phi(k)$ is nonzero only for k of one sign [2,3,18], which disqualifies both $+J(X,T)$ and $-J(X,T)$ as arrival time distributions.

[3]We are not concerned here with the quantum analog of free classical particles with $p_0=0$ that never move (these are eliminated as a source of concern in GRT's theory by their regulation procedure) but with free Bohmian particles that can "turn around" before reaching $x=X$ and never reach that point.

surement which plays a central role in the application of Bohm's theory to measurement in general. The theory was originally constructed so that, to the extent that any measurement is ultimately a position measurement (e.g., that of a pointer), it gives precisely the same statistical prediction for any experimental quantity as conventional quantum mechanics whenever the prediction of the latter is unambiguous. For the usual textbook measurement made at an instant of time selected by the experimentalist (or some device *external* to the system of interest) there is no ambiguity and the statistical distribution of pointer positions predicted by the two theories are identical.[4] However, in a time of arrival measurement the experimentalist selects the point X not the time $T(X)$ at which a given particle arrives [4] and it is not yet evident that the conventional quantum theory of measurement based on a self-adjoint operator for the intrinsic system property of interest can be made to work in this case. It is already clear [2–6,1,7] that, if possible, it will not be an easy task. Perhaps looking at the problem from the point of view of Bohm's theory will provide some useful insight.

Now, given the initial wave function $\psi(x,t=0)$ and particle position $x^{(0)}\equiv x(t=0)$ of an electron, its subsequent motion is uniquely determined by simultaneous integration of the time-dependent Schrödinger equation for $\psi(x,t)$ and the equation of motion for $x(t)$ to obtain the Bohm trajectory $x(x^{(0)},t)$. In Bohm's deterministic theory uncertainty enters only through the probability distribution $|\psi(x^{(0)},0)|^2$ for the unknown initial position $x^{(0)}$ of the particle. The probability distribution for a particle property f that is defined for all trajectories is given by

$$P(f)\equiv\int_{-\infty}^{+\infty}dx^{(0)}|\psi(x^{(0)},0)|^2\delta(f-f(x^{(0)})), \tag{12}$$

where $f(x^{(0)})$ is the value of the property for a particle following the trajectory $x(x^{(0)},t)$. For particle properties that are not defined for some trajectories it is necessary to restrict the range of integration in Eq. (12) to exclude those trajectories and to normalize the resulting distribution accordingly. This is, in general, the case for the arrival time $T(X)$. This is obvious for X on the far side of a barrier. However, even for electrons propagating freely from $t=-\infty$ to $t=+\infty$ there can be trajectories that *never* reach a given point X so that the associated arrival times at X are undefined. When only those particles that actually reach X are included in the analysis the arrival time distribution is given by Eq. (9). This result holds even in the presence of a potential barrier.

The derivation of Eq. (9) is simple when one takes into account the well-known nonintersection property of Bohm trajectories $x(x^{(0)},t)$ with different starting points $x^{(0)}$ [but the same initial wave function $\psi(x,0)$]: If $x^{(0)\prime}\neq x^{(0)}$ then $x(x^{(0)\prime},t)\neq x(x^{(0)},t)$ for any t. This means that only a single Bohm trajectory contributes to the current density $J(X,T)$ at the particular space-time point $(x=X,t=T)$. With this fact in mind let us consider the complete range of starting points $x^{(0)}$ for each of which the associated trajectory $x(x^{(0)},t)$ reaches $x=X$ at least once at some time(s) $T(X;x^{(0)})$ within the temporal range of interest. Because of the nonintersection property the desired range of $x^{(0)}$ must consist of a single continuous interval, say $[x_a^{(0)},x_b^{(0)}]$. The (unnormalized) arrival time distribution is

$$\int_{x_a^{(0)}}^{x_b^{(0)}}dx^{(0)}|\psi(x^{(0)},0)|^2\delta(T-T(X;x^{(0)})). \tag{13}$$

