Electronic-field correlation functions

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In this paper the correlation function of an arbitrary order is derived for an electron field in which the components corresponding to different particle numbers and momenta are mutually uncorrelated. The result is expressed in terms of the complex degree of coherence. The correlation functions clearly demonstrate the antibunching effect. All the computational steps are shown in detail. [S1050-2947(98)08712-5]

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I. INTRODUCTION

Correlation functions characterize the statistical properties of quantum fields. Their theory is richly developed in quantum optics and the correlation functions of arbitrary order are familiar for various types of photon fields, some of them having been measured experimentally [1]. On the other hand, fermion correlation functions have been widely explored neither theoretically nor experimentally and their theory has only been developed more progressively in recent years [2–7]. One of the most important results in this area was published by Silverman [2], namely, the correlation function of the second order for a multiparticle electron state, in which the individual momentum as well as particle-number components of the field are mutually uncorrelated. The correlation function demonstrated antibunching, i.e., the fact that electrons avoid coming into a detector in pairs. Another work, published by Saito *et al.* [3], deals with a correlation function of an arbitrary order for an electron field in a so-called chaotic state. This state is believed to be produced by the most coherent electron source known up to now, the field-emission gun. In the following a detail derivation of the correlation function of an arbitrary order is given for the state introduced by Silverman [2] that could provide a good description of the electron beams used in the experimental electron interferometry.

II. ELECTRON STATE

In the following we consider the case of quasimonochromatic, noninteracting, nonrelativistic, and spin-polarized electrons. The first three assumptions are fully acceptable for electrons with energies of about 5 keV, an energy bandwidth under 1 eV, and emission currents under 1 μA. The last assumption enables us to neglect the spinor structure of the electron field and to treat it as a scalar field. This is no loss of generality, however. It can be shown that the correlation functions for partially polarized or completely unpolarized electrons can be expressed with the help of the analogous functions for polarized electrons and the degree of polarization [8]. We will also suppose that the complete set of electron momentum eigenstates is countable, which allows us to avoid problems with the normalization of the density operator.

According to Silverman [2] we take the electron state as a mixture of Fock states with a Poissonian distribution of the particle number. The probability that there are n electrons in the system is then $P(n)=e^{-\langle n\rangle}\langle n\rangle^n/n!$, where $\langle n\rangle$ is the average number of electrons, and the density operator of the system can be expressed as

$$\hat{\rho}=\sum_{n=0}^{\infty} P(n)\hat{\rho}_n, \tag{1}$$

where $\hat{\rho}_n$ are the density operators corresponding to the components of the field with a definite particle number.

Let us denote the normalized one-electron momentum distribution as $f(p)$. The normalization condition is

$$\sum_p f(p)=1, \tag{2}$$

where the sum is taken over the complete set of momentum eigenvalues. The momentum p should be understood as a three-dimensional vector, which means that the function $f(p)$ contains information about not only the spectral but also the spatial properties of the electron field.

As has been mentioned, the different momentum components of the field are uncorrelated. Therefore the n-particle density operator can be written in the form of an incoherent superposition of momenta eigenstates. The probability that the n electrons have momenta $p_1,p_2,\ldots,p_n$ is given by the product $f(p_1)f(p_2)\cdots f(p_n)$, and the density operator $\hat{\rho}_n$ has therefore the form $\hat{\rho}_n=\hat{A}_n/\mathrm{Tr}(\hat{A}_n)$ with

$$\hat{A}_n=\sum_{p_1,\ldots,p_n}{}' f(p_1)f(p_2)\cdots f(p_n)|p_1,\ldots,p_n\rangle \times\langle p_1,\ldots,p_n|, \tag{3}$$

where $|p_1,\ldots,p_n\rangle$ is the totally antisymmetrical Fock state of n electrons with momenta $p_1,\ldots,p_n$ normalized to 1 (not the δ function) and $\Sigma'_{p_1,\ldots,p_n}$ denotes the sum over all momenta $p_1,p_2\ldots,p_n$ different from each other. The reason why there are no terms in the sum with some p_i,p_j equal

