Vortices in Bose-Einstein-condensed atomic clouds

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The properties of vortex states in a Bose-Einstein-condensed cloud of atoms are considered at zero temperature. Using both analytical and numerical methods, we solve the time-dependent Gross-Pitaevskii equation for the case when a cloud of atoms containing a vortex is released from a trap. In two dimensions we find the simple result that the time dependence of the cloud radius is given by $(1+\omega^2 t^2)^{1/2}$, where ω is the trap frequency. We calculate and compare the expansion of the vortex core and the cloud radius for different numbers of particles and interaction strengths, in both two and three dimensions, and discuss the circumstances under which vortex states may be observed experimentally. [S1050-2947(98)06412-9]

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I. INTRODUCTION

Interest in the properties of atomic clouds was greatly stimulated by the experimental discovery [1–3] of Bose-Einstein condensation in trapped gases of alkali-metal atoms. One of the intriguing features of these condensates that remains to be explored experimentally is their behavior under rotation.

Vortex states in trapped atomic clouds at zero temperature have been considered theoretically by several authors [4–6]. From numerical solutions to the Gross-Pitaevskii equation Dalfovo and Stringari [4] determined the critical angular velocity, which is the lowest angular velocity for which it is favorable for a vortex to enter the cloud. Lundh *et al.* [5] obtained for reasonably large clouds approximate analytical expressions for the critical angular velocity, which agree closely with the numerical results.

For large clouds the presence of a vortex is difficult to detect experimentally since the size of the vortex core is small compared to the size of the cloud. Consequently, the energy of a state in which a vortex is present is nearly equal to the energy of the ground state without the vortex. Rather than measuring the total energy of a trapped cloud it may therefore be advantageous to investigate the density profile of the cloud during a free expansion after the trap potential has been turned off. In this way one may be able to follow how the ''hole'' in the middle of the cloud develops as a function of time, thereby allowing one to distinguish the free expansion of the vortex state from that of the ground state without a vortex. In this paper we shall therefore study the free expansion of a cloud containing a vortex, which is initially trapped in an anisotropic harmonic-oscillator potential and subsequently released.

Apart from the difficulties involved in the detection of a vortex state, one also has to consider its experimental generation [7]. Due to the energy barriers separating rotating and nonrotating states it may be advantageous to generate a vortex state by cooling a rotating cloud in its normal state below the transition temperature rather than rotating the cloud from rest at temperatures below the Bose-Einstein condensation temperature.

There are questions regarding the stability of vortices [6] and the character of the ground state for a fixed angular momentum if the angular momentum per particle is not a multiple of $\hbar$ [8]. However, we shall not address these issues in this paper, but will consider the case of expansion of a cloud containing a vortex with one quantum of circulation.

In the following we shall consider the free expansion of a condensed atomic cloud in the limit when the temperature is sufficiently low that the influence of the normal component is negligible. We assume that the system is dilute in the sense that the scattering length is much less than the interparticle distance. In this case one may neglect the depletion of the condensate and the wave function $\psi(\vec{r},t)$ of the condensed state in an external potential $V(\vec{r})$ satisfies the time-dependent Gross-Pitaevskii equation [9]

$$i\hbar\frac{\partial\psi(\vec{r},t)}{\partial t}=\left[-\frac{\hbar^2}{2m}\nabla^2+V(\vec{r})+U_0|\psi(\vec{r})|^2\right]\psi(\vec{r}). \quad (1)$$

The effective two-body interaction U_0 is given by $U_0=4\pi\hbar^2 a/m$, where a is the scattering length and m is the atomic mass. We shall assume the scattering length to be positive, thereby ensuring the stability of the nonrotating condensed state for any value of the total number of particles N.

The plan of the paper is as follows. In Sec. II we consider a two-dimensional vortex state and show that its development in time is given approximately by a simple, analytical expression for the cloud radius. We also solve the time-dependent Gross-Pitaevskii equation numerically and compare the resulting density profile to the one obtained analytically. In Sec. III we discuss the more realistic three-dimensional case by an approximate variational method as well as by numerical methods. The expansion of the rotating cloud of atoms is compared to that obtained for a nonrotating cloud when the trap potential is turned off. Section IV is a brief conclusion.

II. VORTICES IN TWO DIMENSIONS

In the following we consider a two-dimensional geometry, where the vortex is uniform along its axis, which we take to be the z axis. The harmonic-oscillator potential is assumed to be isotropic in the xy plane and given by

$$V=\frac{1}{2}m\omega^2\rho^2, \tag{2}$$

where $\rho^2=x^2+y^2$.