Again because of the nonintersection property, there is one and only one value of $x^{(0)}$ in the interval $[x_a^{(0)},x_b^{(0)}]$ for which the trajectory $x(x^{(0)},t)$ reaches X at a particular value of T within the range of interest. In addition, of course, even if that trajectory reaches X more than once only one of its arrival times is equal to the specified value of T. Hence

$$\begin{aligned}\delta(x(x^{(0)},t)-X)|_{t=T}&=\left.\frac{\delta(t-T(X;x^{(0)}))}{|dx(x^{(0)},t)/dt|}\right|_{t=T}\\&=\left.\frac{\delta(t-T(X;x^{(0)}))}{|v(x(x^{(0)},t),t)|}\right|_{t=T}\end{aligned} \tag{14}$$

contains only a single term and Eq. (13) becomes

$$|v(X,T)|\int_{x_a^{(0)}}^{x_b^{(0)}}dx^{(0)}|\psi(x^{(0)},0)|^2\delta(x(x^{(0)},T)-X). \tag{15}$$

The integral is just the probability density $|\psi(X,T)|^2$ and Eq. (15) reduces to

$$|v(X,T)||\psi(X,T)|^2=|J(X,T)|, \tag{16}$$

using Eq. (11). It is important to note that the modulus sign is not added by hand but emerges naturally via the standard formula for changing the argument of a Dirac δ function. Normalization gives Eq. (9), which has the nice property that particles that never reach X do not contribute to $J(X,t)$ at any time t and are automatically excluded from the arrival time distribution. The denominator of Eq. (9) is the fraction of particles in the ensemble that reach X if and only if each of these particles reaches X just once. Now, it follows from Eq. (11) for the velocity field $v(x,t)$ that $J(X,T)>0$ corresponds to a particle arriving at $x=X$ at $t=T$ from the left and $J(X,T)<0$ corresponds to a particle arriving at X at time T from the right, leading immediately to the decomposition (10).

Neither Eq. (9) nor Eq. (10) necessarily follows for an ensemble of classical particles because a positive (or negative) current at (X,T) can in general have contributions from both left-going and right-going particles. Hence, contrary to the case in (pure state) Bohmian mechanics, it is possible in classical mechanics to have exactly zero particle current at (X,T) with $\pi(T,X)$ nonzero, even large.[5] This is not a problem in classical mechanics because the particle trajectories can be used to decompose the current at X into left-going and

[4]The microscopic interpretation of the distribution of pointer positions can, however, be very different in the two theories.

[5]The second case study in Sec. III shows that according to GRT's theory this is also possible for a pure state within conventional quantum mechanics.

right-going components at any instant of time T. On the other hand, within conventional quantum mechanics the right-going and left-going components, J_+ and J_-, respectively, of the probability current density are in general ill-defined quantities and some would regard attempts [19–21] to carry out such a decomposition as a meaningless exercise.

III. COMPARISON OF TWO ARRIVAL TIME THEORIES

Grot, Rovelli, and Tate stated without explicit proof that, for the extended definition of time of arrival, a free $[V(x)=0]$ particle in one dimension always reaches an arbitrary position X so that $T(X)$ is never complex. If this at first sight very plausible claim is true then it should follow from an exact theory of arrival times based on conventional quantum mechanics. It seems instead to be a basic premise of GRT's theory so that the most that one can hope for is to show self-consistency. The theory of Muga, Brouard, and Maciás [5], based on their perfectly absorbing complex potential model for a particle detector, applies only for X located sufficiently far to the left or right of the initial wave packet that $J(X,t)$ does not change sign for the range $t\geqslant 0$ of interest. Hence this theory is not general enough to check GRT's claim. In any case, Bohmian mechanics provides an internally consistent possible scenario in which their claim is not in general upheld. This is now shown explicitly for two simple choices of initial wave function $\psi_0(x)$, the first with $\phi(0)$ very small but nonzero and the second with $\phi(0)$ exactly zero.

First consider the free evolution of the minimum-uncertainty-product initial wave function investigated in detail by GRT

$$\psi(x,0)=\frac{1}{[2\pi(\Delta x)^2]^{1/4}}\exp\left[-\left(\frac{x-a}{2\Delta x}\right)^2+iKx\right], \quad (17)$$

with centroid $a<0$ and mean wave number $\langle k\rangle=K>0$ for definiteness. Integration of the time-dependent Schrödinger equation and the equation of motion gives

$$x(x^{(0)},t)=a+\hbar Kt/m+(x^{(0)}-a)\left(1+\frac{\hbar^2t^2}{4m^2(\Delta x)^4}\right)^{1/2} \quad (18)$$