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is that in this case the state $|p_1, \ldots, p_n\rangle$ does not have any physical meaning due to the Pauli principle. The trace of $\hat{A}_n$ is equal to

$$\mathrm{Tr}(\hat{A}_n) = \sum_{p_1, \ldots, p_n}{}' f(p_1)f(p_2)\cdots f(p_n)\mathrm{Tr}\{|p_1, \ldots, p_n\rangle \times \langle p_1, \ldots, p_n|\} = \sum_{p_1, \ldots, p_n}{}' f(p_1)f(p_2)\cdots f(p_n), \tag{4}$$

because the trace of the projection operator $|p_1, \ldots, p_n\rangle\langle p_1, \ldots, p_n|$ is equal to unity. Now the only reason why we introduced the momentum discretization was to avoid the problems with the normalization of the density operator. We can therefore assume that the number N of states contributing significantly to the sum in Eq. (2) is very large, as in the case of a quasicontinuous momentum spectrum. It can be shown that the normalized difference

$$\frac{\sum_{p_1, \ldots, p_n} f(p_1)f(p_2)\cdots f(p_n) - \sum_{p_1, \ldots, p_n}{}' f(p_1)f(p_2)\cdots f(p_n)}{\sum_{p_1, \ldots, p_n} f(p_1)f(p_2)\cdots f(p_n)}$$

goes to zero for large N. This follows from the fact that the relative number of combinations $p_1, \ldots, p_n$ with $p_i = p_j$ for some i,j $(i \neq j)$ with respect to the number of all combinations[1] behaves like $1/N$ and hence goes to zero for large N. Therefore we can use an approximation $\Sigma'_{p_1, \ldots, p_n} \to \Sigma_{p_1, \ldots, p_n}$ in the following. From Eqs. (2) and (4) and this approximation, it follows immediately that $\mathrm{Tr}(\hat{A}_n) = 1$, and we can write $\hat{\rho}_n = \hat{A}_n$, which will also be used in the following.

III. CORRELATION FUNCTION OF THE *k*TH ORDER

The correlation function of the kth order is defined as follows:

$$G^{(k)}(r_1,t_1,r_2,t_2, \ldots, r_k,t_k) = \mathrm{Tr}\{\hat{\rho}\hat{\psi}^\dagger(r_1,t_1)\cdots\hat{\psi}^\dagger(r_k,t_k)\hat{\psi}(r_k,t_k)\cdots\hat{\psi}(r_1,t_1)\}, \tag{5}$$

where $\hat{\psi}(r_i,t_i)$ and $\hat{\psi}^\dagger(r_i,t_i)$ are the field operators annihilating and creating an electron at the space-time point (r_i,t_i), respectively. The correlation function expresses the probability density of finding k electrons at points $r_1, r_2, \ldots, r_k$ and times $t_1, t_2, \ldots, t_k$, and it should be noted that in general it is not proportional to the probability of detecting k electrons at the corresponding points and times. The latter is in fact connected with the particle fluxes rather than densities, which is the reason why some authors (e.g., [2,3]) define the correlation function with the help of flux operators. For quasimonochromatic electrons, however, the detection probability is proportional to the correlation function (5) by the factor $\Pi_{i=1}^n \alpha_i v S_i$, where α_i and S_i are the quantum efficiency and cross section of the ith detector, respectively, and v is the mean velocity of the electrons.