In the stationary case the condensate wave function depends on time only through its phase according to

$$\psi(\vec{r},t)=\psi(\vec{r})e^{-i\mu t/\hbar}, \tag{3}$$

where μ is the chemical potential. In the Thomas-Fermi approximation the chemical potential is obtained by inserting Eq. (3) into Eq. (1) and neglecting the kinetic-energy term. This results in the following expression for the particle density:

$$|\psi|^2=\frac{1}{U_0}\left(\mu-\frac{1}{2}m\omega^2\rho^2\right), \tag{4}$$

provided ρ is less than $\rho_{\max}=(2\mu/m\omega^2)^{1/2}$. For ρ greater than $\rho_{\max}$, the density is zero.

The number of particles ν per unit length along the z axis is

$$\nu=2\pi\int_0^{\rho_{\max}}d\rho\rho|\psi|^2=\frac{\pi\mu^2}{U_0 m\omega^2}. \tag{5}$$

In terms of the dimensionless parameter γ, which is defined by

$$\gamma=\nu a, \tag{6}$$

the chemical potential is then seen to be given by

$$\mu=2\hbar\omega\gamma^{1/2}. \tag{7}$$

The Thomas-Fermi approximation becomes exact in the limit of large γ. Since the chemical potential is given in terms of the total energy per unit length E by the equation $\mu=\partial E/\partial\nu$ we conclude that $\mu=3E/2\nu$, and therefore

$$E=\frac{4}{3}\nu\hbar\omega\gamma^{1/2}. \tag{8}$$

Since we shall use Gaussian trial functions later on in discussing the time evolution of the vortex state, it is instructive to compare Eqs. (7) and (8) with the result of calculating the ground-state energy variationally with a trial function of the form

$$\psi(x,y)=Ae^{-\rho^2/2b^2}, \tag{9}$$

where b is a variational parameter. Writing $b=\alpha a_{\rm osc}$, where $a_{\rm osc}=(\hbar/m\omega)^{1/2}$ is the oscillator length, one finds

$$E=\nu\hbar\omega\left(\frac{1}{2\alpha^2}+\frac{1}{2}\alpha^2+\frac{\gamma}{\alpha^2}\right). \tag{10}$$

This expression has a minimum for $\alpha=\alpha_0=(1+2\gamma)^{1/4}$ and therefore the variational estimate of the energy of the ground state is

$$E=\nu\hbar\omega\sqrt{2\gamma+1}. \tag{11}$$

In the limit of large γ this variational approximation to the energy exceeds the asymptotically exact result (8) by only 6%. In the opposite limit, for small γ, it is exact to order γ.

We now consider a vortex state corresponding to one quantum h/m of circulation around the z axis. The corrections to the Thomas-Fermi energy (8) consist of two terms: the vortex energy per unit length and the kinetic energy associated with the rounding of the density profile in the vicinity of the surface of the cloud, which is of the same order of magnitude as the vortex energy. For large clouds the vortex energy per unit length is [5]

$$E_v=\pi|\psi(0)|^2\frac{\hbar^2}{m}\ln\left(\frac{0.888\rho_{\max}}{\xi_0}\right)\approx\frac{\nu\hbar\omega}{2\gamma^{1/2}}\ln(3.55\gamma^{1/2}), \tag{12}$$

where the coherence length ξ_0 is defined by

$$\frac{\hbar^2}{2m\xi_0^2}=U_0|\psi(0)|^2. \tag{13}$$

Note that the ratio of the coherence length to the cloud radius in the Thomas-Fermi limit is given by

$$\frac{\xi_0}{\rho_{\max}}=\frac{\hbar\omega}{2\mu}=\frac{1}{4\gamma^{1/2}}. \tag{14}$$

The kinetic energy associated with the surface is [5]

$$E_k=\nu\frac{\hbar\omega}{4\gamma^{1/2}}\ln(2.43\gamma^{1/3}). \tag{15}$$

Summing up the various contributions, the total energy per unit length for the cloud in the vortex state is therefore

$$E=\nu\hbar\omega\left[\frac{4}{3}\gamma^{1/2}+\frac{1}{3\gamma^{1/2}}\ln(13.0\gamma)\right] \tag{16}$$

for sufficiently large clouds ($\gamma\gg1$). To leading order in γ the energy of the vortex state is thus the same as that of the ground state. We may compare the result (16) for the energy to the result of using a variational trial function of the form

$$\psi(\rho,\phi)=A\rho e^{i\phi}e^{-\rho^2/2b^2}, \tag{17}$$

where ϕ is the azimuthal angle, yielding

$$E=\nu\hbar\omega\sqrt{2\gamma+4}. \tag{18}$$

The variational estimate of the vortex energy is obtained by subtracting Eq. (11) from Eq. (18) and gives the value $\nu\hbar\omega 3\sqrt{2}/4\gamma^{1/2}$, which is to be compared with Eq. (12). This shows that the variational calculation gives a poor estimate of the energy of the vortex for strong coupling. The reason for this is that the trial wave function has only one length scale, which determines both the size of the cloud and the size of the vortex core.