for the Bohm trajectory with $x=x^{(0)}$ at $t=0$. A selection of such trajectories is shown in Fig. 1. The position x of a particle following the trajectory with $x^{(0)}=x_-^{(0)}\equiv a-2(\Delta x)^2K$ increases monotonically from $x=-\infty$ at $t=-\infty$ to $x=a$ at $t=+\infty$, while that of a particle following the trajectory with $x^{(0)}=x_+^{(0)}\equiv a+2(\Delta x)^2K$ increases monotonically from $x=a$ at $t=-\infty$ to $x=+\infty$ at $t=+\infty$. These two special trajectories are shown as dotted lines in the figure. They act as bifurcation lines, together separating the trajectories into three distinct groups. Those with $x^{(0)}<x_-^{(0)}$ start at $x=-\infty$ at $t=-\infty$ and end at $x=-\infty$ at $t=+\infty$. For $a<0$, the case under consideration, it follows from the non-intersection property of Bohm trajectories that a particle following a trajectory with $x^{(0)}<x_-^{(0)}$ never reaches $x=X=0$. Trajectories with $x_-^{(0)}<x^{(0)}<x_+^{(0)}$ start at $x=-\infty$ at $t=-\infty$ and end at $x=+\infty$ at $t=+\infty$ passing through $x=X=0$ once and only once. Trajectories with $x^{(0)}>x_+^{(0)}$ start at $x=+\infty$ at

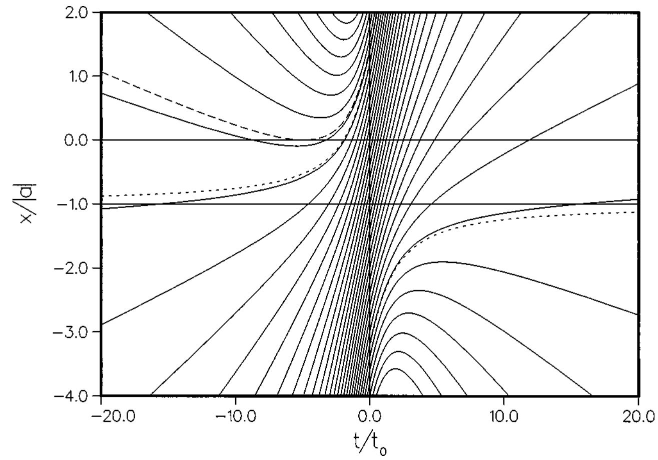

FIG. 1. Bohm trajectories for the Gaussian "initial" wave function $\psi(x,0)$ of Eq. (17) with $a=-10$ Å, $\langle k\rangle=K=1$ Å^{-1}, and $\Delta k=(2\Delta x)^{-1}=0.15$ Å^{-1}. The "initial" positions for the set of trajectories shown by the continuous curves are $x^{(0)}=-26,-24,\ldots,0,\ldots,42,44$ Å. The dotted curves show the special trajectories discussed in the text with $x^{(0)}=x_+^{(0)}$ and $x_-^{(0)}$ and the dashed curve the one with $x^{(0)}=x_0^{(0)}$. [$t_0\equiv|a|/(\hbar K/m)$.]

$t=-\infty$ and end at $x=+\infty$ at $t=+\infty$. For $a<0$, the member of this group with $x^{(0)}=x_0^{(0)}\equiv a+[a^2+4K^2(\Delta x)^4]^{1/2}$ just reaches $x=X=0$ where it turns around and heads back in the direction of $x=+\infty$. This special trajectory, shown as a dashed line in Fig. 1, divides the third group into those that pass through $x=X=0$ twice and those that never reach $x=0$. For the latter, the solution $T(X=0)$ of $x(x^{(0)},T)=0$ is complex valued.

More generally, for the initial wave function (17) with a and K arbitrary and Δx not infinite, there is no choice of X for which $T(X)$ is never complex. However, for $\Delta k\ll K$, the regime in which GRT applied their theory to Eq. (17), the overwhelming number of particles have $x^{(0)}$ in the range $(x_-^{(0)},x_+^{(0)})$, of width $4(\Delta x)^2K=2(K/\Delta k)\Delta x\gg 2\Delta x$ centered on a, where Bohm trajectories pass precisely once through $x=X=0$. We now consider an initial wave function $\psi_0(x)$ for which $\phi(k=0)=0$ and no Bohm trajectory reaches $x=X$ when $X=0$ and the probability of a particle (of unknown $x^{(0)}$) reaching $x=X$ is less than 1/2 for any value of X.

