Persistent currents in a toroidal trap

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Using elementary microscopic methods, we theoretically study persistent flow of an alkali-metal vapor Bose-Einstein condensate around a tight toroidal trap. The angular velocity of a persistent current must be smaller than the angular frequency of the lowest condensate excitation. A supercurrent may be excited by rotating a perturbing potential that is strong enough to cut the toroidal condensate. [S1050-2947(98)07907-4]

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Experimental studies of Bose-Einstein condensates (BEC) in alkali-metal vapors are now well under way [1]. While experimental data are not yet available, the potential of superfluidity and persistent currents in these systems has been recognized from the outset. Theoretical effort [2] has focused on vortex states of a weakly interacting Bose gas bound to a harmonic trap, states in which the entire condensate rotates in accordance with a quantized value of angular momentum. Studies of excitations of such vortices [3,4] have led to arguments that a vortex cannot be stable [4] in a trap with the minimum of the confining potential at the center.

Unimpeded by the no-go rule [4], a persistent current may be stable in other types of traps that can pin the vortex. We consider a rather extreme case, a trap that confines a condensate to a torus. With the additional assumption that transverse confinement is tight, the motion of the condensate along the torus is amenable to a simple microscopic treatment. We point out that superfluid flow may be stable as long as the angular velocity at which the condensate circulates around the torus is lower than the angular frequencies of the elementary excitations of the condensate. Creating a persistent current also proves to be a nontrivial task. We develop an approach whereby the condensate is stirred up by a rotating potential strong enough to cut the torus.

Specifically, we take the radius of the condensate torus R to be much larger than the transverse dimensions of a cut across the condensate. Two simplifications ensue from such an assumption. First, the frequencies of the excitations involving the transverse coordinates, call them y and z, tend to be much higher than the excitation frequencies of the motion in the direction along the ring coordinate x. In what follows we assume that the transverse motion is frozen to a wave function $\psi(y,z)$. Second, as has been anticipated in our notation already, in our mathematics we straighten the torus and treat the motion along the ring as linear translation. As a vestige of the original topology we impose periodic boundary conditions over the circumference of the torus $2\pi R$.

For atom-atom interactions we adopt the conventional δ function pair potential characterized by the s wave scattering length a. As it comes to the interactions, the only relevant parameter of the transverse motion is the length scale l defined by $l^{-2}=\int dydz|\psi(y,z)|^4$. We embody atom-atom interactions into the dimensionless parameter $\xi=2aR/l^2$. It will frequently prove convenient to discuss the fluid in a rotating coordinate system in which a stationary condensate would rotate like a wheel at the angular velocity $-\omega$. We employ the dimensionless parameter $\nu=mR^2\omega/\hbar$ for the angular velocity. Without restricting the generality, in the following we assume that $\nu\geqslant 0$. Finally, we use R as the unit of length, the atomic mass m as the unit of mass, and $\hbar^2/(mR^2)$ as the unit of energy.

All told, the atoms move in the interval $x\in[-\pi,\pi]$ with periodic boundary conditions. In the basis of the plane waves $u_k(x)=1/\sqrt{2\pi}\,e^{ikx}$ with $k=0,\pm 1,\ldots,$ the second-quantized many-body Hamiltonian in the rotating frame reads

$$H=\sum_k\left(\frac{k^2}{2}-\nu k\right)b_k^\dagger b_k+\sum_{p,q}\tilde{V}(p-q)b_p^\dagger b_q+\frac{1}{2}\xi\sum_{k,p,q}b_{k+q}^\dagger b_{p-q}^\dagger b_p b_k. \tag{1}$$

We allow for a potential $V(x)$ in the direction of the torus, and $\tilde{V}(k)=(1/2\pi)\int_{-\pi}^{\pi}dxe^{-ikx}V(x)$ are the Fourier coefficients of the potential. The corresponding Gross-Pitaevskii equation [5,6] (GPE) for a system of N atoms is

$$\left(-\frac{1}{2}\frac{\partial^2}{\partial x^2}+i\nu\frac{\partial}{\partial x}+V+2\pi N\xi|\psi|^2\right)\psi=\epsilon\psi. \tag{2}$$

In the absence of the potential V, the plane waves u_k are still the eigenstates of the GPE, though the energy (chemical potential) depends on both rotation and atom-atom interactions:

$$\epsilon_k(\nu)=k^2/2-\nu k+N\xi. \tag{3}$$

It is a peculiarity of the transformation to the rotating frame that atoms in the rotating-frame eigenstate u_k still have the velocity $v=k$ with respect to the *stationary* frame.

