Amplifying an atomic wave signal using a Bose-Einstein condensate

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We investigate a matter wave amplifier model which is capable of enhancing an external input atomic wave signal. This amplifier makes use of a cavity QED interaction with a trapped Bose-Einstein condensate. We examine the atomic wave equations and describe the gain performance. [S1050-2947(98)04412-6]

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Bose-Einstein condensation (BEC) of dilute atomic gases [1–3] has become an important source of macroscopically coherent matter. Exploring new applications of matter waves is a timely topic for current research. In this paper we address an interesting question: If information is embedded in the number of atoms in a weak atomic wave, how do we make an amplifier to enhance the signal? The concept of an amplifier can generally be defined by the relation

$$N_{\rm out}=GN_{\rm in}+N_n, \tag{1}$$

where $N_{\rm in}$ and $N_{\rm out}$ are the atom numbers of input and output waves. The parameter G is the gain coefficient, and N_n describes the possible noise contributions. Although there have been many investigations of matter wave amplification [4–6], most of these studies focus on the buildup of atomic waves in a trap [5] and the output properties [6]. The problems of achieving Eq. (1) in the context of signal amplification have not been fully explored [7].

In this paper we report an interaction in which a BEC coupled with an optical cavity field can realize Eq. (1) as a matter wave amplifier. Given an input atomic wave as a signal, the amplifier can produce an output atomic wave proportional to the input number (apart from the noise). The scheme of our model is shown in Fig. 1. We consider a Bose-Einstein condensate trapped in an optical cavity. Each atom in the condensate has the same internal state $|a\rangle$ responsible for the trapping. An input atomic wave with a different internal state $|b\rangle$ propagates through the condensate. We assume that the state $|b\rangle$ does not interact with the trap. The condensate and the input wave are Raman coupled by an external laser with a frequency ω_l, and an optical cavity field with a frequency ω_c [8]. We assume that the external laser field is well described by a classical field in the form of a plane wave with a constant amplitude, and we treat the cavity field as a fully quantized field. We note that the quantization of the cavity field is not crucial to our treatment. However, a quantized description allows for a more complete analysis in that the effects of quantum noise can be naturally included. In this paper we also assume that the condensate is sufficiently dilute so that effects of collisions between the input atoms and condensate atoms can be ignored.

As shown in Fig. 1, an atom in the condensate can change its internal state from $|a\rangle$ to $|b\rangle$ by absorbing a laser photon and emitting a photon into the optical cavity [9]. Using the second quantized annihilation atomic field operators $\hat{\Psi}_a$ for the BEC and $\hat{\Psi}_b$ for the input wave, the Hamiltonian of the system is given by

$$\mathcal{H}=\mathcal{H}_{\rm BEC}+\hbar\omega_c\hat{C}^\dagger\hat{C}+\int \hat{\Psi}_b^\dagger(\vec{x})\left[-\frac{\hbar^2}{2M}\nabla^2\right]\times\hat{\Psi}_b(\vec{x})d^3x+\mathcal{H}_{\rm int}, \tag{2}$$

where $\mathcal{H}_{\rm BEC}$ denotes the (bare) Hamiltonian of the condensate in the absence of the optical fields. In writing Eq. (2), $\hat{C}$ and $\hat{C}^\dagger$ are annihilation and creation operators of the cavity field, and we have assumed that there is only one cavity mode resonantly involved in the Raman interaction. The kinetic energy of the input wave is given by the third term of Eq. (2) where M is the atomic mass. The matter fields are coupled by the photon-atom interaction $\mathcal{H}_{\rm int}$,

$$\mathcal{H}_{\rm int}=\hbar g e^{-i\omega_l t}\hat{C}^\dagger\int \hat{\Psi}_b^\dagger(\vec{x})\hat{\Psi}_a(\vec{x})u(\vec{x})e^{i\vec{k}_l\cdot\vec{x}}d^3x+\text{H.c.}, \tag{3}$$

where g is the coupling parameter controlled by the amplitude of the external laser (with wave vector $\vec{k}_l$), and $u(\vec{x})$ is

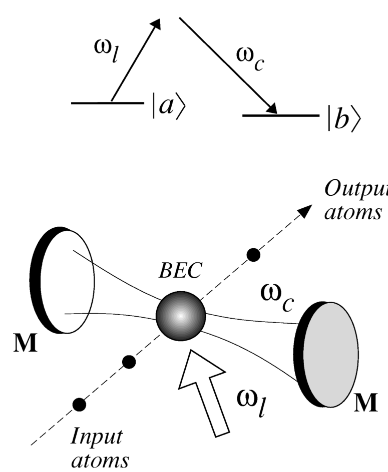

FIG. 1. A scheme of the matter wave amplifier. The input atoms are prepared in state $|b\rangle$, and the BEC atoms are in state $|a\rangle$. The atomic transitions are shown at the top of the figure.

the mode function of the optical cavity. Interaction (3) can be derived by applying the usual adiabatic elimination procedure commonly used in single-atom cavity QED systems [10]. In this work the frequency (or the ac Stark) shifts terms are omitted.