A. Free expansion

We shall now use Gaussian trial functions also for treating the time development of the density when the cloud is released from the trap at a certain instant of time $t=0$. We assume the solution to be homologous in the sense that the local velocity is proportional to the distance from the axis and therefore employ a trial function of the form

$$\psi(\rho,\phi,t)=A\rho e^{i\phi}e^{-\rho^2/2R^2}e^{i\beta\rho^2/2}, \tag{19}$$

where $A=\sqrt{\nu/\pi}R^{-2}$ is the time-dependent normalization constant found from conserving the number of particles per unit length. The radial velocity is given by the derivative of the phase with respect to ρ. The quantities R and β depend on time. We wish to evaluate the Lagrangian

$$L=\int_0^\infty d\rho 2\pi\rho\left[\frac{i\hbar}{2}\left(\psi^*\frac{\partial\psi}{\partial t}-\psi\frac{\partial\psi^*}{\partial t}\right)-\frac{\hbar^2}{2m}|\nabla\psi|^2-\frac{U_0}{2}|\psi|^4\right] \tag{20}$$

following the method used in Ref. [10]. Performing the integration over ρ, we obtain the Lagrangian as a function of the variables β and R and their time derivatives $\dot\beta$ and $\dot R$:

$$L=-\nu\left(\frac{\hbar^2}{mR^2}(1+\beta^2R^4)+\nu\frac{U_0}{8\pi R^2}+\dot\beta\hbar R^2\right). \tag{21}$$

From the Lagrange equations for the two independent variables β and R we obtain

$$\beta=\frac{m\dot R}{\hbar R} \tag{22}$$

and

$$\dot\beta=\frac{\hbar}{mR^4}-\frac{\hbar\beta^2}{m}+\frac{\nu U_0}{8\pi\hbar R^4}. \tag{23}$$

When Eq. (22) for β is inserted in Eq. (23) we arrive at the acceleration equation

$$m\ddot R=-\frac{\partial U(R)}{\partial R}, \tag{24}$$

where

$$U(R)=\frac{\hbar^2}{2mR^2}\left(1+\frac{\gamma}{2}\right). \tag{25}$$

Note that the contributions to the effective potential energy U from the kinetic term due to the zero point motion and the interaction energy both scale in the same way as a function of the size of the cloud. This result is peculiar to two dimensions. Equation (24) has the solution

$$R^2(t)=R^2(0)+v_0^2t^2, \tag{26}$$

where the velocity v_0 is given by

$$v_0^2=\frac{\hbar^2}{m^2R(0)^2}\left(1+\frac{\gamma}{2}\right). \tag{27}$$

Using the result of the minimization in the stationary case with the trial function (17), one finds

$$R(0)^4=a_{\rm osc}^4\left(1+\frac{\gamma}{2}\right) \tag{28}$$

and thus

$$v_0=a_{\rm osc}\omega\left(1+\frac{\gamma}{2}\right)^{1/4}. \tag{29}$$

This implies that the result (26) may be written in the form

$$R(t)^2=R(0)^2(1+\omega^2t^2). \tag{30}$$

This simple result is a consequence of the fact that the effective potential energy varies as R^{-2}, which, as we remarked above, is a special feature of two dimensions.

We note that the root-mean-square (rms) radius $\rho_{\rm rms}$ is $\sqrt{2}R$, so the final value of the rms velocity is $v_{\rm rms}=\sqrt{2}v_0$. Figure 1 shows how the final value of $v_{\rm rms}$ varies with γ. For a nonrotating cloud, the corresponding analysis is easily carried out with the result $v_{\rm rms}=v_0=a_{\rm osc}\omega(1+2\gamma)^{1/4}$. This is included in Fig. 1. As we shall see from the numerical solutions presented below, the simple result (30) yields an accurate description of the expanding vortex state.

B. Numerical results

For the numerical study it is convenient to work in terms of scaled quantities and we introduce the variables $f=\psi/e^{i\phi}(U_0/\mu)^{1/2}$, $\rho_1=\rho/a_{\rm osc}$, and $\mu_1=\mu/\hbar\omega$. The stationary Gross-Pitaevskii equation may then be written in the dimensionless form

$$-\frac{1}{2}\left(\frac{\partial^2 f}{\partial\rho_1^2}+\frac{1}{\rho_1}\frac{\partial f}{\partial\rho_1}-\frac{1}{\rho_1^2}f\right)+\frac{1}{2}\rho_1^2f+\mu_1f^3=\mu_1f. \tag{31}$$

The equation for a nonrotating cloud is similar if the scaled wave function is defined by $f=\psi/(U_0/\mu)^{1/2}$ and the centrifugal term $f/2\rho_1^2$ (which comes from the phase $e^{i\phi}$ of the wave function in the case of a vortex state) is omitted.

