QED commutation relations for inhomogeneous Kramers-Kronig dielectrics

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Recently a quantization scheme for the phenomenological Maxwell theory of the full electromagnetic field in an inhomogeneous three-dimensional, dispersive, and absorbing dielectric medium has been developed and applied to a system consisting of two infinite half-spaces with a common planar interface (H.T. Dung, L. Knöll, and D.-G. Welsch, Phys. Rev. A **57**, 3931 (1998)). Here we show that the scheme, which is based on the classical Green-tensor integral representation of the electromagnetic field, applies to any inhomogeneous medium. For this purpose we prove that the fundamental equal-time commutation relations of QED are preserved for an arbitrarily space-dependent, Kramers-Kronig consistent permittivity. Further, an extension of the quantization scheme to linear media with bounded regions of amplification is given, and the problem of anisotropic media is briefly addressed. [S1050-2947(98)07807-X]

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I. INTRODUCTION

Quantization of the phenomenological Maxwell theory of the full electromagnetic field in an inhomogeneous three-dimensional, dispersive, and absorbing dielectric medium of given permittivity necessarily requires a concept that is consistent with the principle of causality and the dissipation-fluctuation theorem, and necessarily yields the fundamental equal-time commutation relations of QED. Recently it has been shown [1] that the classical Green-tensor integral representation of the electromagnetic field in a medium with space-dependent, complex permittivity can be quantized, in agreement with the conditions mentioned, introducing operator noise current and charge densities and expressing them in terms of a continuous set of bosonic fields. The quantization scheme generalizes previous work on dispersive and absorbing bulk material [2] and one-dimensional slablike systems with stepwise constant, complex permittivity [2–7].

In particular, the ordinary vacuum QED is recognized in the limit when the permittivity approaches unity, and the frequently used approximate quantization schemes for radiation in dispersionless and lossless inhomogeneous media (see, e.g., [8–10]) and purely dispersive media (see, e.g., [11–13]) are recognized in the narrow-bandwidth limit. Further, the concept is in full agreement with the Huttner-Barnett approach [14] to quantization of the electromagnetic field in bulk material. In this scheme, which is based on the Hopfield model [15] of a homogeneous dielectric, the electromagnetic field is coupled to a harmonic-oscillator polarization field that interacts with a continuous set of harmonic-oscillator reservoir fields. All couplings are assumed to be bilinear, and the Hamiltonian of the total system is diagonalized by using a Fano-type technique [16].

The proof of the consistence with QED of the quantization scheme developed in [1] requires the calculation of some frequency integral of the (classical) Green tensor in order to verify the fundamental equal-time commutation relation between the electric and magnetic fields. So far the proof for a three-dimensional inhomogeneous medium has been based on the explicit expression of the Green tensor for a system that consists of two dispersive and absorbing bulk dielectrics with a common planar interface. Although it is the simplest inhomogeneous system, the involved form of the Green tensor requires performing a rather lengthy calculation, and the question about the validity of the theory for more complicated three-dimensional systems may arise. In this paper we show that the fundamental equal-time commutation relation between the electric and magnetic fields is satisfied for any inhomogeneous three-dimensional, dispersive, and absorbing dielectric medium, without making use of a particular form of the Green tensor. This enables us to show that the theory applies to arbitrary inhomogeneous, linear media including media with bounded regions of amplification. Finally, we briefly address the extension of the theory to anisotropic media.

The paper is organized as follows. The quantization scheme is outlined in Sec. II. In Sec. III from the partial differential equation for the Green tensor an integral equation is derived, and general properties of the Green tensor are studied. Section IV presents the proof of the fundamental commutation relation between the electric and magnetic fields, and in Sec. V it is shown that the scheme also applies to media with both absorption and (in bounded regions of space) amplification. Finally, a summary and some concluding remarks are given in Sec. VI.

II. QUANTIZATION SCHEME

Following [1], we spectrally decompose the (Schrödinger) electric and magnetic field operators as

$$\hat{\mathbf{E}}(\mathbf{r}) = \int_0^\infty d\omega \underline{\hat{\mathbf{E}}}(\mathbf{r},\omega) + \text{H.c.} \tag{1}$$

and

$$\hat{\mathbf{B}}(\mathbf{r}) = \int_0^\infty d\omega \underline{\hat{\mathbf{B}}}(\mathbf{r},\omega) + \text{H.c.}, \tag{2}$$

respectively, where $\underline{\hat{\mathbf{E}}}(\mathbf{r},\omega)$ and $\underline{\hat{\mathbf{B}}}(\mathbf{r},\omega)$ satisfy Maxwell's equations

$$\nabla\cdot\underline{\hat{\mathbf{B}}}(\mathbf{r},\omega)=0, \tag{3}$$

$$\nabla\cdot[\epsilon_0\epsilon(\mathbf{r},\omega)\underline{\hat{\mathbf{E}}}(\mathbf{r},\omega)]=\underline{\hat{\rho}}(\mathbf{r},\omega), \tag{4}$$

$$\nabla\times\underline{\hat{\mathbf{E}}}(\mathbf{r},\omega)=i\omega\underline{\hat{\mathbf{B}}}(\mathbf{r},\omega), \tag{5}$$

$$\nabla\times\underline{\hat{\mathbf{B}}}(\mathbf{r},\omega)=-i\frac{\omega}{c^2}\epsilon(\mathbf{r},\omega)\underline{\hat{\mathbf{E}}}(\mathbf{r},\omega)+\mu_0\underline{\hat{\mathbf{j}}}(\mathbf{r},\omega). \tag{6}$$