Let us momentarily ignore both atom-atom interactions and the potential $V(x)$, and work in the nonrotating laboratory frame with $\nu=0$. A state with all the N atoms in any one-particle state u_k is evidently an eigenstate of the Hamiltonian (1). By the translational symmetry, such a state should be a good first approximation to an eigenstate of the Hamiltonian (1) even in the presence of atom-atom interactions.

Here we are merely restating the observation that plane waves are the solutions to the GPE (2), the Hartree-Fock equation for the Boson system. For $k \neq 0$ we have a putative persistent current. In accordance with standard arguments about quantization of circulation [5], the flow velocity of persistent current is quantized in units of $\Delta v = \Delta k = 1$. To eliminate inessential complications in the notation we take $k \geqslant 0$, unless the context implies otherwise.

We next address the stability of the circulating current, so far keeping $V(x) = 0$. While in the usual textbook analyses [5] of Bogoliubov theory $k = 0$ is assumed, the calculations can equally easily be carried out for a nonzero flow velocity of the condensate; cf. [7]. The frequency of an excitation of the condensate with wave number $k+q$ turns out to be

$$\omega_{k+q} = kq + |q|\sqrt{c^2 + q^2/4}. \tag{4}$$

Here $c = \sqrt{N\xi}$ is the speed of zero sound in the condensate. The sign ambiguity in Bogoliubov theory, $\pm|q|\sqrt{\ldots}$, has been resolved by requiring that in the limit of a stationary noninteracting gas, $k \to 0$ and $c \to 0$, excitation energies should be positive. For $v = k > c$, Eq. (4) gives a soft mode and negative excitation frequencies. We take these as signs of instability. For instance, slightest nonidealities of the torus or even quantum fluctuations pump the soft mode, and the circulating condensate is destroyed.

Equation (4) makes an explicit verification of the well-known argument [5] about the excitation spectrum of a moving fluid based on Galilean transformations. The square root term corresponds to the excitation energy of a stationary condensate, and the kq term is akin to a Doppler shift. We also have the familiar [5,8] condition that a rotating condensate may be stable only if the speed of the fluid is slower than the speed of zero sound. As there have recently been scores of discussions of elementary excitations of a condensate [9–12], another viewpoint may be illuminating. By our assumptions excitations across the torus are frozen out, leaving excitations along the ring. The lowest mode clearly is a sound wave that completes one wavelength around the torus, giving the angular frequency c. The stability condition $v < c$ is the same as requiring that in stable circulation the angular velocity must be smaller than the angular frequency of the lowest-energy condensate excitation. This observation underlines the difference between persistent currents and elementary excitations.

Another implication of the condition $v < c$ follows because on one hand the lowest possible persistent current speed is $v = 1$, and on the other hand the speed of sound satisfies $c = \sqrt{N\xi}$. Superfluidity requires a minimum strength from atom-atom interactions, namely $N\xi > 1$. A similar condition was found in Ref. [4].

Of course, for a perfect torus conservation of angular momentum alone guarantees a persistent current. The crux here is that our argument may apply even if a perturbation $V(x)$ breaks circular symmetry. Then the plane waves u_k are no longer exact eigenstates of the one-body Hamiltonian, and neither is the excitation spectrum of a flowing condensate given exactly by Eq. (4). Nonetheless, if $V(x)$ is ''small enough,'' Eq. (4) should be a useful approximation, and shows that a persistent current still is a local energy minimum for the N-atom system. In microscopic terms, this is the essence of superfluidity [8].