IV. CALCULATION OF THE CORRELATION FUNCTION

We begin the calculation with the general relation that holds for the annihilation and creation operators

$$\hat{\psi}(r,t) = \sum_p K(r,t|p)\hat{a}(p), \qquad \hat{\psi}^\dagger(r,t) = \sum_p K^*(r,t|p)\,\hat{a}^\dagger(p). \tag{6}$$

Here $\hat{a}(p)$ and $\hat{a}^\dagger(p)$ are the annihilation and creation operators of an electron with momentum p and $K(r,t|p)$ is the electron propagator in the momentum-coordinate representation. The operators $\hat{a}(p)$, $\hat{a}^\dagger(p)$ and $\hat{\psi}(r,t)$, $\hat{\psi}^\dagger(r,t)$ satisfy the standard fermion anticommutation relations

$$\hat{a}^\dagger(p)\hat{a}(p') + \hat{a}(p')\hat{a}^\dagger(p) = \delta(p,p'), \qquad \hat{\psi}^\dagger(r,t)\hat{\psi}(r',t) + \hat{\psi}(r',t)\hat{\psi}^\dagger(r,t) = \delta(r-r'), \tag{7}$$

where $\delta(p,p')$ denotes the Kronecker delta, while $\delta(r-r')$ is the three-dimensional Dirac delta function. If we denote the field operators $\hat{\psi}(r_i,t_i)$, $\hat{\psi}^\dagger(r_i,t_i)$ and the propagator $K(r_i,t_i|p)$ as $\hat{\psi}_i$, $\hat{\psi}_i^\dagger$, and $K_i(p)$, respectively, we can simplify the notation as follows:

$$\hat{\psi}_i = \sum_p K_i(p)\,\hat{a}(p), \quad \hat{\psi}_i^\dagger = \sum_p K_i^*(p)\,\hat{a}^\dagger(p). \tag{8}$$

With the help of the relations (8) we can express the product of the field operators from Eq. (5) in the following way:

$$\hat{\psi}_1^\dagger\cdots\hat{\psi}_k^\dagger\hat{\psi}_k\cdots\hat{\psi}_1 = \sum_{\{q\},\{q'\}} K_1^*(q_1)\cdots K_k^*(q_k)K_k(q_k')\cdots K_1(q_1')\hat{a}^\dagger(q_1)\cdots\hat{a}^\dagger(q_k)\hat{a}(q_k')\cdots\hat{a}(q_1'). \tag{9}$$

Here $\Sigma_{\{q\},\{q'\}}$ expresses the summation over $q_1, \ldots, q_k$ and $q_1', \ldots, q_k'$. Using the form (1) of the density operator $\hat{\rho}$, we can write

[1] Of course, the momenta belong now to the set of the corresponding N eigenvalues only.

$$G^{(k)}=\sum_{n=0}^{\infty} P(n)\mathrm{Tr}\{\hat{\rho}_n\hat{\psi}_1^{\dagger}\cdots\hat{\psi}_k^{\dagger}\hat{\psi}_k\cdots\hat{\psi}_1\}=\sum_{n=0}^{\infty} P(n)G_n^{(k)}, \tag{10}$$

where $G_n^{(k)}$ is the correlation function corresponding to the n-electron density operator $\hat{\rho}_n$. We have omitted the arguments r_i, t_i for simplicity. The correlation function corresponding to the density operator $\hat{\rho}_n$ can be now expressed with the help of Eqs. (3) and (9) and the approximation $\hat{\rho}_n=\hat{A}_n$ as

$$\begin{aligned}G_n^{(k)}=&\sum_{\{q\},\{q'\}}\sum_{p_1,\ \ldots\ ,p_n}{}' f(p_1)\cdots f(p_n)K_1^*(q_1)\cdots K_k^*(q_k)K_k(q_k')\cdots K_1(q_1')\mathrm{Tr}\{|p_1,\ \ldots\ ,p_n\rangle\\ &\times\langle p_1,\ \ldots\ ,p_n|\hat{a}^{\dagger}(q_1)\cdots\hat{a}^{\dagger}(q_k)\hat{a}(q_k')\cdots\hat{a}(q_1')\}.\end{aligned} \tag{11}$$