Here, the complex-valued permittivity

$$\epsilon(\mathbf{r},\omega)=\epsilon_R(\mathbf{r},\omega)+i\epsilon_I(\mathbf{r},\omega) \tag{7}$$

is a function of frequency and space, with

$$\epsilon(\mathbf{r},\omega)\to 1 \quad \text{if} \quad \omega\to\infty. \tag{8}$$

For chosen $\mathbf{r}$ the real part (responsible for dispersion) and the imaginary part (responsible for absorption) are related to each other according to the Kramers–Kronig relations, because of causality. This also implies that $\epsilon(\mathbf{r},\omega)$ is a holomorphic function in the upper complex frequency plane,

$$\frac{\partial}{\partial\omega^*}\epsilon(\mathbf{r},\omega)=0 \quad (\omega_I>0). \tag{9}$$

The dependence on $\mathbf{r}$ of $\epsilon(\mathbf{r},\omega)$ indicates that the dielectric properties spatially change in general.

In order to be consistent with the dissipation-fluctuation theorem, in Eqs. (4) and (6), respectively, an operator noise charge density $\underline{\hat{\rho}}(\mathbf{r},\omega)$ and an operator noise current density $\underline{\hat{\mathbf{j}}}(\mathbf{r},\omega)$ have been introduced, which fulfill the equation of continuity

$$\nabla\cdot\underline{\hat{\mathbf{j}}}(\mathbf{r},\omega)=i\omega\underline{\hat{\rho}}(\mathbf{r},\omega). \tag{10}$$

Eventually, $\underline{\hat{\mathbf{j}}}(\mathbf{r},\omega)$ is related to a bosonic vector field $\hat{\mathbf{f}}(\mathbf{r},\omega)$ as

$$\underline{\hat{\mathbf{j}}}(\mathbf{r},\omega)=\omega\sqrt{\frac{\hbar\epsilon_0}{\pi}\epsilon_I(\mathbf{r},\omega)}\hat{\mathbf{f}}(\mathbf{r},\omega), \tag{11}$$

$$[\hat{f}_i(\mathbf{r},\omega),\hat{f}_j^\dagger(\mathbf{r}',\omega')]=\delta_{ij}\delta(\mathbf{r}-\mathbf{r}')\delta(\omega-\omega'), \tag{12}$$

$$[\hat{f}_i(\mathbf{r},\omega),\hat{f}_j(\mathbf{r}',\omega')]=0=[\hat{f}_i^\dagger(\mathbf{r},\omega),\hat{f}_j^\dagger(\mathbf{r}',\omega')]. \tag{13}$$

The fields $\hat{\mathbf{f}}(\mathbf{r},\omega)$ for all ω can be regarded as basic variables of an overall system that consists of the electromagnetic field, the polarization field, and the reservoir fields and whose Hamiltonian reads

$$\hat{H}=\int d^3\mathbf{r}\int_0^\infty d\omega tw\hat{\mathbf{f}}^\dagger(\mathbf{r},\omega)\cdot\hat{\mathbf{f}}(\mathbf{r},\omega). \tag{14}$$

The quantization scheme implies that the electromagnetic field operators can be expressed in terms of $\hat{\mathbf{f}}(\mathbf{r},\omega)$. From Maxwell's equations it is seen that $\underline{\hat{\mathbf{E}}}(\mathbf{r},\omega)$ satisfies the partial differential equation

$$\nabla\times\nabla\times\underline{\hat{\mathbf{E}}}(\mathbf{r},\omega)-\frac{\omega^2}{c^2}\epsilon(\mathbf{r},\omega)\underline{\hat{\mathbf{E}}}(\mathbf{r},\omega)=i\mu_0\omega\underline{\hat{\mathbf{j}}}(\mathbf{r},\omega), \tag{15}$$

so that

$$\underline{\hat{E}}_i(\mathbf{r},\omega)=i\mu_0\int d^3\mathbf{s}\omega G_{ik}(\mathbf{r},\mathbf{s},\omega)\underline{\hat{j}}_k(\mathbf{s},\omega), \tag{16}$$

where $\underline{\hat{j}}_k(\mathbf{s},\omega)$ is given by Eq. (11), and $G_{ik}(\mathbf{r},\mathbf{s},\omega)$ is the tensor-valued Green function of the classical problem. Here and in the following we adopt the convention of summation over repeated vector-component indices. Combining Eqs. (5) and (16), the corresponding expression for $\underline{\hat{B}}_i(\mathbf{r},\omega)$ is easily derived. The integral representations of $\hat{E}_i(\mathbf{r})$ and $\hat{B}_i(\mathbf{r})$ are then found from Eqs. (1) and (2), respectively, from which the (equal-time) commutation relations are derived to be

$$[\hat{E}_i(\mathbf{r}),\hat{E}_k(\mathbf{r}')]=[\hat{B}_i(\mathbf{r}),\hat{B}_k(\mathbf{r}')]=0 \tag{17}$$

and

$$[\hat{E}_i(\mathbf{r}),\hat{B}_k(\mathbf{r}')]=\frac{\hbar}{\pi\epsilon_0}\epsilon_{kmj}\partial_m^{r'}\int_{-\infty}^{+\infty}d\omega\frac{\omega}{c^2}G_{ij}(\mathbf{r},\mathbf{r}',\omega) \tag{18}$$