Consider the initial wave function

$$\psi(x,0)=N\left[\exp\left(-\frac{(x-a)^2}{4(\Delta x)^2}+iKx\right)-\exp\left(-\frac{(x+a)^2}{4(\Delta x)^2}-iKx\right)\right], \quad (19)$$

where

$$N=\left\{2^{3/2}\pi^{1/2}\Delta x\left[1-\exp\left(-2(\Delta x)^2K^2-\frac{a^2}{2(\Delta x)^2}\right)\right]\right\}^{-1/2}. \quad (20)$$

Its Fourier transform is

$$\phi(k)=N2\pi^{1/2}\Delta x\{\exp[-(k-K)^2(\Delta x)^2-i(k-K)a] -\exp[-(k+K)^2(\Delta x)^2+i(k+K)a]\} \quad (21)$$

and has the important properties that $\phi(0)=0$ and $\phi(\pm\infty)=0$. Solution of the time-dependent Schrödinger equation [with $V(x)=0$] gives

$$\psi(x,t)=2N\Delta x[(\Delta x)^2-i\hbar t/2m]^{1/2}\eta^{1/2}\exp[-K^2(\Delta x)^2+iKa]\exp[\beta_0+\beta_2x^2+i(\gamma_0+\gamma_2x^2)]\times[\exp(\delta x+i\epsilon x)-\exp(-\delta x-i\epsilon x)] \quad (22)$$

and

$$J(x,t)=(\hbar/m)2N^2(\Delta x)^2\eta^{1/2}\exp[-2K^2(\Delta x)^2]\times\exp[2(\beta_0+\beta_2x^2)]\{4\gamma_2x[\cosh(2\delta x)-\cos(2\epsilon x)]+2\epsilon\sinh(2\delta x)-2\delta\sin(2\epsilon x)\}, \quad (23)$$

where

$$\eta\equiv[4(\Delta x)^4+(\hbar t/m)^2]^{-1}, \quad (24)$$

$$\beta_0\equiv\eta[4(\Delta x)^6K^2-(\Delta x)^2a^2-2Ka(\Delta x)^2\hbar t/m], \quad (25)$$

$$\beta_2\equiv-\eta(\Delta x)^2, \quad (26)$$

$$\gamma_0\equiv-\eta\{4Ka(\Delta x)^4+[4K^2(\Delta x)^4-a^2]\hbar t/2m\}, \quad (27)$$

$$\gamma_2\equiv\eta\hbar t/2m, \quad (28)$$

$$\delta\equiv2\eta(\Delta x)^2(a+K\hbar t/m), \quad (29)$$

$$\epsilon\equiv2\eta[2K(\Delta x)^4-a\hbar t/2m]. \quad (30)$$

The probability density $|\psi(x,t)|^2$ is zero at $x=0$ for all t. Within (nonrelativistic) Bohmian mechanics a particle can reach a point $x=X$ at a time $t=T$ when $|\psi(X,T)|^2=0$ only if $v(X,T)=\pm\infty$ [22]. From Eqs. (22) and (23), respectively, it follows that as $x\to0$, for any t, $|\psi(x,t)|^2\to0$ as x^2 and $J(x,t)\to0$ as x^3 so that $v(x,t)\to0$ as x. [The coefficient multiplying x depends on t allowing $v(x,t)$ to vanish at points (x,t) with $x\neq0$ so that particle trajectories can turn around before reaching $x=0$.] Hence, according to Bohm's theory, a particle with the wave function (22) for all time never reaches the point $x=X=0$, i.e., $T(0)$ is complex for all possible starting points $x^{(0)}$. Furthermore, if such a particle has $x^{(0)}<0$ then it never reaches a point $x=X>0$. On the other hand, some of the particles with $x^{(0)}>0$ reach $x=X>0$ twice, once from the right and once from the left. This behavior is completely different from the prediction of GRT's theory that every free particle in a state with $\phi(0)=0$ is certain to arrive at any point X at some (real) time T.

Since, according to Bohmian mechanics, no particles reach $X=0$ for the initial wave function (19) the arrival time distribution is undefined for this special case. Accordingly, Fig. 2 compares the $\pm$ components of the arrival time distributions $\pi(T;X)$ of the two theories for the small but finite value $X=a/100<0$. According to the calculation based on Bohm's theory, for this particular value of X only about 1 out of every 88 particles in the ensemble reaches X, first from the left and then, after turning around, from the right. According to GRT's theory, all of the particles in the ensemble reach $x=X$, half from the right and half from the left, with those from the left arriving slightly earlier on average because $X/|a|$ is small and negative.