We can identify the interaction Hamiltonian (3) as a form of parametric down conversion: An atom in the BEC is *down converted* into a cavity photon, and an atom with the internal state $|b\rangle$ [7]. In the language of nonlinear optics, the BEC acts as a pump, the input atomic wave acts as a signal, and the optical cavity field acts as an idler. We emphasize that it is the coherence feature of BEC that makes such an analogy feasible, as does the coherent pump field in photonic systems. However, there are some differences which arise for atomic systems. The most apparent one is that atomic motion typically occurs on a much longer time scale than optical time scales. Since our system consists of both photons and atoms, there are well separated time scales. For example, the storage time of cavity photons can be much shorter than the atomic transit time. In the following we shall derive the atomic wave equation and examine the possibility of amplification.

With Hamiltonian (2) and the commutation relations $[\hat{\Psi}_\mu(\vec{x},t),\hat{\Psi}_\nu(\vec{x}',t)]=0$ and $[\hat{\Psi}_\mu(\vec{x},t),\hat{\Psi}^\dagger_\nu(\vec{x}',t)]=\delta_{\mu\nu}\delta(\vec{x}-\vec{x}')$, the Heisenberg equations of motion for $\hat{\Psi}_b$ and $\hat{C}$ are obtained:

$$i\frac{\partial\hat{\Psi}_b(\vec{x},t)}{\partial t}=-\frac{\hbar\nabla^2}{2M}\hat{\Psi}_b(\vec{x},t)+g\hat{C}^\dagger\hat{\Psi}_a(\vec{x},t)u(\vec{x})e^{i\vec{k}_l\cdot\vec{x}}e^{-i\omega_l t}, \tag{4}$$

$$i\frac{d\hat{C}}{dt}=(\omega_c-i\kappa)\hat{C}+ge^{-i\omega_l t}\int\hat{\Psi}^\dagger_b(\vec{x},t)\times\hat{\Psi}_a(\vec{x},t)u(\vec{x})e^{i\vec{k}_l\cdot\vec{x}}d^3x+\hat{f}(t). \tag{5}$$

Here κ is the decay rate of the optical cavity field, and $\hat{f}(t)$ is the Langevin noise operator. In this paper we assume that the correlation functions of $\hat{f}(t)$ are governed by a zero-temperature bath, so that

$$\langle\hat{f}^\dagger(t)\hat{f}(t')\rangle=0,\quad\langle\hat{f}(t)\hat{f}^\dagger(t')\rangle=2\kappa\delta(t-t'). \tag{6}$$

Note that we have not shown the equation of motion for the condensate field operator $\hat{\Psi}_a$. This is because we shall restrict the system to the weakly interacting regime where the change of $\hat{\Psi}_a$ can be neglected. In this regime, we may replace $\hat{\Psi}_a$ by its mean-field value

$$\hat{\Psi}_a(\vec{x},t)\approx\sqrt{N}\phi(\vec{x})e^{-i\mu t}. \tag{7}$$

Here N is the number of atoms in the condensate, $\phi(\vec{x})$ is the condensate wave function determined by the mean-field theory, and $\hbar\mu$ is the mean-field energy of the condensate including the energy of the internal level. Approximation (7) is equivalent to the treatment of the pump field as a nondepleted classical coherent field in parametric down conversion.