(ϵ_{kmj}, Levi-Civita tensor; $\partial_m^{r'}\equiv\partial/\partial x_m'$). On the other hand, from QED it is well known that

$$[\hat{E}_i(\mathbf{r}),\hat{B}_k(\mathbf{r}')]=-\frac{i\hbar}{\epsilon_0}\epsilon_{ikm}\partial_m^r\delta(\mathbf{r}-\mathbf{r}'), \tag{19}$$

which reveals that the quantization scheme is in full agreement with QED, if in Eq. (18) the integral over ω yields

$$\epsilon_{kmj}\partial_m^{r'}\int_{-\infty}^{+\infty}d\omega\frac{\omega}{c^2}G_{ij}(\mathbf{r},\mathbf{r}',\omega)=\epsilon_{kmj}\partial_m^{r'}i\pi\delta_{ij}\delta(\mathbf{r}-\mathbf{r}'). \tag{20}$$

It should be pointed out that this is also the condition for obtaining the correct commutation relations for the potentials and canonically conjugated momenta.

Apart from scalar electrodynamics for slablike systems [2–7], Eq. (20) has been proved correct for bulk material and two infinite half-spaces with a common planar interface [1,2] by making use of the explicit form of the Green function. In what follows we show that the quantization scheme yields the correct commutation relations for arbitrary inhomogeneous dielectrics, i.e., for any permittivity $\epsilon(\mathbf{r},\omega)$. For this purpose, let us first consider some general properties of the Green function.

III. GREEN FUNCTION

From Eqs. (15) and (16) it is easily seen that the tensor-valued Green function [matrix elements of the fundamental solution of Eq. (15)] satisfies the equation

$$\left[\partial_i^r\partial_k^r-\delta_{ik}\left(\Delta^r+\frac{\omega^2}{c^2}\epsilon(\mathbf{r},\omega)\right)\right]G_{kj}(\mathbf{r},\mathbf{s},\omega)=\delta_{ij}\delta(\mathbf{r}-\mathbf{s}). \tag{21}$$

This equation and the boundary condition at infinity determine the Green function uniquely. Similarly to Eq. (15) [together with Eq. (16)] there are no nontrivial solutions of the homogeneous problem. Let us consider absorbing bulk material. Since the Green function must vanish at infinity, absorption obviously prevents one from constructing a solution of the homogeneous equation that is different from zero at finite space points. When the dielectric material extends only over a finite region of space, we may assume that $\epsilon(\mathbf{r},\omega)\to 1$ for $\mathbf{r}\to\infty$. To preserve the analytical properties of $\epsilon(\mathbf{r},\omega)$, the limit $\mathbf{r}\to\infty$ must be performed first, thus keeping a (small) imaginary part $\epsilon_I(\mathbf{r},\omega)$ in the permittivity, which again implies that there is only the trivial (zero) solution of the homogeneous equation.

It is well known that the Fourier transform of a response function that describes a causal relation between two physical quantities is a holomorphic function in the upper complex frequency half-plane (see, e.g., [17–19]). A typical example is the causal relation between the averages of polarization and electric-field strength. Obviously, $\underline{D}_{ij}(\mathbf{r},\mathbf{s},\omega)=i\mu_0\omega G_{ij}(\mathbf{r},\mathbf{s},\omega)$ as a function of ω is nothing but the Fourier transform of the tensor response function $D_{ij}(\mathbf{r},\mathbf{s},\tau)$ that causally relates the electric field $E_i(\mathbf{r},t)$ observed at space-point $\mathbf{r}$ and time t to an external (pointlike) current $j_j^{\mathrm{ext}}(\mathbf{r},t)=J_j^{\mathrm{ext}}(\mathbf{s},t)\delta(\mathbf{r}-\mathbf{s})$ at space-point $\mathbf{s}$ and time $t-\tau(\tau\geqslant 0)$ [cf. Eq. (16)]:

$$E_i(\mathbf{r},t)=\int_0^\infty d\tau D_{ij}(\mathbf{r},\mathbf{s},\tau)J_j^{\mathrm{ext}}(\mathbf{s},t-\tau). \tag{22}$$

Hence,

$$i\mu_0\omega G_{ij}(\mathbf{r},\mathbf{s},\omega)=\underline{D}_{ij}(\mathbf{r},\mathbf{s},\omega)=\int_0^\infty d\tau e^{i\omega\tau}D_{ij}(\mathbf{r},\mathbf{s},\tau) \tag{23}$$

is a holomorphic function of ω in the upper complex half-plane, i.e.,

$$\frac{\partial}{\partial\omega^*}\omega G_{kj}(\mathbf{r},\mathbf{s},\omega)=0 \quad (\omega_I>0), \tag{24}$$

with

$$\omega G_{kj}(\mathbf{r},\mathbf{s},\omega)\to 0 \quad \text{if} \quad |\omega|\to\infty. \tag{25}$$

Note that Eq. (24) is in full agreement with the differential equation (21) [together with the boundary condition at infinity]. In this equation the frequency ω is a parameter, and we may assume that $G_{ij}(\mathbf{r},\mathbf{s},\omega)$ as a function of ω is differentiable with respect to ω in the upper complex half-plane. Applying $\partial/\partial\omega^*$ to Eq. (21), we easily see that

$$\left[\partial_i^r\partial_k^r-\delta_{ik}\left(\Delta^r+\frac{\omega^2}{c^2}\epsilon(\mathbf{r},\omega)\right)\right]\frac{\partial}{\partial\omega^*}\omega G_{kj}(\mathbf{r},\mathbf{s},\omega)=0 \quad (\omega_I>0), \tag{26}$$

because of Eq. (9). Since there is no nontrivial solution of the homogeneous problem, we see that $\omega G_{kj}(\mathbf{r},\mathbf{s},\omega)$ satisfies the Cauchy-Riemann equations (24).