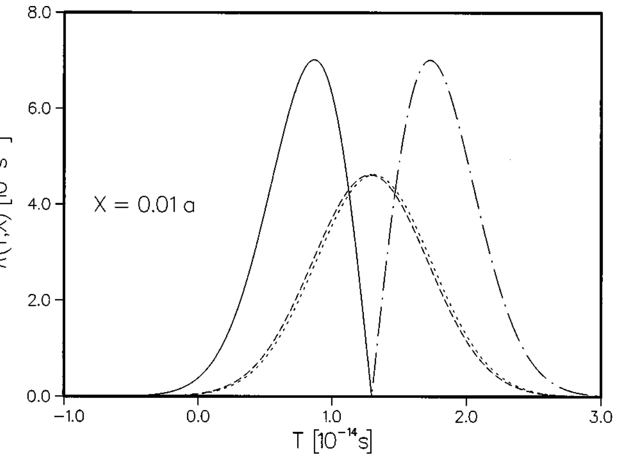

FIG. 2. Arrival time distribution components $\pi_\pm(T;X)$ for those free particles that actually reach the position $x=X=a/100$ at some time(s) T in the range $[-\infty,+\infty]$ calculated for the initial wave function (19) with $K=1$ Å^{-1}, $\Delta k=(2\Delta x)^{-1}=0.01$ Å^{-1}, and $a=-150$ Å. The $\pm$ components of GRT's distribution are shown by the long-dashed and short-dashed curves, respectively; those of the distribution based on Bohmian mechanics are shown by the solid and dash-dotted curves, respectively.

Now consider an entirely different situation in which half of the particles in the ensemble have an initial wave function that is just the first component of Eq. (19) [i.e., Eq. (17)] and half have an initial wave function that is the second component, both suitably renormalized. For $|K\Delta x|\gg1$ the fraction of the particles that, according to Bohmian mechanics, turn around before reaching $X=0$ (or $X=a/100$) is negligible and the results for the distributions $\pi_\pm(T;X)$ for the entire mixed ensemble will be very close to the corresponding results of GRT's theory. These in turn will be virtually indistinguishable from GRT's results for the ensemble described by the pure state (19) because the first term of Eq. (19) is dominated by Fourier components with $k\sim K>0$ and contributes almost exclusively to $\pi_+(T;X)$ while the second contributes almost exclusively to $\pi_-(T;X)$. For the theory based on Bohmian mechanics, interference between the two time-evolved components of the pure state (19) has a dramatic effect on the distribution of arrival times at X for $|X/a|\ll1$, while it has virtually no effect according to GRT's theory, which decouples the $k>0$ and $k<0$ contributions to $\pi(T;X)$. This decoupling follows from the ''key point'' that the regulated operator $\hat{T}_\epsilon$ commutes with the operator $\widehat{\mathrm{sgn}}$ that gives the sign of k. I find it a cause for concern that this important ingredient of the theory apparently emerges as a consequence of regulation. Now

$$[\hat{T}_\epsilon,\widehat{\mathrm{sgn}}]F(k)=-i(2m/\hbar)\epsilon^{-2}kF(k)\delta(k), \quad (31)$$

while

$$[\hat{T},\widehat{\mathrm{sgn}}]F(k)=-i(2m/\hbar k)F(k)\delta(k), \quad (32)$$

so that regulation has considerably enlarged the class of functions $F(k)$ for which the above-mentioned commutator is zero for all k, including $k=0$. In the absence of regulation $F(k)$ must approach zero faster than k as $k\to 0$ while regulation allows $F(k)$ to diverge provided that it does so more slowly than k^{-1}.

It should be mentioned at this point that in a recent paper [23] Delgado and Muga, using a very different approach that involves neither operator ordering difficulties nor regulation, obtained an arrival time distribution that can be shown to be identical to that derived by Grot, Rovelli, and Tate. However, the stated region of applicability of Delgado and Muga's theory is narrower. In particular, $\phi(k)$ must be nonzero only for k of one sign so that either π_+ or π_- is zero. Hence their theory is not applicable to either of the cases considered in this section.