It can be shown on the basis of anticommutation relations that the trace in Eq. (11) is equal to

$$\begin{aligned}&\mathrm{Tr}\{|p_1,\ \ldots\ ,p_n\rangle\langle p_1,\ \ldots\ ,p_n|\hat{a}^{\dagger}(q_1)\cdots\hat{a}^{\dagger}(q_k)\hat{a}(q_k')\cdots\hat{a}(q_1')\}\\ &\quad=\langle p_1,\ \ldots\ ,p_n|\hat{a}^{\dagger}(q_1)\cdots\hat{a}^{\dagger}(q_k)\hat{a}(q_k')\cdots\hat{a}(q_1')|p_1,\ \ldots\ ,p_n\rangle\\ &\quad=\sum_{\mathrm{P}\in\mathcal{P}}\mathrm{sign}(\mathrm{P})\ \delta(q_1,q'_{\mathrm{P}(1)})\cdots\delta(q_k,q'_{\mathrm{P}(k)})\sum_{\substack{i_1,\ \ldots\ ,i_k=1\\ \text{all } i \text{ different}}}^{n}\delta(p_{i_1},q_1)\cdots\delta(p_{i_k},q_k).\end{aligned} \tag{12}$$

Here P, $\mathrm{P}^{(j)}$, and sign(P) denote a permutation of k indexes $1,2,\ \ldots\ ,k$, and the jth element and sign of this permutation, respectively. The first sum is made over the set $\mathcal{P}$ of all permutations. The fermion nature of the electrons is demonstrated by the factor sign(P).

We make our notation more clear in an example with $k=3$. Then the set of all permutations is

$$\mathcal{P}=\{(1,2,3),(3,1,2),(2,3,1),(1,3,2),(3,2,1),(2,1,3)\}$$

and if we choose the permutation $\mathrm{P}=(2,1,3)$, we have $\mathrm{P}^{(1)}=2$, $\mathrm{P}^{(2)}=1$, $\mathrm{P}^{(3)}=3$, and $\mathrm{sign}(\mathrm{P})=-1$ because the permutation is odd.

Substituting Eq. (12) into Eq. (11) we obtain

$$\begin{aligned}G_n^{(k)}=&\sum_{\{q\},\{q'\}}K_1^*(q_1)\cdots K_k^*(q_k)K_k(q_k')\cdots K_1(q_1')\sum_{\mathrm{P}\in\mathcal{P}}\mathrm{sign}(\mathrm{P})\ \delta(q_1,q'_{\mathrm{P}(1)})\cdots\delta(q_k,q'_{\mathrm{P}(k)})\\ &\times\sum_{\substack{i_1,\ \ldots\ ,i_k=1\\ \text{all } i \text{ different}}}^{n}\ \sum_{p_1,\ \ldots\ ,p_n}{}' f(p_1)\cdots f(p_n)\ \delta(p_{i_1},q_1)\cdots\delta(p_{i_k},q_k).\end{aligned} \tag{13}$$

In Eq. (13) we calculate first the sum over the n momenta $p_1,\ \ldots\ ,p_n$. It is useful to note that for every possible combination $i_1,\ \ldots\ ,i_k$ with all i different, it holds due to the normalization condition (2) of the function $f(p)$

$$\sum_{p_1,\ \ldots\ ,p_n}{}' f(p_1)\cdots f(p_n)\ \delta(p_{i_1},q_1)\cdots\delta(p_{i_k},q_k)=f(q_1)\cdots f(q_k), \tag{14}$$

that is, the sum does not depend on the indexes $i_1,\ \ldots\ ,i_k$. We have used here again the approximation $\Sigma'\to\Sigma$. Therefore all the $n(n-1)\cdots(n-k+1)$ terms [we will denote this product as $V(k,n)$ in the following] in the sum over $i_1,\ \ldots\ ,i_k$ different from each other give the same result, and we obtain

$$\sum_{\substack{i_1,\ \ldots\ ,i_k=1\\ \text{all } i \text{ different}}}^{n}\ \sum_{p_1,\ \ldots\ ,p_n}{}' f(p_1)\cdots f(p_n)\ \delta(p_{i_1},q_1)\cdots\delta(p_{i_k},q_k)=V(k,n)\ f(q_1)\cdots f(q_k). \tag{15}$$