From the theory of partial differential equations it is known (see, e.g., [20]) that there exists an equivalent formulation of the problem in terms of an integral equation. As shown in Appendix A, $G_{ij}(\mathbf{r},\mathbf{s},\omega)$ satisfies the integral equation

$$G_{ij}(\mathbf{r},\mathbf{s},\omega)=G_{ij}^{(0)}(\mathbf{r},\mathbf{s},\omega)+\int d^3\mathbf{v}K_{ik}(\mathbf{r},\mathbf{v},\omega)G_{kj}(\mathbf{v},\mathbf{s},\omega), \tag{27}$$

where

$$G_{ij}^{(0)}(\mathbf{r},\mathbf{s},\omega)=[\delta_{ij}-\partial_i^r\partial_j^s q^{-2}(\mathbf{s},\omega)]g(|\mathbf{r}-\mathbf{s}|,\omega) \tag{28}$$

and

$$K_{ik}(\mathbf{r},\mathbf{v},\omega)=[\partial_k^v\ln q^2(\mathbf{v},\omega)][\partial_i^r g(|\mathbf{r}-\mathbf{v}|,\omega)] +[q^2(\mathbf{v},\omega)-q_0^2(\omega)]g(|\mathbf{r}-\mathbf{v}|,\omega)\delta_{ik}. \tag{29}$$

Here, the function

$$g(|\mathbf{r}|,\omega)=\frac{e^{iq_0(\omega)|\mathbf{r}|}}{4\pi|\mathbf{r}|}=\int\frac{d^3\mathbf{k}}{(2\pi)^3}\frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{k^2-q_0^2(\omega)} \tag{30}$$

is introduced, where

$$q^2(\mathbf{r},\omega)=\frac{\omega^2}{c^2}\epsilon(\mathbf{r},\omega) \tag{31}$$

and

$$q_0^2(\omega)=\frac{\omega^2}{c^2}\epsilon_0(\omega), \tag{32}$$

$\epsilon_0(\omega)\equiv\overline{\epsilon(\mathbf{r},\omega)}_{\mathbf{r}}$ being an appropriately space-averaged reference permittivity [for the integral equation with an $\mathbf{s}$-dependent reference permittivity $\epsilon_0(\mathbf{s},\omega)$, see Appendix A]. Obviously, $G_{ij}^{(0)}(\mathbf{r},\mathbf{s},\omega)$ is the Green function for a homogeneous medium with permittivity $\epsilon(\mathbf{r},\omega)\equiv\epsilon_0(\omega)$. The second term on the right-hand side in Eq. (27) essentially arises from the inhomogeneities. Note that according to the Fredholm alternative the solution of the integral equation (27) is unique, because of the nonexistence of nontrivial solutions of the homogeneous problem. From Eqs. (30)–(32) and Eq. (8) it follows that the integral kernel $K_{ik}(\mathbf{r},\mathbf{v},\omega)$, Eq. (29), is a holomorphic function of ω in the upper complex half-plane, with

$$K_{ik}(\mathbf{r},\mathbf{v},\omega)\to 0 \quad \text{if} \quad |\omega|\to\infty, \tag{33}$$

where $K_{ik}(\mathbf{r},\mathbf{v},\omega)$ decreases as does $\epsilon(\mathbf{r},\omega)-1$.

Let us write the integral equation (27) in the compact form

$$G=G^{(0)}+\mathcal{K}G, \tag{34}$$

where

$$\mathcal{K}G\equiv(\mathcal{K}G)_{ij}(\mathbf{r},\mathbf{s},\omega)=\int d^3\mathbf{v}K_{ik}(\mathbf{r},\mathbf{v},\omega)G_{kj}(\mathbf{v},\mathbf{s},\omega). \tag{35}$$

Assuming that G can be found by iteration, we may write

$$G=G^{(0)}+\sum_{n=1}^{\infty}\mathcal{K}^nG^{(0)}. \tag{36}$$

From Eq. (28) it is seen that $G_{ij}^{(0)}(\mathbf{r},\mathbf{s},\omega)$ has a cubic singularity $|\mathbf{r}-\mathbf{s}|^{-3}$ for $\mathbf{r}\to\mathbf{s}$, and Eq. (29) reveals that the kernel $K_{ik}(\mathbf{r},\mathbf{v},\omega)$ is only weakly singular (the singularity is weaker than the spatial dimension). Hence, at least after the third iteration step the result is perfectly regular at $\mathbf{r}=\mathbf{s}$.