The prediction of Bohmian mechanics that it is possible for some freely propagating electrons to come to rest momentarily and then change their direction of motion is certainly counterintuitive from the point of view of classical mechanics and, if one allows oneself to think of such things, perhaps also of quantum mechanics. However, in Bohm's theory a particle with $V(x)=0$ is not really free because it is always under the guiding influence of the wave function. Now consider a particle prepared in the initial state (17) with $|a|\gg\Delta x$ and incident on the infinite potential step $V(x)=V_0\Theta(x)$, with $V_0=\infty$. In this case the prediction of Bohmian mechanics that the particle will turn around without ever reaching the region $x>0$ is no surprise. However, for $x\leqslant 0$ and for all t the wave function $\psi(x,t)$ of this particle is identical, aside from an unimportant normalization factor, to Eq. (22) and the corresponding trajectories $x(x^{(0)},t)$ for $x^{(0)}<0$ are also identical. With this in mind, the turning around of the freely propagating particles described by Eq. (22) does not seem so strange.

When $V(x)=0$ the wave-number distribution $|\phi(k;t)|^2$ for the ensemble of particles is, of course, independent of t. Now, if one *assumes* that each particle in the ensemble moves with a time-independent velocity $v=\hbar k/m$, with v varying only from particle to particle [consistent with $|\phi(k)|^2$], then it follows that every particle with $v\neq 0$ must reach an arbitrary position X once and only once at some time T between $-\infty$ and $+\infty$. However, there is, in my opinion, no justification for this assumption within standard quantum mechanics.[6] If this is in fact the case, then there is a corresponding lack of justification for regarding the possible turning around of free particles in Bohm's theory as unphysical.

Suppose that $\phi(k)$ is nonzero only for $k>0$. Then GRT's theory predicts that free particles arrive at $x=X$ only from the left, i.e., $\pi_-(T,X)=0$. However, it is well known [2,3,18] that $\phi(k)$ being nonzero only for positive k does not guarantee that $J(x,t)$ be non-negative for all x and t. Now, if $J(X,T)$ is negative for $T_1<T<T_2$ then the prediction that no particles will be found to arrive at X from the right during this time interval, during which the integrated probability density for the region $[-\infty,X]$ is monotonically increasing, must be regarded as counterintuitive. It is certainly contrary to the simple self-consistent picture provided by Bohm's theory of particles arriving at X from the right whenever $J(X;t)$ is negative.

IV. DISCUSSION AND CONCLUDING REMARKS

Suppose that one were to base a calculation of the distribution of arrival times on a Hamiltonian describing an otherwise freely propagating electron that is coupled at $x=X$ to the quantum analog of a classical stopwatch that indicates the time $T(X)$ at which a classical particle arrives at $x=X$.[7] Also suppose that a single microscopic stopwatch "pointer" variable $\theta(X)$ when amplified to the macroscopic level with negligible loss in resolution gives the experimental value $T_{\rm expt}(X)$ for an individual run. Since the calculated distribution of $\theta(X)$ values for an ensemble of coupled electron-stopwatch systems is common to both conventional quantum mechanics and Bohmian mechanics the two theories will give precisely the same predictions for the distribution of $T_{\rm expt}(X)$ values. The possibility of disagreement sets in when it is assumed that it is possible, in principle at least, to design a quantum stopwatch (to say nothing of the amplification apparatus) that reveals the intrinsic value of $T(X)$ so that one can predict the measured distribution of arrival times by calculating the distribution of arrival times in the absence of the microscopic stopwatches. The theoretical expression for the distribution of such intrinsic arrival times derived within conventional quantum mechanics by Grot, Rovelli, and Tate using the particular operator ordering (2) is not the same as the expression derived within Bohmian mechanics. A question that immediately arises is whether or not a different choice of operator ordering in GRT's theory could remove the discrepancy between the two theories. Another is whether or not the two theories are timing the same quantum entities. A short digression at this point on a related problem will hopefully clarify the subsequent discussion.