We integrate this equation using the Runge-Kutta method to find the wave function f as a function of ρ_1 corresponding to a given dimensionless parameter μ_1. The time-independent wave functions for $\mu=2.5\hbar\omega$ and $\mu=10\hbar\omega$ are shown in Fig. 2. From normalization, we find that these correspond to $\gamma=0.55$ and 23.6, respectively.

To get a feeling for the orders of magnitude, note that for the realistic case of a system with an axial dimension of $10a_{\rm osc}$ and with $a_{\rm osc}/a=100$, the total number of particles is $\sim10^3\gamma$. In the experiment performed at JILA [1], $a_{\rm osc}/a=700$, $N=2000$, and the axial dimension of the cloud was of the same order of magnitude as $a_{\rm osc}$. In the Ioffe-Pritchard trap at MIT [11], $a_{\rm osc}/a=400$, $N=10^6$, and the length in the z direction is $\sim80a_{\rm osc}$, giving $\gamma\sim30$. The case $\gamma=23.6$ is thus quite close to typical experimental conditions.

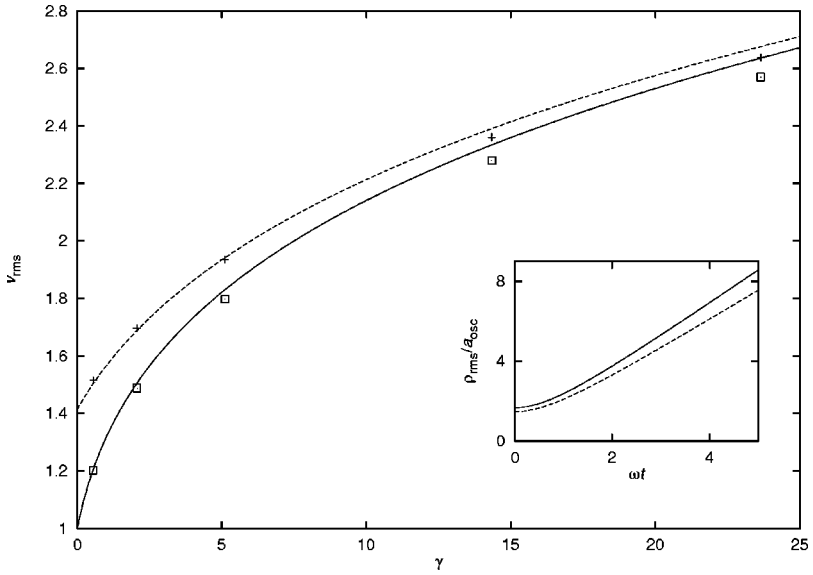

FIG. 1. Final root-mean-square velocity for a two-dimensional cloud in a free expansion as a function of the dimensionless coupling parameter $\gamma = \nu a$. The full line corresponds to the vortex state in the variational treatment and the dashed line to the ground state. Numerical solutions to the full time-dependent problem are indicated by crosses (vortex) and squares (ground state). The inset shows the time evolution of the rms radius in units $a_{\rm osc}$ for a vortex (full line) and a ground-state cloud (dashed line), for $\gamma = 2.07$.

When released from the trap, the wave function evolves according to the time-dependent Gross-Pitaevskii equation (1), which in dimensionless form is

$$i\frac{\partial f}{\partial t_1} = -\frac{1}{2}\left(\frac{\partial^2 f}{\partial \rho_1^2} + \frac{1}{\rho_1}\frac{\partial f}{\partial \rho_1} - \frac{1}{\rho_1^2} f\right) + \mu_1 f^3, \qquad (32)$$

where $t_1 = \omega t$. As before, the nonrotating case is obtained by dropping the centrifugal term in Eq. (32). This equation is integrated with respect to time using the Crank-Nicholson method [12], with the initial value given by the stationary wave functions obtained above. During the expansion, we compute the rms radius $\rho_{\rm rms}$ of the system. We find it to follow Eq. (30) to a good degree of accuracy. At large times $\rho_{\rm rms}$ does indeed grow linearly with time, and the final velocities for a few different values of γ are plotted in Fig. 1. The inset in Fig. 1 shows the evolution of $\rho_{\rm rms}$ for $\gamma = 2.07$, corresponding to the chemical potential $\mu = 3.54\hbar\omega$ for the vortex state and $\mu = 3.07\hbar\omega$ for the ground state.

The Crank-Nicholson method generates for each time step an explicit table of the real and imaginary parts of the wave function. On comparing the wave function at later times with the initial one, we have been able to find that the system does indeed undergo a nearly homologous expansion, i.e., the wave function does not change its shape, but only flattens out with time. We shall discuss the physical reasons for such behavior, in both two and three dimensions, at the end of Sec. III.