IV. COMMUTATION RELATION

The results given in Sec. III now enable us to prove Eq. (20) for arbitrary inhomogeneous dielectrics. For this purpose we first decompose the Green function into two parts,

$$G_{ij}(\mathbf{r},\mathbf{s},\omega)=(G_1)_{ij}(\mathbf{r},\mathbf{s},\omega)+(G_2)_{ij}(\mathbf{r},\mathbf{s},\omega), \tag{37}$$

where $(G_1)_{ij}(\mathbf{r},\mathbf{s},\omega)$ and $(G_2)_{ij}(\mathbf{r},\mathbf{s},\omega)$ satisfy the integral equations

$$G_\mu=G_\mu^{(0)}+\mathcal{K}G_\mu \quad (\mu=1,2), \tag{38}$$

with [cf. Eq. (28)]

$$(G_1)_{ij}^{(0)}(\mathbf{r},\mathbf{s},\omega)=\delta_{ij}g(|\mathbf{r}-\mathbf{s}|,\omega) \tag{39}$$

and

$$(G_2)_{ij}^{(0)}(\mathbf{r},\mathbf{s},\omega)=-\partial_j^s\partial_i^r q^{-2}(\mathbf{s},\omega)g(|\mathbf{r}-\mathbf{s}|,\omega). \tag{40}$$

It is easily seen that $(G_2)_{ij}(\mathbf{r},\mathbf{s},\omega)$ can be given by

$$(G_2)_{ij}(\mathbf{r},\mathbf{s},\omega)=\partial_j^s\Gamma_i(\mathbf{r},\mathbf{s},\omega), \tag{41}$$

where Γ is the solution of the integral equation

$$\Gamma=\Gamma^{(0)}+\mathcal{K}\Gamma, \tag{42}$$

with

$$\Gamma_i^{(0)}(\mathbf{r},\mathbf{s},\omega)=-\partial_i^r q^{-2}(\mathbf{s},\omega)g(|\mathbf{r}-\mathbf{s}|,\omega). \tag{43}$$

Both $\omega(G_1)_{ij}(\mathbf{r},\mathbf{s},\omega)$ and $\omega(G_2)_{ij}(\mathbf{r},\mathbf{s},\omega)$ are holomorphic functions of ω in the upper complex half-plane, with $\omega(G_\mu)_{ij}(\mathbf{r},\mathbf{s},\omega)\to 0$ if $|\omega|\to\infty$. Note that $\omega(G_2)_{ij}(\mathbf{r},\mathbf{s},\omega)$ may be singular at $\omega=0$. Nevertheless, when substituting $G_{ij}(\mathbf{r},\mathbf{s},\omega)$ from Eq. (37) [together with Eq. (41)] back into Eq. (16), we can integrate by parts and use the equation of continuity (10) to obtain

$$\underline{\hat{E}}_i(\mathbf{r},\omega)=i\mu_0\int d^3\mathbf{s}\,\omega(G_1)_{ik}(\mathbf{r},\mathbf{s},\omega)\underline{\hat{j}}_k(\mathbf{s},\omega) \tag{44}$$

$$+\mu_0\int d^3\mathbf{s}\,\omega^2\Gamma_i(\mathbf{r},\mathbf{s},\omega)\underline{\hat{\rho}}(\mathbf{s},\omega). \tag{45}$$

Hence, $i\mu_0\omega(G_1)_{ik}(\mathbf{r},\mathbf{s},\omega)$ and $\mu_0\omega^2\Gamma_i(\mathbf{r},\mathbf{s},\omega)$ are the Fourier transforms of the response functions relating the electric-field strength to the (noise) current density $\underline{\hat{j}}_k(\mathbf{s},\omega)$ and the (noise) charge density $\underline{\hat{\rho}}(\mathbf{s},\omega)$ separately. Obviously, $\omega^2\Gamma_i(\mathbf{r},\mathbf{s},\omega)$ is not singular at $\omega=0$.

Combining Eqs. (37) and (41), we easily see that the term on the right-hand side in Eq. (18) can be rewritten as, on recalling that $\epsilon_{kmj}\partial_m^{r'}\partial_j^{r'}(\cdots)=0$,

$$\epsilon_{kmj}\partial_m^{r'}\int_{-\infty}^{+\infty}d\omega\frac{\omega}{c^2}G_{ij}(\mathbf{r},\mathbf{r}',\omega)$$

$$=\epsilon_{kmj}\partial_m^{r'}\int_{-\infty}^{+\infty}d\omega\frac{\omega}{c^2}(G_1)_{ij}(\mathbf{r},\mathbf{r}',\omega). \tag{46}$$

Thus, only the noise-current response function $\sim\omega(G_1)_{ij}(\mathbf{r},\mathbf{r}',\omega)$ contributes to the commutator (18). We now substitute in Eq. (46) for $(G_1)_{ij}(\mathbf{r},\mathbf{r}',\omega)$ the integral equation (38) ($\mu=1$) to obtain

$$\int_{-\infty}^{+\infty}d\omega\frac{\omega}{c^2}(G_1)_{ij}(\mathbf{r},\mathbf{r}',\omega)=i\pi\delta_{ij}\delta(\mathbf{r}-\mathbf{r}')$$

$$+\int_{-\infty}^{+\infty}d\omega\int d^3\mathbf{v}\frac{\omega}{c^2}K_{ik}(\mathbf{r},\mathbf{v},\omega)$$

$$\times(G_1)_{kj}(\mathbf{v},\mathbf{r}',\omega), \tag{47}$$

where the (bulk-material) relation [2]

$$\int_{-\infty}^{+\infty}d\omega\frac{\omega}{c^2}(G_1)_{ij}^{(0)}(\mathbf{r},\mathbf{r}',\omega)=i\pi\delta_{ij}\delta(\mathbf{r}-\mathbf{r}') \tag{48}$$

has been used. Hence it remains to prove that the second term on the right-hand side in Eq. (47) vanishes [compare Eqs. (46) and (47) with Eq. (20)].