The mean dwell time $\tau_D(x_1,x_2)$ is defined as the average time spent in the region $x_1<x<x_2$ subsequent to $t=0$ by an ensemble of electrons prepared in the initial state $\psi(x,0)$ in the presence of an arbitrary potential $V(x)$. Jaworsky and Wardlaw [24] postulated the following widely, but not universally, accepted expression for this quantity:

$$\tau_D(x_1,x_2)=\int_0^\infty dt\int_{x_1}^{x_2}dx|\psi(x,t)|^2. \tag{33}$$

This is identical to the expression for the intrinsic dwell time derived [25,16] using Bohmian mechanics. The stationary-state-scattering limit of Eq. (33) is identical to the expression for the mean dwell time $\tau_D(x_1,x_2;k)$ derived [26] by very weakly coupling the scattering particle while "in" the region $[x_1,x_2]$ to a calibrated version of the quantum "stopwatch" of Salecker and Wigner [27]. In this case the thought experiment accurately reveals the average value of the intrin-

[6]There is no implication here that GRT made this assumption. It is intended only as a scenario that would support their claim.

[7]We are now considering the usual arrival time problem in which the particle is timed from $t=0$ so that we do not have to worry about initializing the stopwatch at $t=-\infty$.

sic time of interest. Unfortunately, this agreement between the intrinsic quantity calculated within Bohmian mechanics and the corresponding result of the thought experiment breaks down when one attempts to derive mean transmission and reflections times τ_T and τ_R by analyzing the Salecker-Wigner (SW) clock readings separately for the two subensembles distinguished by whether the scattered particle is found to be transmitted or reflected by $V(x)$, respectively.

Brouard, Sala, and Muga [28] *assumed* the correctness of Eq. (33) within conventional quantum mechanics and, writing it in the equivalent form

$$\tau_D(x_1,x_2)=\int_0^\infty dt\int_{-\infty}^\infty dx\,\psi^*(x,t)\hat{D}(x_1,x_2)\psi(x,t), \tag{34}$$

where the projector $\hat{D}(x_1,x_2)$ is defined by $\hat{D}(x_1,x_2)\psi(x,t)=\Theta(x-x_1)\Theta(x_2-x)\psi(x,t)$, investigated the decomposition of τ_D into parts associated with transmission and reflection. Their approach was based on the projectors $\hat{T}$ and $\hat{R}$ defined by $\hat{T}\psi(x,t)=\psi_T(x,t)$ and $\hat{R}\psi(x,t)=\psi_R(x,t)$, where ψ_T and ψ_R are the (to be) transmitted and reflected components of ψ respectively. Using $\hat{T}+\hat{R}=\hat{1}$, $\hat{D}^n=\hat{D}$ $(n=1,2,\dots)$, and the ambiguity in operator ordering resulting from the fact that $\hat{D}$ does not commute with $\hat{T}$ and $\hat{R}$, they showed that $\hat{D}$ in Eq. (34) could be replaced by an infinite number of equivalent operators, e.g., $(\hat{T}\hat{D}+\hat{R}\hat{D})$ or $(\hat{D}\hat{T}+\hat{D}\hat{R})$, leading to an infinity of different T-R decompositions of τ_D. The simplest of these corresponded to well-known expressions for τ_T and τ_R already existing in the literature and derived by a variety of methods. However, it was shown [21] that no choice of operator ordering could lead to intrinsic mean transmission and reflection times in agreement with the (unique) results of Bohmian mechanics. It was concluded that the basic difference between the two approaches is simply that they clock fundamentally different entities: Schrödinger waves in the projector approach and (postulated) pointlike particles in the approach based on Bohmian mechanics. The predicted mean transmission and reflection times ''measured'' by the SW quantum clock are identical to the intrinsic mean transmission and reflection times obtained in the projector approach from the particular decomposition $\hat{D}=(1/2)[\hat{T}\hat{D}+\hat{D}\hat{T}]+(1/2)[\hat{R}\hat{D}+\hat{D}\hat{R}]$. This might lead one to reject the intrinsic times based on Bohmian mechanics. However, it has been shown [26] that there are situations in which the SW quantum clock results for τ_T or τ_R are clearly unphysical (e.g., $\tau_R<0$), casting doubt on the interpretation of the T- and R-subensemble averaged clock readings in terms of particle times in the general case.