One might rotate the entire apparatus while the atoms are cooled through the BEC phase transition, which is likely to leave behind a rotating condensate. However, we wish to consider less drastic measures to stir up superfluid flow. The question of creating a persistent current is nontrivial; the stationary condensate is a persistent *zero* current, and resists change in its own right. We shall investigate the effects of a rotating potential $V(x - \nu t)$ in the co-rotating frame using Eqs. (1) and (2).

We begin with a perturbative treatment of the potential $V(x)$, first ignoring atom-atom interactions. In this case we essentially adapt the argument of Ref. [13] as follows. The unperturbed states are the plane waves u_k with energies $\epsilon_k(\nu) = k^2/2 - k\nu$. The states k and $k+1$ become degenerate at the angular velocity of rotation $\nu = k + 1/2$, when the velocity of the moving potential is halfway between the atomic velocities of the two states. If the matrix element $\tilde{V} \equiv \tilde{V}(k=1)$ is nonzero, the perturbation lifts the degeneracy to first order in $\tilde{V}$. The crossing of the states k and $k+1$ then turns into an avoided crossing. When the angular velocity ν is ramped up slowly enough, the avoided crossings are traversed adiabatically as $k = 0 \to 1 \to 2 \to \cdots$. This establishes a persistent current.

In the next step we incorporate atom-atom interactions, but continue to assume that only two plane-wave states $i \equiv k$ and $f \equiv k+1$ participate in the crossing. The relevant Hamiltonian is

$$H = \nu b_i^\dagger b_i + \tilde{V}(b_f^\dagger b_i + b_i^\dagger b_f) + \xi b_f^\dagger b_f b_i^\dagger b_i . \tag{5}$$

We have made a few inconsequential simplifications, in particular we have ignored a polynomial of the conserved total particle number $b_i^\dagger b_i + b_f^\dagger b_f$. The exchange energy $\propto \xi$ is what remains of atom-atom interactions. This time we take $\tilde{V}$ to be a perturbation in comparison with the exchange interaction.

The zeroth-order eigenstates are the number states $|n_i n_f\rangle$ with $n_i + n_f = N$, and the corresponding energies are $E(n_i) = n_i\nu + \xi n_i(N - n_i)$. The states n_i and $n_i + 1$ cross at $\nu = \xi(2n_i + 1 - N)$. Since the perturbation $\propto \tilde{V}$ has a matrix element between the states n_i and $n_i + 1$, the avoided-crossing scenario emerges once more, but with a twist. For ordinary condensates the scattering length and hence the parameter ξ are positive, and the type of chain $n_i = N \to N-1 \to \cdots$ that transfers the atoms from the initial state k to the final state $k+1$ ensues when the rotation frequency is swept *downward*, $\dot{\nu} < 0$. The somewhat bizarre prescription emerges that one first sweeps up the angular velocity of the perturbation $V(x)$ through the transition $k \to k+1$ diabatically, then backtracks adiabatically to effect the desired transition $k \to k+1$, and finally sweeps back through the transition diabatically to continue to the next k.

Unfortunately, there is a common uncertainty in these perturbative schemes. Consider the crossing of states k and $k+1$. In the absence of atom-atom interactions the nearest many-body state with all atoms in a single one-particle state

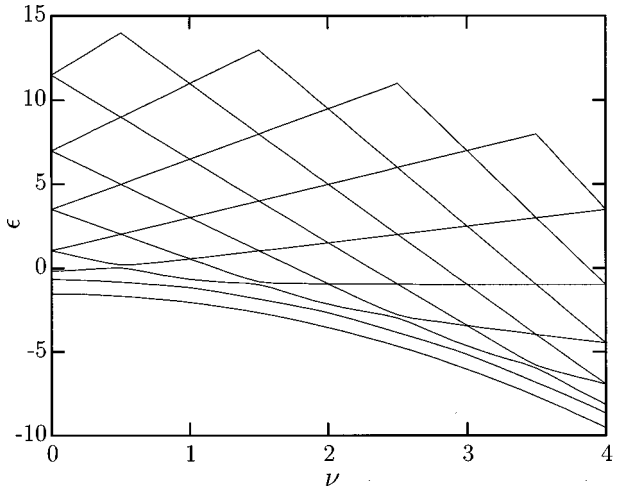

FIG. 1. Energies of the 11 lowest-energy eigenstates of the Gross-Pitaevskii equation (2) as a function of the angular velocity of rotation ν of the potential $V(x)$ in the corotating frame. The figure is for the noninteracting condensate with $\xi=0$, and the potential is $V(x)=-2\cos^2(x/2)$.