Since $K_{ik}(\mathbf{r},\mathbf{v},\omega)$ and $\omega(G_1)_{kj}(\mathbf{v},\mathbf{s},\omega)$ are holomorphic functions of ω in the upper complex half-plane, the ω integral can be calculated by contour integration along a large half-circle (with $|\omega|=R,\quad R\to\infty$). To calculate this integral, we recall that for $|\omega|\to\infty$ both $K_{ik}(\mathbf{r},\mathbf{v},\omega)$ and $\omega(G_1)_{kj}(\mathbf{v},\mathbf{s},\omega)$ approach zero at least as ω^{-1}, and $\omega K_{ik}(\mathbf{r},\mathbf{v},\omega)(G_1)_{kj}(\mathbf{v},\mathbf{s},\omega)$ approaches zero at least as ω^{-2}. Hence, for $R\to\infty$ the contour integral vanishes at least as R^{-1}, and the second term on the right-hand side in Eq. (47) indeed equals zero, i.e.,

$$\int_{-\infty}^{+\infty}d\omega\frac{\omega}{c^2}(G_1)_{ij}(\mathbf{r},\mathbf{r}',\omega)=i\pi\delta_{ij}\delta(\mathbf{r}-\mathbf{r}'), \tag{49}$$

which [together with Eq. (46)] shows, that Eq. (20) is valid for arbitrary space-dependent permittivity. In other words, the application of the quantization scheme to arbitrary inhomogeneous dielectrics yields the correct QED commutation relation (19).

V. EXTENSIONS OF THE QUANTIZATION SCHEME

The developed concept of quantization of the electromagnetic field in a dispersive and absorbing background medium that is described in terms of a spatially varying, complex permittivity essentially rests on the following assumptions and principles. (i) The permittivity $\epsilon(\mathbf{r},\omega)$ of the dielectric medium and the Green tensor $G_{ij}(\mathbf{r},\mathbf{s},\omega)$ of the classical Maxwell equations are holomorphic functions of ω in the upper complex frequency half-plane, because of causality. (ii) There is no nontrivial solution of the homogeneous Maxwell equations that satisfies the boundary condition at infinity, i.e., the electric and magnetic fields are uniquely determined by their integral representations. (iii) To be consistent with the dissipation-fluctuation theorem, noise current and noise charge densities must be introduced even if there are no additional sources embedded in the dielectric medium. (iv) Quantization then requires the integral representations to be regarded as relations between operator-valued fields, where the operator noise current density satisfies the commutation relation

$$[\underline{\hat{j}}_i(\mathbf{r},\omega),\underline{\hat{j}}_j^\dagger(\mathbf{r}',\omega')]=\omega^2\frac{\hbar\epsilon_0}{\pi}\epsilon_I(\mathbf{r},\omega)\,\delta_{ij}\delta(\mathbf{r}-\mathbf{r}')\times\delta(\omega-\omega'). \tag{50}$$

So far we have assumed that $\epsilon_I(\mathbf{r},\omega)>0$ $(\omega>0)$, as is the case for absorbing media. Obviously, the statements (i)–(iv) remain valid, if (in agreement with the Kramers-Kronig relations) $\epsilon_I(\mathbf{r},\omega)<0$ $(\omega>0)$ in a bounded region of space, which corresponds to the presence of an amplifying medium in that region. Obviously, the commutation relation (50) is obtained if (in that region) the operator noise current density $\underline{\hat{\mathbf{j}}}(\mathbf{r},\omega)$ is related to the bosonic field $\hat{\mathbf{f}}(\mathbf{r},\omega)$ as

$$\underline{\hat{\mathbf{j}}}(\mathbf{r},\omega)=\omega\sqrt{-\frac{\hbar\epsilon_0}{\pi}\epsilon_I(\mathbf{r},\omega)}\hat{\mathbf{f}}^\dagger(\mathbf{r},\omega), \tag{51}$$

which reflects the well-known fact that amplification requires the roles of the noise creation and destruction operators to be exchanged (see, e.g., [21,22]). From inspection of Eqs. (11) and (51), we see that the two equations can be combined to express the operator noise current density associated with damping and amplification in terms of the bosonic field as

$$\underline{\hat{\mathbf{j}}}(\mathbf{r},\omega)=\omega\sqrt{\frac{\hbar\epsilon_0}{\pi}|\epsilon_I(\mathbf{r},\omega)|}[\Theta(\epsilon_I)\hat{\mathbf{f}}(\mathbf{r},\omega)+\Theta(-\epsilon_I)\hat{\mathbf{f}}^\dagger(\mathbf{r},\omega)], \tag{52}$$

with $\Theta(x)$ being the unit step function $[\Theta(x)=1$ for $x>0$, and $\Theta(x)=0$ elsewhere]. Note that the operator noise charge density $\underline{\hat{\rho}}(\mathbf{r},\omega)$ is given by Eq. (10), with $\underline{\hat{\mathbf{j}}}(\mathbf{r},\omega)$ from Eq. (52).