Returning to the intrinsic arrival time problem, there is no obvious reason why a different ordering of the operators $\hat{x}_0$ and $\hat{p}_0^{-1}$ within GRT's approach should lead to the arrival time distribution (9) of Bohmian mechanics. In fact, this appears most unlikely as long as the decoupling of $k<0$ and $k>0$ contributions to $\pi(T;X)$ is maintained in GRT's theory. Moreover, regardless of which ordering is chosen GRT's arrival time distribution will involve the squared modulus of an integral over k, while that based on Bohmian mechanics will involve the modulus of a product of two different integrals over k [obtained by Fourier transforming $\psi^*(x,t)$ and $\psi(x,t)$ in the expression for $J(x,t)$]. In general, these will not give equivalent results for $\pi(T;X)$. Again, a fundamental difference between the two approaches for calculating intrinsic arrival times is that they clock fundamentally different entities. On the other hand, if one introduces a particle detector ''at'' $x=X$ and measures the time at which it fires then conventional quantum mechanics and Bohmian mechanics are presumably concerned with the same pointlike particle *at the instant of detection* and they should agree in their predictions for the distribution of firing times for the detector, provided (a model of) the detector is included in the Hamiltonian. GRT have based their approach on the expectation that their expression (or another one based on a different operator ordering) for the distribution of intrinsic arrival times will give the distribution of firing times for an ideal detector. Hopefully, this issue will be resolved experimentally in the not too distant future. In Bohmian mechanics, on the other hand, there is no general expectation that the intrinsic particle property of interest will be revealed in a quantum-mechanical measurement. For the case of arrival times, before making such a claim it would, of course, be necessary to show that the presence of the detector has negligible effect on the calculated trajectories at least until the actual instant of detection.

In conclusion, Grot, Rovelli, and Tate's theory for the distribution of intrinsic arrival times for freely propagating quantum particles can in some circumstances lead to results that are very different from those based on Bohmian mechanics; both theories contain counterintuitive features; GRT's theory contains some basic assumptions (beyond their particular choice of operator ordering and regulation) that need to be justified, in particular the assumption that $k>(<)0$ corresponds to arrival from the left (right). The important question is whether it is possible to perform sufficiently noninvasive experiments to meaningfully test the predictions of these theories. Unfortunately, the case study that exhibited a dramatic difference between the predictions of the two theories depended on the coherent interference between the two components of the wave function at the position X where one would have to insert the particle detector to perform the experiment. It could turn out that the two theories will agree within experimental error whenever it is possible to perform a meaningful measurement of the arrival time distribution reasonably unperturbed by the presence of the detector.[8] This would at least reconcile the present status of the arrival time problem with statements such as the following [29]: ''As an intellectual apparatus that allows us to figure out what will happen in all conceivable kinds of situations, quantum mechanics works just fine and tells us whatever answers we need to know.'' Clearly, further work is needed to determine whether or not this optimistic assessment of conventional quantum mechanics is justified within the context of arrival times.

Notes added

Recently I came across three relevant works prior to publication, two of which have since been published.

[8]The restrictions placed by Delgado and Muga on the initial state for the applicability of Eq. (8) within their theory is a step in this direction.

In the first [30], Delgado reformulates the arrival time theory of Delgado and Muga [23] in terms of an operator for the modulus of the current. Changing Delgado's notation for this operator from $\hat{J}^{+}(X)$ to $|\hat{J}(X)|$, one can contrast the result for the arrival time distribution obtained using Bohmian mechanics with that obtained by Delgado and Muga (within the restricted range of applicability of their theory) very succintly: $|\langle\hat{J}\rangle|$ versus $\langle|\hat{J}|\rangle$, respectively.

In the second article, Halliwell and Zafiris [31] apply the decoherent histories generalization of quantum mechanics to the arrival time problem. Of particular relevance to the present paper is their brief review of the work of Yamada and Takagi [32], who showed that within the decoherent histories approach the notion of the time of arrival of a quantum particle at a point $x=X$ is rarely meaningful. For the very special case in which the initial state $\psi(x,0)$ is antisymmetric about $x=0$, they proved that the probability of the freely evolving particle crossing $x=X=0$ is well defined but equal to zero. Now the initial wave function (19) of the second case study of the present paper is antisymmetric about $x=0$ and hence the decoherent histories approach and that based on Bohmian mechanics are in complete agreement that no particle in the ensemble described by Eq. (19) ever arrives at $X=0$ (for $X\neq 0$ the decoherent histories approach cannot consistently assign a probability for crossing $x=X$).

In the third paper, Aharonov, Oppenheim, Popescu, Reznik, and Unruh [33] argue but do not prove that the time of arrival cannot be precisely defined and measured in quantum mechanics. I do not see any necessary inconsistency between their conclusions and my own. Although the time of arrival can be precisely defined within Bohmian mechanics, measurement of its intrinsic value is just as problematic as in the conventional approach.

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