Next we substitute Eq. (15) into Eq. (13) and evaluate the sum over the momenta $q_1',\ \ldots\ ,q_k'$:

$$G_n^{(k)}=V(k,n)\sum_{\{q\},\{q'\}} f(q_1)\cdots f(q_k)\,K_1^*(q_1)\cdots K_k^*(q_k)K_k(q_k')\cdots K_1(q_1')\sum_{\mathrm{P}\in\mathcal{P}}\mathrm{sign(P)}\,\delta(q_1,q'_{\mathrm{P}(1)})\cdots\delta(q_k,q'_{\mathrm{P}(k)})$$

$$=V(k,n)\sum_{\mathrm{P}\in\mathcal{P}}\mathrm{sign(P)}\sum_{q_1,\ \cdots\ ,q_k} f(q_1)\cdots f(q_k)\,K_1^*(q_1)\cdots K_k^*(q_k)K_{\mathrm{P}(1)}(q_1)\cdots K_{\mathrm{P}(k)}(q_k). \tag{16}$$

Gathering the terms with equal q_i, we can write the result in the form

$$G_n^{(k)}=V(k,n)\sum_{\mathrm{P}\in\mathcal{P}}\mathrm{sign(P)}\sum_q f(q)\,K_1^*(q)K_{\mathrm{P}(1)}(q)\cdots\sum_q f(q)\,K_k^*(q)K_{\mathrm{P}(k)}(q). \tag{17}$$

We see that the correlation functions for different electron numbers differ from each other only by the multiplicative factor $V(k,n)$. Therefore due to Eq. (10) the evaluation of the ''total'' correlation function $G^{(k)}$ reduces now to the averaging of this factor for the Poissonian distribution. It is easy to verify that for this distribution $\langle V(k,n)\rangle=\langle n(n-1)\cdots(n-k+1)\rangle=\langle n\rangle^k$ holds and for $G^{(k)}$ we get

$$G^{(k)}=\sum_{n=0}^{\infty}P(n)G_n^{(k)}=\langle n\rangle^k\sum_{\mathrm{P}\in\mathcal{P}}\mathrm{sign(P)}\sum_q f(q)\,K_1^*(q)K_{\mathrm{P}(1)}(q)\cdots\sum_q f(q)\,K_k^*(q)K_{\mathrm{P}(k)}(q). \tag{18}$$

If we denote $\Gamma_{ij}=\langle n\rangle\Sigma_q f(q)\,K_i^*(q)K_j(q)$, this result can be written in the form of a determinant

$$G^{(k)}=\begin{vmatrix}\Gamma_{11} & \Gamma_{12} & \cdots & \Gamma_{1k}\\ \Gamma_{21} & \Gamma_{22} & \cdots & \Gamma_{2k}\\ \vdots & \vdots & & \vdots\\ \Gamma_{k1} & \Gamma_{k2} & \cdots & \Gamma_{kk}\end{vmatrix}. \tag{19}$$

It is very useful to note at this point that $\Gamma_{ij}=\mathrm{Tr}\{\hat{\rho}\hat{\psi}^\dagger(r_i,t_i)\hat{\psi}(r_j,t_j)\}$ (for the proof see Appendix A). Thus the elements of the determinant in Eq. (19) are in fact the correlation functions of the first order with unequal arguments (the field operators correspond to two different space-time points) referred to as the cross-correlation functions in quantum optics [1]. We also introduce the complex degree of coherence $\gamma_{ij}=\Gamma_{ij}/\sqrt{\Gamma_{ii}\Gamma_{jj}}$, which is the normalized cross-correlation function. The complex degree of coherence expresses the mutual coherence of the electron field at the space-time points P_i and P_j [P_i is an abbreviated notation for the point (r_i,t_i)] and contains information about both the temporal and spatial coherence of the field. From the definition of Γ_{ij}, γ_{ij} and from the normalization condition (2) of the function $f(p)$, it follows that for all i,j $0\leqslant|\gamma_{ij}|\leqslant 1$ holds.[2] The case $|\gamma_{ij}|=1$ corresponds to the complete mutual coherence of the field at the points P_i,P_j, while $|\gamma_{ij}|=0$ corresponds to the complete incoherence. With the help of the determinant algebra and the fact that $\gamma_{ii}=1$ for all i, the correlation function can be written as