The quantization scheme based on Eq. (52) can be regarded as the extension of the concept for amplifying, one-dimensional slablike systems [23,24] to arbitrary inhomogeneous media that contain bounded regions in which amplification is realized. From the derivation given in Sec. IV it is clearly seen that the fundamental QED commutation relation (19) is satisfied independently of the sign of $\epsilon_I(\mathbf{r},\omega)$. For an absorbing medium the poles of $\omega(G_1)_{ij}(\mathbf{r},\mathbf{s},\omega)$ as a function of ω are in the lower complex half-plane. When the gain owing to amplification (e.g., in a resonatorlike equipment) tends to compensate for the losses, then the poles may approach the real axis and sharply peaked resonances may be observed. Obviously, if there are poles on the (real) axis, the ω integral must be performed along the axis $\omega+i\varepsilon$, $\varepsilon\to 0$. Note that in such a case the model of linear amplification may fail, because of nonlinear saturation.

Another possible extension of the quantization scheme is the inclusion in the theory of anisotropic media, for which the permittivity is a symmetric, complex tensor function of ω,

$$\epsilon_{ij}(\mathbf{r},\omega)=\epsilon_{ji}(\mathbf{r},\omega), \tag{53}$$

which also varies with space in general. As we plan to show in a forthcoming article, the quantization scheme also applies to the electromagnetic field in anisotropic Kramers-Kronig dielectrics. The calculation relies on a symmetry relation satisfied by the Green function according to the Lorentz reciprocity theorem [25–27], and the same integral relation (18) for the fundamental commutator between the electric and magnetic field operators can be derived.

VI. CONCLUSIONS

We have studied quantization of the full electromagnetic field in linear, isotropic, inhomogeneous Kramers-Kronig dielectrics, using the formalism of Green-tensor integral representation of the electromagnetic field, in which the electromagnetic field operators are related to bosonic basic fields via the Green tensor of the classical problem. The formalism can be regarded as a natural extension of the mode concept, which only applies—apart from vacuum QED—to narrow-bandwidth fields. Basing on very general properties of the (classical) Green tensor, we have shown that the formalism yields exactly the QED equal-time commutation relations between the fundamental electromagnetic fields for any linear, isotropic dielectric medium.

For this purpose we have derived an integral equation for the Green tensor, the kernel function of which describes the effect of spatially varying permittivity. From the holomorphic properties of the Green tensor and the integral kernel as functions of frequency it then follows that the QED equal-time commutation relation (19) between the electric and magnetic fields is preserved, independently of the dependence on space of the permittivity. Since the holomorphic properties are observed for absorbing media as well as amplifying media, the quantization scheme applies to any linear, isotropic, causal medium. The only condition is that amplification, which gives rise to a negative imaginary part of the permittivity, extends over bounded regions of space—a condition that is physically always fulfilled. It is worth noting that the scheme can also be extended to anisotropic media, as will be shown in a forthcoming paper in detail.

In order to show that the quantization scheme is consistent with QED, we have restricted our attention to the equal-time commutation relations. Clearly, the results can also be used for determining the commutation relations of the

(Heisenberg) electromagnetic field operators at different times. Recalling that $\hat{\mathbf{f}}(\mathbf{r},\omega,t)=\hat{\mathbf{f}}(\mathbf{r},\omega)e^{-i\omega t}$, it can easily be derived that inclusion of $\cos[\omega(t-t')]$ in the integral on the right-hand side of Eq. (18) yields the commutator $[\hat{E}_i(\mathbf{r},t),\hat{B}_k(\mathbf{r}',t')]$. Decomposing the Green tensor as shown in Eq. (37) [together with Eq. (41)], we find that

$$[\hat{E}_i(\mathbf{r},t),\hat{B}_k(\mathbf{r}',t')]$$

$$=\frac{\hbar}{\pi\epsilon_0}\epsilon_{kmj}\partial_m^{r'}\int_{-\infty}^{+\infty}d\omega\,\frac{\omega}{c^2}(G_1)_{ij}(\mathbf{r},\mathbf{r}',\omega)\cos[\omega(t-t')], \tag{54}$$

where $(G_1)_{ij}(\mathbf{r},\mathbf{r}',\omega)$ satisfies the integral equation (38), with $(G_1)_{ij}^{(0)}(\mathbf{r},\mathbf{r}',\omega)$ from Eq. (39). This result can be regarded as a natural generalization of the well-known result of vacuum QED (see, e.g., [28]).

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APPENDIX: DERIVATION OF THE INTEGRAL EQUATION

In order to derive an integral equation equivalent to the differential equation (21), we formally write in Eq. (21)

$$\epsilon(\mathbf{r},\omega)=\epsilon(\mathbf{r},\omega)+\epsilon_0(\mathbf{s},\omega)-\epsilon_0(\mathbf{s},\omega), \tag{A1}$$

where $\epsilon_0(\mathbf{s},\omega)$ is an appropriately chosen reference permittivity, which also satisfies the Kramers-Kronig relations. Hence we may rewrite Eq. (21) as

$$[\Delta^r+q_0^2(\mathbf{s},\omega)]G_{ij}(\mathbf{r},\mathbf{s},\omega)=[q_0^2(\mathbf{s},\omega)-q^2(\mathbf{r},\omega)]G_{ij}(\mathbf{r},\mathbf{s},\omega)$$
$$+\partial_i^r\partial_k^r G_{kj}(\mathbf{r},\mathbf{s},\omega)-\delta_{ij}\delta(\mathbf{r}-\mathbf{s}), \tag{A2}$$

where the abbreviations

$$q^2(\mathbf{r},\omega)=\frac{\omega^2}{c^2}\epsilon(\mathbf{r},\omega),\quad q_0^2(\mathbf{s},\omega)=\frac{\omega^2}{c^2}\epsilon_0(\mathbf{s},\omega) \tag{A3}$$

have been used. Now we introduce the Green function

$$g(|\mathbf{r}|,\mathbf{s},\omega)=\frac{e^{iq_0(\mathbf{s},\omega)|\mathbf{r}|}}{4\pi|\mathbf{r}|} \tag{A4}$$