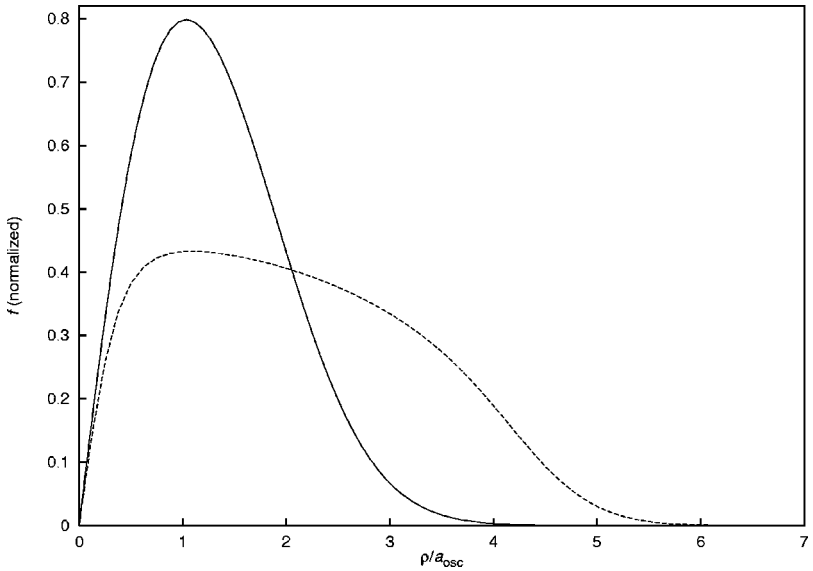

FIG. 2. Radial wave function [solution to the stationary Gross-Pitaevskii equation (31)] for the two-dimensional vortex state. The full line is the wave function with chemical potential $\mu_1 = 2.5$, and the dashed line corresponds to $\mu_1 = 10$. Note that the units on the y axis are such that the wave functions are normalized to unity, i.e., they are not the same as in Eq. (31).

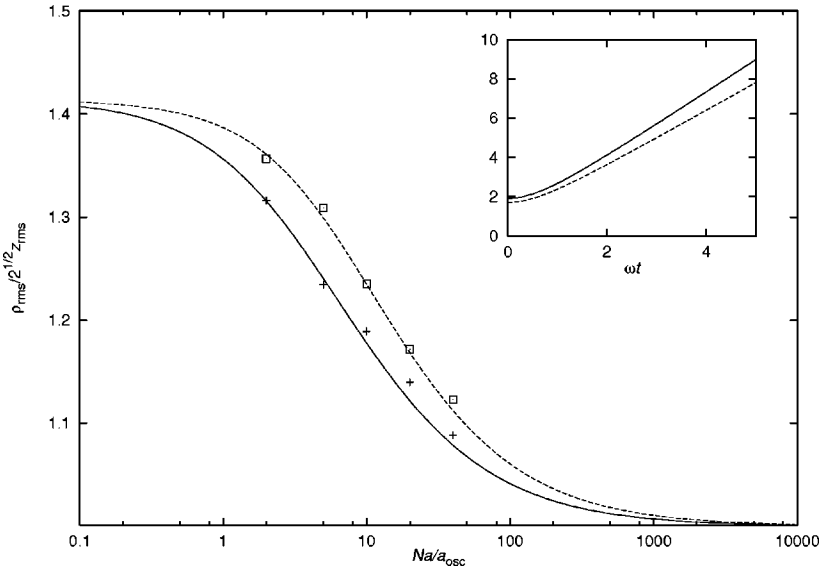

FIG. 3. Aspect ratio for a vortex system in an isotropic harmonic trap, before (dashed line) and after (full line) free expansion. The crosses and boxes are computed using the seminumerical scheme described in the text. The inset shows the expansion for a vortex in the radial (full line) and axial (dashed) directions, for $Na/a_{\rm osc}=20$.

III. VORTICES IN THREE DIMENSIONS

A. Simple variational approach

The clouds studied in experiments do not possess the translational invariance in one direction that was assumed in our calculations above. It is therefore of interest to explore the consequences of motion in the third direction on the development of a vortex state. Our discussion for the case of two dimensions may easily be generalized to three dimensions by employing the trial function

$$\psi(\rho,\phi,z,t)=\frac{N^{1/2}}{\pi^{3/4}Z^{1/2}R^2}\rho e^{i\phi}e^{-\rho^2/2R^2}e^{i\beta\rho^2/2}e^{-z^2/2Z^2}e^{i\gamma z^2/2}, \tag{33}$$

which describes a cloud undergoing expansion in both the radial and the z directions. The result is the two coupled equations

$$\ddot{R}=\frac{\hbar^2}{m^2R^3}+\frac{NU_0}{4(2\pi)^{3/2}m}\frac{1}{R^3Z} \tag{34}$$

and

$$\ddot{Z}=\frac{\hbar^2}{m^2Z^3}+\frac{NU_0}{2(2\pi)^{3/2}m}\frac{1}{R^2Z^2}. \tag{35}$$