$$G^{(k)}=\Gamma_{11}\Gamma_{22}\cdots\Gamma_{kk}\begin{vmatrix}1 & \gamma_{12} & \cdots & \gamma_{1k}\\ \gamma_{21} & 1 & \cdots & \gamma_{2k}\\ \vdots & \vdots & & \vdots\\ \gamma_{k1} & \gamma_{k2} & \cdots & 1\end{vmatrix}. \tag{20}$$

V. ANTIBUNCHING

In Appendix B it is shown that the determinant in Eq. (20) cannot exceed 1. Using this and the fact that $\Gamma_{ii}=G^{(1)}(r_i,t_i)$, which can be easily verified by substituting $k=1$ into Eq. (19), we obtain a very important inequality,

$$G^{(k)}(r_1,t_1,\ \ldots\ ,r_k,t_k)\leqslant G^{(1)}(r_1,t_1)G^{(1)}(r_2,t_2)\cdots\times G^{(1)}(r_k,t_k). \tag{21}$$

The case of equality in Eq. (21) corresponds to the situation when $\gamma_{ij}=0$ for all i,j, $i\neq j$ (see Appendix B), i.e., when the degree of coherence assigned to any pair of points P_i,P_j is zero. This means that if the field at all the k points P_i is mutually completely incoherent, the probability that we will find an electron at each of them is simply equal to the product of the probabilities of finding an electron at the individual points, which is an expected result.

On the other hand, if the field is mutually coherent, at least at one pair of the points P_i,P_j, the inequality (21) is sharp. Therefore the probability that at each of the k space-time points we will find an electron is less than the product of the probabilities of finding an electron at the individual points, which demonstrates the electron antibunching. It is useful to note that this result is independent of the positions of points P_i, and antibunching is therefore quite a universal phenomenon. If some two points P_i,P_j approach each other, which results in $|\gamma_{ij}|$ going to unity, the ''intensity'' of antibunching increases and in the limiting case when P_i,P_j become identical, the ith and jth lines of the determinant ma-

[2]Alternatively, the inequality $|\gamma_{ij}|\leqslant 1$ follows from the Schwarz inequality for operators (see Ref. [1], p. 593).

trix (19) become equal. This results in the annulling of the determinant itself, and therefore the probability that we will find two electrons at the same point is equal to zero, which is in fact nothing other than the Pauli principle, however.

VI. CONCLUSION

We have derived the correlation function of an arbitrary order for an electron state in which the momenta as well as the particle-number components of the field are mutually uncorrelated. We have expressed the result in a compact form (20) in terms of the complex degrees of coherence. Using the features of matrices and determinants we have proved that if the electron field is mutually coherent at the points for which the correlation function is expressed, the electrons show antibunching.