$[q_0(\mathbf{s},\omega)=(\omega/c)\sqrt{\epsilon_0(\mathbf{s},\omega)}]$, which is easily proved to satisfy the differential equation

$$[\Delta^r+q_0^2(\mathbf{s},\omega)]g(|\mathbf{r}|,\mathbf{s},\omega)=-\delta(\mathbf{r}). \tag{A5}$$

The Green function $g(|\mathbf{r}|,\mathbf{s},\omega)$ enables us to convert Eq. (A2) into the integral equation

$$G_{ij}(\mathbf{r},\mathbf{s},\omega)=-\int d^3\mathbf{v}\,g(|\mathbf{r}-\mathbf{v}|,\mathbf{s},\omega)$$
$$\times\{[q_0^2(\mathbf{s},\omega)-q^2(\mathbf{v},\omega)]G_{ij}(\mathbf{v},\mathbf{s},\omega)$$
$$+\partial_i^v\partial_k^v G_{kj}(\mathbf{v},\mathbf{s},\omega)-\delta_{ij}\delta(\mathbf{v}-\mathbf{s})\}. \tag{A6}$$

Next we apply ∂_i^r on Eq. (21) to obtain

$$\partial_i^r q^2(\mathbf{r},\omega)G_{ij}(\mathbf{r},\mathbf{s},\omega)=-\partial_j^r\delta(\mathbf{r}-\mathbf{s}), \tag{A7}$$

from which we find that

$$\partial_i^r G_{ij}(\mathbf{r},\mathbf{s},\omega)=-q^{-2}(\mathbf{r},\omega)\partial_j^r\delta(\mathbf{r}-\mathbf{s})$$
$$-[\partial_i^r\ln q^2(\mathbf{r},\omega)]G_{ij}(\mathbf{r},\mathbf{s},\omega). \tag{A8}$$

Substituting in Eq. (A6) for $\partial_i^v\partial_k^v G_{kj}(\mathbf{v},\mathbf{s},\omega)$ the result of Eq. (A8), integrating by parts, and performing the integrals with δ functions, we derive

$$G_{ij}(\mathbf{r},\mathbf{s},\omega)=G_{ij}^{(0)}(\mathbf{r},\mathbf{s},\omega)+\int d^3\mathbf{v}K_{ik}(\mathbf{r},\mathbf{v},\mathbf{s},\omega)G_{kj}(\mathbf{v},\mathbf{s},\omega). \tag{A9}$$

Here,

$$G_{ij}^{(0)}(\mathbf{r},\mathbf{s},\omega)=[\delta_{ij}-\partial_i^r\partial_j^s q^{-2}(\mathbf{s},\omega)]g(|\mathbf{r}-\mathbf{s}|,\mathbf{s},\omega)$$
$$+\partial_i^r[\partial_j^s q^{-2}(\mathbf{v},\omega)g(|\mathbf{r}-\mathbf{v}|,\mathbf{s},\omega)]|_{\mathbf{v}=\mathbf{s}}, \tag{A10}$$

and the integral kernel reads

$$K_{ik}(\mathbf{r},\mathbf{v},\mathbf{s},\omega)=[\partial_k^v\ln q^2(\mathbf{v},\omega)][\partial_i^r g(|\mathbf{r}-\mathbf{v}|,\mathbf{s},\omega)]$$
$$+[q^2(\mathbf{v},\omega)-q_0^2(\mathbf{s},\omega)]g(|\mathbf{r}-\mathbf{v}|,\mathbf{s},\omega)\delta_{ik}. \tag{A11}$$

It should be pointed out that the reference permittivity $\epsilon_0(\mathbf{s},\omega)$ can be chosen freely in principle, since the exact solution of the integral equation (A9) does not depend on $\epsilon_0(\mathbf{s},\omega)$. In practice, however, it may be advantageous to choose $\epsilon_0(\mathbf{s},\omega)$ such that $G_{ij}^{(0)}(\mathbf{r},\mathbf{s},\omega)$ gives a sufficiently good zeroth-order approximation of $G_{ij}(\mathbf{r},\mathbf{s},\omega)$ for an approximate solution of Eq. (A9).

In the simplest case $\epsilon_0(\mathbf{s},\omega)$ may be chosen to be independent of $\mathbf{s}$, e.g., by averaging $\epsilon(\mathbf{r},\omega)$ over space,

$$\epsilon_0(\mathbf{s},\omega)\to\epsilon_0(\omega)=\overline{\epsilon(\mathbf{r},\omega)}^{\,\mathbf{r}}. \tag{A12}$$

Obviously, in this case the Green funcion (A4) and the integral kernel (A11) become independent of $\mathbf{s}$ and Eqs. (A9)–(A11) reduce to Eqs. (27)–(29) $[g(|\mathbf{r}|,\mathbf{s},\omega)\to g(|\mathbf{r}|,\omega)$, $K_{ik}(\mathbf{r},\mathbf{v},\mathbf{s},\omega)\to K_{ik}(\mathbf{r},\mathbf{v},\omega)]$ together with Eqs. (30)–(32).

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