would be the one with the atoms in the state $k-1$. This is separated from the state with all atoms in the state k by the energy N. When atom-atom interactions are taken into account in the system of states k and $k+1$ using Eq. (5), exchange interactions are seen to move the ensuing many-body states around by as much as $N^2\xi/4$. If $N^2\xi/4 \gtrsim N$, it may no longer be permissible to treat the crossing $k \to k+1$ without including the state $k-1$. Perturbative arguments are probably valid only for $N\xi \lesssim 1$, when there is *no* superfluidity. The qualitative results are not necessarily incorrect in the presence of superfluidity, but without an elaborate further analysis one cannot tell which are reliable and which are not. That is why we defer details of the perturbative arguments to a future publication [14].

Instead, we turn to a nonperturbative method to excite a persistent current. We begin by noting that if $\phi(x)$ is a solution with energy ϵ to the GPE (2) for the nonrotating potential $V(x)$, then the momentum-translated state $e^{i\nu x}\phi(x)$ with the energy $\epsilon-\frac{1}{2}\nu^2$ is a formal solution in the corotating frame for the same potential rotated at angular velocity ν. We say ''formal'' because, unless ν is an integer, in general $e^{i\nu x}\phi(x)$ does not satisfy the correct periodic boundary conditions. A bound state is a notable exception. Assuming that the minimum of the potential $V(x)$ is at $x=0$, for a bound state $\phi(x)$ nearly vanishes as $x \to \pm\pi$ [15], and so does $e^{i\nu x}\phi(x)$. Therefore $e^{i\nu x}\phi(x)$ may be a good approximation to the true eigenstate in the rotating frame. When the angular velocity is increased, bound states should drop in unison in energy, as $-\frac{1}{2}\nu^2$.

Many concepts we have brought up are demonstrated in Fig. 1. We choose a potential $V(x)=-2\cos^2(x/2)$ such that at $\nu=0$ there are three negative-energy eigenstates. The potential is peculiar in that it only has nonzero Fourier components at $k=0,\pm1$, so that state k is directly coupled only to states $k\pm1$. We set $\xi=0$, and solve the GPE by integrating the corresponding time-dependent GPE in imaginary time using the split-operator method [11]. Figure 1 shows the energies of eleven lowest-lying eigenstates as a function of the angular velocity of rotation ν.

Were there no potential $V(x)$, at $\nu=0$ the energies would simply be $k^2/2$ with $k=0,\pm1,\pm2,\ldots$, all states except $k=0$ being doubly degenerate. The potential $V(x)$ has coupled strongly the states $k=0$ and $k=\pm1$. The resulting three states are pulled to negative energies, i.e., below the top of the potential, and are bound to a varying degree. Other states are affected little at $\nu=0$. With the rotation, the relatively unaffected state doublets split according to $k^2/2 \pm |k|\nu$. Where states with $\Delta k=1$ meet we see anticrossings, but otherwise most of the states practically cross when ν is varied. The bound states are a notable exception. In particular, the energy of the ground state varies with ν very accurately as $-\frac{1}{2}\nu^2$.

One may solve the GPE in the same way for nonvanishing interatomic interactions, $\xi\neq0$. Deep potentials, $V_{\max}-V_{\min} \gg 1$, strong atom-atom interactions, $N\xi \gg 1$, and level crossings are the makings of numerical problems, but we have been able to verify the same qualitative behavior as in Fig. 1 well into the domain of superfluidity.

A strategy to produce a persistent current is now obvious. First one either condenses the atoms with the potential $V(x)$ in place, or first condenses and then turns on $V(x)$ adiabatically. Next one accelerates the potential adiabatically. After the desired rate of rotation is achieved, the strength of the rotating potential $V(x)$ is slowly turned to zero. One may want to choose the final angular velocity ν close to an integer to avoid the mixing of states that happens at half-integer values of ν. The ground state in the rotating frame in the presence of the potential $V(x)$ then evolves into the ground state for the potential $V(x)=0$, a persistent current with k equal to the integer nearest to ν. ''Slowly'' or ''adiabatically'' mean slowly on the time scales of the elementary excitations of the condensate.