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APPENDIX A: DERIVATION OF THE CROSS-CORRELATION FUNCTION

With the help of the relations (8) we express the operator $\hat{\psi}_i^\dagger \hat{\psi}_j = \hat{\psi}^\dagger(r_i,t_i)\hat{\psi}(r_j,t_j)$ as follows:

$$\hat{\psi}_i^\dagger \hat{\psi}_j = \sum_{q,q'} K_i^*(q) K_j(q') \hat{a}^\dagger(q) \hat{a}(q'). \tag{A1}$$

Then the n-electron cross correlation function is

$$\begin{aligned} \mathrm{Tr}(\rho_n \hat{\psi}_i^\dagger \hat{\psi}_j) &= \sum_{q,q'} \sum_{p_1,\ \ldots\ ,p_n}{}' f(p_1)\cdots f(p_n) K_i^*(q) K_j(q')\, \mathrm{Tr}\{|p_1,\ \ldots\ ,p_n\rangle\langle p_1,\ \ldots\ ,p_n| \hat{a}^\dagger(q)\, \hat{a}(q')\} \\ &= \sum_{q,q'} \sum_{p_1,\ \ldots\ ,p_n}{}' f(p_1)\cdots f(p_n)\, K_i^*(q) K_j(q')\, \delta(q,q') \sum_{s=1}^{n} \delta(q,p_s). \end{aligned} \tag{A2}$$

After the evaluation of the sums over $p_1, \ldots, p_n$ and q' in a way analogous to the one used in Sec. IV, we obtain

$$\mathrm{Tr}(\rho_n \hat{\psi}_i^\dagger \hat{\psi}_j) = n \sum_q f(q)\, K_i^*(q) K_j(q), \tag{A3}$$

and with the help of Eq. (1) we get the desired result

$$\mathrm{Tr}(\rho \hat{\psi}_i^\dagger \hat{\psi}_j) = \langle n \rangle \sum_q f(q)\, K_i^*(q) K_j(q) = \Gamma_{ij}. \tag{A4}$$

APPENDIX B: UPPER BOUND OF THE DETERMINANT IN EQ. (20)

The matrix

$$A = \begin{pmatrix} 1 & \gamma_{12} & \cdots & \gamma_{1k} \\ \gamma_{21} & 1 & \cdots & \gamma_{2k} \\ \vdots & \vdots & & \vdots \\ \gamma_{k1} & \gamma_{k2} & \cdots & 1 \end{pmatrix} \tag{B1}$$

is Hermitian because $\gamma_{ij} = \gamma_{ji}^*$ for all i,j, which follows from the definition of γ_{ij} and Γ_{ij}. Therefore it is possible to transform it into the diagonal form with a unitary transformation, i.e., there exists a unitary matrix U for which the matrix $B = UAU^{-1}$ is diagonal. This transformation changes neither the determinant of the matrix, because it is a similarity transformation, nor the trace of the matrix, because it is a unitary transformation. If we denote the diagonal elements of the matrix B as b_i, then $\mathrm{Tr}(A) = \mathrm{Tr}(B) = \Sigma_{i=1}^k b_i$ and $\det(A) = \det(B) = \Pi_{i=1}^k b_i$ evidently holds. At the same time, from (B1) it follows that $\mathrm{Tr}(A) = k$. Moreover, it is possible to show on the basis of the Schwarz inequality for operators (see Ref. [1], pp. 585 and 593) that the matrix A is nonnegative definite, and hence all its eigenvalues b_i are nonnegative. Thus, we look for the set of non-negative real numbers b_i for which $\Pi_{i=1}^k b_i$ is maximal under the condition $\Sigma_{i=1}^k b_i = k$. This problem can be easily solved using the inequality between the arithmetical and geometrical averages. The arithmetical average of the numbers b_i is $\Sigma_{i=1}^k b_i/k = 1$ and the geometrical average $\sqrt[k]{\Pi_{i=1}^k b_i}$, which is an ascending function of $\Pi_{i=1}^k b_i$, is hence always less than or equal to unity, the equality taking place when $b_1 = b_2 = \cdots = b_k = 1$. In this case the matrix B is a unity matrix, from which it follows that A is also a unity matrix. Then $\gamma_{ij} = 0$ for $i \neq j$ and $\det(A) = \det(B) = 1$. Thus, $\det(A) \leqslant 1$ holds and the equality takes place when all the nondiagonal elements of the matrix A vanish.

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