The inset in Fig. 3 shows the results for the radial and axial expansion (in units $a_{\rm osc}$) for the case $Na/a_{\rm osc}=20$. For large times the velocity becomes constant, just as for the two-dimensional case. In Fig. 3 the aspect ratios of a cloud in an isotropic trap before and after expansion are shown as a function of the coupling parameter $Na/a_{\rm osc}$. The aspect ratio A is defined as the ratio between the root-mean-square distances $A=\rho_{\rm rms}/\sqrt{2}z_{\rm rms}$, where the factor of $\sqrt{2}$ takes into account the fact that there are two coordinates perpendicular to z. With this normalization, the aspect ratio is unity if the rms values of x, y, and z are equal. For the variational wave function we employ, one finds $\rho_{\rm rms}=\sqrt{2}R$ and $z_{\rm rms}=Z$, so the aspect ratio is simply R/Z. An aspect ratio differing from unity in an isotropic trap would be a clear signal for the presence of a vortex, but from Fig. 3 we find that there are deviations of the aspect ratio from unity of more than about 5% only for a number of particles less than $100a_{\rm osc}/a$.

One may ask whether it is possible to improve the signal for the presence of a vortex by changing the external potential. We have therefore computed the aspect ratio for a system in an anisotropic external potential $V(\vec{r})=\frac{1}{2}m\omega^2(\rho^2+\lambda^2z^2)$ as a function of the anisotropy parameter λ. An oblate trap corresponds to $\lambda>1$ and a prolate one to $\lambda<1$. The computation was carried out for $Na/\bar{a}_{\rm osc}=100$, where $\bar{a}_{\rm osc}=(\hbar/m\omega\lambda^{1/3})^{1/2}$ is the geometrical mean of the oscillator lengths in the spatial directions. Figure 4 displays the ratio of the aspect ratio of a vortex A_v to that of a nonrotating system A_n as a function of λ, both in the stationary case and after a long expansion time.

We see that for strong oblate anisotropy, the aspect ratio of a cloud expanding in a vortex state is initially close to that for the nonrotating state, but following expansion they are significantly different. For clouds expanding from a prolate trap, the ratio of aspect ratios for the vortex and the nonrotating state is enhanced both before and after expansion.

B. Core structure

In the variational calculations described above, the radial wave function is described in terms of a single length scale R, which determines both the size of the cloud and the size of the vortex core. This is unrealistic when particle interactions are important since the size of the core is then determined by the coherence length ξ, which is a function of the density of particles. To study the dynamics of the core we employ a more general trial wave function

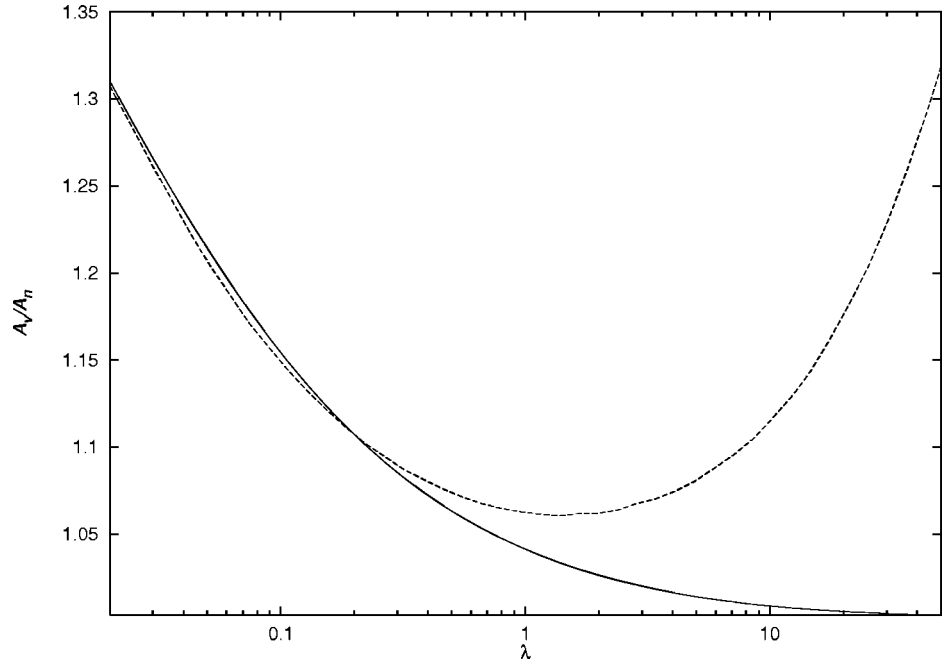

FIG. 4. Ratio of the aspect ratio for a cloud containing a vortex A_v to that for a nonrotating cloud A_n in the stationary case (full line) and after a long time of expansion (dashed line), plotted as a function of the trap anisotropy λ for $Na/\bar{a}_{\rm osc}=100$. We see that after expansion the difference in the aspect ratios is enhanced for a strong anisotropy, compared to an isotropic trap.