Our thesis is that robust driving of a persistent current is possible using a potential $V(x)$ that confines the condensate, i.e., cuts the torus. Repulsive atom-atom interactions lead to an expansion of the condensate and make the confinement more difficult, but the argument applies just the same.

A toroidal trap could be prepared by making a radially symmetric trap and then punching the center out with a blue-detuned laser beam [16]. Extrapolating from current experiments [1], the dimensions could be $R\sim0.05$ mm and $l\sim0.01$ mm. Together with the scattering length $a\sim10^{-8}$ m, these give the estimate $\xi\sim10^{-2}$. At present atom numbers reach up to about 10^6 and we estimate $N\xi\sim10^4$, deep in the regime of superfluidity. Using the mass of a Na atom, we have the frequency scale for superfluid rotation $\hbar/(mR^2)\sim2\pi\times0.1$ Hz. The frequency of the lowest elementary excitation is $\sqrt{N\xi}$ times that, the familiar order of magnitude $2\pi\times10$ Hz. In the Thomas-Fermi limit, large atom number, one may think of condensate excitations as acoustic resonances of zero sound [17]. The frequencies of excitations involving transverse directions should thus be of the order $2\pi R/l$ higher than the frequencies of longitudinal excitations, $\sim2\pi\times300$ Hz in our example. If one operates over time scales long enough to avoid longitudinal excitations, by implication one does not drive transverse excitations either. The perturbing potential $V(x)$ could be produced, e.g., by dipole forces of light. Superfluid motion might be detected by erecting suddenly a stationary (in the laboratory frame)

potential barrier. An asymmetry in the condensate should result, depending on from which side the fluid runs into the barrier.

Thanks to the very low temperature, this type of fluid is mesoscopic even at near-millimeter sizes. Novel phenomena might ensue. For instance, what happens to the stability arguments at $c \sim 1$, in which case the quantization of both persistent current modes and elementary excitations have to be considered more carefully than we have done? Another aspect we want to emphasize is that alkali-metal vapor condensates are clean, well-characterized, and highly controllable systems. Here we have the opportunity and the incentive to discuss superfluidity under a variety of conditions *ab initio*, using microscopic methods. Research projects immediately suggest themselves. For instance, superfluid flow is stable for a sufficiently small perturbation $V(x)$. What happens if $V(x)$ is not so small?

Notably absent from our argument is spontaneously broken gauge symmetry [18]. It has been shown earlier that a number of staples of broken symmetry, such as Josephson effect and interference of two condensates, may be analyzed without ever invoking symmetry breaking [19]. Now, these examples have dealt with the *global* phase of the condensate, and superfluidity may seem different; superfluid flow velocity is attributed to the gradient of the phase of the superfluid order parameter [5], a *local* reference to the phase. Nonetheless, we have in effect [18] managed to predict characteristic consequences of superfluidity without invoking a condensate or superfluid phase or order parameter. In fact, we believe that no broken symmetry, order parameter, or global or local condensate phase is in principle needed for a description of superfluidity. Bloch [8] has some time ago presented a analysis of persistent currents in toroidal geometry that is in complete agreement with our no-phase philosophy. However, Ref. [8] builds on abstract properties of many-body wave functions, and is of limited utility in practical calculations. When one wants predictions in complicated condensed-matter systems, phenomenology in terms of an order parameter may still be the best, or all, one has. Our thrust is that with alkali-metal vapor condensates one could, and should, think of methods that do not rely on an order parameter.

We have advocated the toroidal trap as an advantageous configuration for studies of superfluidity and persistent currents, discussed the stability of superfluid flow, and devised a method to excite a persistent current. Apart from these technical items, we have a broader agenda: We wish to promote low-temperature alkali-metal vapors as clean, mesoscopic systems for experimental and theoretical, microscopic, *ab initio* investigations of superfluidity.

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