$$\psi(\rho,z,\phi,t)=\sqrt{N}f(\rho,t)\frac{1}{\pi^{1/4}Z^{1/2}}e^{-z^2/2Z^2+i\beta z^2}e^{i\phi}, \quad (36)$$

with the parameters β and Z depending on time. The radial wave function $f(\rho,t)$, which is normalized to unity, is to be determined variationally.

Minimizing the action obtained from the Lagrangian (20) with respect to Z, $\beta(t)$ and $f(\rho,t)$ yields the coupled set of equations

$$\beta(t)=\frac{m}{2\hbar}\frac{\dot{Z}}{Z}, \quad (37)$$

$$\ddot{Z}=\frac{\hbar^2}{m^2Z^3}+\frac{NU_0}{\sqrt{2\pi}mZ^2}\langle|f|^4\rangle, \quad (38)$$

$$i\hbar\frac{\partial f}{\partial t}=-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}-\frac{1}{\rho^2}\right)f + \left[\frac{\hbar^2}{2mZ^2}+\frac{NU_0}{4\sqrt{2\pi}Z}\langle|f|^4\rangle\right]f+\frac{NU_0}{\sqrt{2\pi}Z}|f|^2f, \quad (39)$$

where $\langle|f|^4\rangle\equiv\int d^2\rho|f(\rho)|^4$. These equations may now be solved numerically, at each time step simultaneously taking a Crank-Nicholson step for Eq. (39) and a Runge-Kutta step for Eq. (38). To find the initial conditions, we have to consider the static case with an external potential. We choose to consider the expansion of a cloud initially confined in an isotropic potential. The relevant equations are obtained from Eqs. (37)–(39) for a potential $V(\vec{r})=\frac{1}{2}m\omega^2(\rho^2+z^2)$, by adding to the right-hand side of Eq. (39) the term $\frac{1}{2}m\omega^2\rho^2f$ and to the left-hand side of Eq. (38) the term ω^2Z. Note that when the particle interaction vanishes, the wave function is that of the lowest state with unit angular momentum of a particle in the harmonic-oscillator potential. This corresponds to putting $Z=a_{\rm osc}$ and $f(\rho)=\pi^{-1/2}a_{\rm osc}^{-2}\rho e^{-\rho^2/2a_{\rm osc}^2}$ in our variational trial function. Since the Crank-Nicholson method preserves the normalization of the wave function and f is to be normalized to unity for any value of the coupling NU_0, we may produce any stationary wave function of the system by starting out with this one-particle wave function and then integrating the static equations of motion discussed above with respect to time, while adiabatically increasing the coupling constant up to the desired value [13]. The wave functions we obtain by this method are in good semiquantitative agreement with those obtained by Dalfovo and Stringari [4] from numerical solutions of the three-dimensional time-independent Gross-Pitaevskii equation.

Regarding the time development of the outer radius of the cloud, we find that the results of the seminumerical scheme closely follow those found using the simpler approach, Eqs. (34) and (35), especially for weak coupling, as the data points in Fig. 3 show. We have also considered the time dependence of the core size, which we characterize by the radius ρ_i at which the particle density first reaches e^{-1} times its maximum value. Results for $\rho_i/\rho_{\rm rms}$ as a function of time are shown in Fig. 5. If the radial wave function were of the form of the first excited oscillator state $\sim\rho e^{-\alpha\rho^2}$, where α is a constant, the ratio $\rho_i/\rho_{\rm rms}$ would be approximately 0.282. This is indeed what we find for $Na/a_{\rm osc}\lesssim 1$. For larger values of $Na/a_{\rm osc}$ the inner radius is a smaller fraction of $\rho_{\rm rms}$ initially, but the ratio increases with time. The qualitative behavior of the ratio $\rho_i/\rho_{\rm rms}$ with time for $Na/a_{\rm osc}\gg 1$ may be understood in terms of two sorts of processes. Initially the

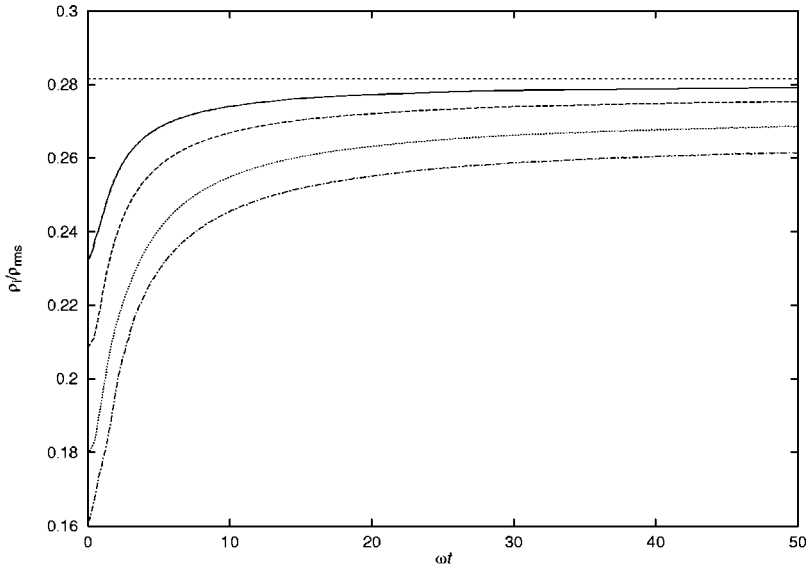

FIG. 5. Evolution in time of the ratio of the vortex core radius to the total system radius for $Na/a_{osc}=5$ (full line), 10 (dashed line), 20 (dotted line), and 30 (dot-dashed line), respectively. The straight line shows the value $\rho_i/\rho_{rms}=0.282$, corresponding to the free-particle limit $Na/a_{osc}=0$.

characteristic time for adjustment of the core size of the vortex $\tau_{ad}\sim\hbar/nU_0$ is small compared to the expansion time $\tau_{ex}\sim R/v_s$, where v_s is the sound velocity of the cloud at $t=0$. Under those conditions the core of the vortex can adjust essentially instantaneously to the local density and thus one expects ρ_i to scale as the coherence length $\rho_i\propto\xi_0\propto(R^3/Na)^{1/2}$ or $\xi_0\sim R(R/Na)^{1/2}$. We take the density n to be $N/R(t)^3$ since any anisotropy is quite unimportant here as long it is of order unity. This behavior ceases at a decoupling time t_d at which $\tau_{ad}=\tau_{ex}$. In three dimensions, this means

$$R(t)\sim R(0)^{3/4}(Na)^{1/4},$$

which gives us

$$\omega t_d\sim(Na/a_{osc})^{1/5}.$$

Here we have assumed that $R(t)\simeq v_s t$, which holds when $\omega t\gg 1$. A more careful treatment would result in a somewhat larger decoupling time. In two dimensions, $\tau_{ad}=\tau_{ex}$ implies that

$$\frac{R(t)}{R(0)}\sim(\nu a)^{1/2}.$$

Since $R(t)=R(0)\omega t$ for ωt large, the decoupling time is given by

$$\omega t_d\sim(\nu a)^{1/2}$$

in two dimensions. At larger times the potential energy will play essentially no role and the evolution of the cloud will be the same as for free particles. To the extent that the cloud is expanding homologously at the decoupling time, we expect that ρ_i/ρ_{rms} will remain constant and equal to its value at the decoupling time. The results of the numerical calculations exhibited in Fig. 5 are consistent with the assumption of homologous expansion at larger times. We conclude that in two dimensions the expansion is homologous to a good approximation during all of the expansion, while in three dimensions, the relative size of the core grows until the decoupling time, whereafter it approaches a constant value.

Our results indicate that by allowing a cloud containing a vortex to expand freely, the structure will increase in size more rapidly than the size of the cloud. In this way one may thus facilitate optical detection of the structure associated with the vortex core.

IV. CONCLUSIONS

In summary, we have described the time evolution of a freely expanding Bose-Einstein-condensed atomic cloud with a singly quantized vortex in two and three dimensions using both analytical approximations and direct numerical calculation. We have found that simple variational estimates describe very well the time evolution of the radius of the cloud.

In the limit of weak coupling, the aspect ratio of the cloud with a vortex differs from that of the nonrotating ground state both before and after expansion. This effect may be enhanced by using an anisotropic trapping potential.

In two dimensions, the cloud is seen to undergo a nearly homologous expansion, while in three dimensions, the vortex core expands faster than the size of the cloud during the initial stage of expansion, thus making it easier to detect experimentally. After a decoupling time t_d, estimated to be $t_d=\omega^{-1}(Na/a_{osc})^{1/5}$ for expansion in three dimensions, the size of the core will increase linearly with time.

We conclude that a vortex is most easily observed experimentally in the weak coupling regime, i.e., for small values of the parameter Na/a_{osc}, which is attained either for small numbers of particles or for shallow traps since the relative importance of the vortex to the energy, system size, and velocity of expansion is larger in this case.

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