Given that the input atomic wave has an average momentum $\hbar\vec{k}_0$, it is convenient to define the slowly varying variables

$$\hat{\psi}_b(\vec{x},t)\equiv\hat{\Psi}_b(\vec{x},t)e^{i\varepsilon t}, \tag{8}$$

$$\hat{c}(t)\equiv\hat{C}(t)e^{i\omega_c t}, \tag{9}$$

where $\varepsilon=\hbar k_0^2/2M$. By tuning the fields frequencies at the Raman resonance

$$\omega_c-\omega_l+\varepsilon-\mu=0, \tag{10}$$

Eqs. (4) and (5) can be reduced to

$$i\frac{\partial\hat{\psi}_b(\vec{x},t)}{\partial t}=-\left(\frac{\hbar\nabla^2}{2M}+\varepsilon\right)\hat{\psi}_b(\vec{x},t)+g\sqrt{N}\hat{c}^\dagger\eta(\vec{x}), \tag{11}$$

$$i\frac{d\hat{c}}{dt}=-i\kappa\hat{c}+g\sqrt{N}\int\hat{\psi}^\dagger_b(\vec{x},t)\eta(\vec{x})d^3x+\hat{f}(t)e^{i\omega_c t}. \tag{12}$$

Here

$$\eta(\vec{x})\equiv\phi(\vec{x})u(\vec{x})e^{i\vec{k}_l\cdot\vec{x}} \tag{13}$$

can be regarded as an effective mode function which is a product of the BEC wave function and the mode functions of the optical fields.

The coupled operator equations (11) and (12) determine the dynamics of the fields under the nondepleted assumption (7). It is obvious that if $\hat{c}$ can be treated as a constant number (i.e., if the cavity field is also a constant classical field), then atoms can be added to the output mode according to the solution of Eq. (11). However, this trival case does not meet our goal of signal amplification. This is because in this case the gain in the output does not depend on the input at all. Atoms are added to the output whether the input is present or not. In order to achieve signal amplification, the cavity field has to be a dynamical variable. Here we consider a simple situation where the cavity is initially in the vacuum state. We assume that the cavity decay rate κ is the fastest time scale of the system such that $\kappa\gg g\sqrt{N}$. In this ''bad-cavity'' limit, the number of cavity photons is very small. By using the adiabatic approximation, we have

$$\hat{c}(t)\approx-i\frac{g\sqrt{N}}{\kappa}\int\hat{\psi}^\dagger_b(\vec{x},t)\eta(\vec{x})d^3x-i\frac{\hat{f}(t)e^{i\omega_c t}}{\kappa}. \tag{14}$$

We can eliminate $\hat{c}$ and $\hat{c}^\dagger$ and obtain a wave equation for $\hat{\psi}_b(\vec{x},t)$:

$$i\frac{\partial\hat{\psi}_b(\vec{x},t)}{\partial t}=-\left(\frac{\hbar\nabla^2}{2M}+\varepsilon\right)\hat{\psi}_b(\vec{x},t)+i\frac{|g|^2N}{\kappa}\eta(\vec{x})\int\hat{\psi}_b(\vec{x}',t)\eta^*(\vec{x}')d^3x'+i\frac{g\sqrt{N}}{\kappa}\hat{f}^\dagger(t)e^{-i\omega_c t}\eta(\vec{x}). \tag{15}$$

We point out that apart from the noise part (the third term), the same equation (15) can also be derived for weak classical (but dynamical) cavity fields. Our quantized field treatment allows us to include quantum noises in the wave equation naturally.

Equation (15) is the basic equation describing an atomic wave propagating through a BEC under the cavity-assisted Raman interaction. The integral term in the wave equation (15) indicates a ''nonlocal'' dependence. This is because we have assumed that there is only one cavity mode effectively involved in the interaction, and therefore the effects of light propagation through the sample are neglected.

The general solution of the wave equation (15) can be obtained by solving the propagator $U(\vec{x},t;\vec{x}',0)$, which is defined by

$$i\frac{\partial U(\vec{x},t;\vec{x}',0)}{\partial t}=-\left(\frac{\hbar\nabla^2}{2M}+\varepsilon\right)U(\vec{x},t;\vec{x}',0)+i\frac{|g|^2N}{\kappa}\eta(\vec{x})\int \eta^*(\vec{x}'')U(\vec{x}'',t,x',0)d^3x'', \tag{16}$$

and the condition $U(\vec{x},0;\vec{x}',0)=\delta(\vec{x}-\vec{x}')$. Given an initial field $\hat{\psi}_b(\vec{x},0)$, the field operator $\hat{\psi}_b(\vec{x},t)$ at a later time t is given by

$$\hat{\psi}_b(\vec{x},t)=\hat{\psi}_b^{(s)}(\vec{x},t)+\hat{\psi}_b^{(n)}(\vec{x},t), \tag{17}$$

where $\hat{\psi}_b^{(s)}(\vec{x},t)$ is the signal part and $\hat{\psi}_b^{(n)}(\vec{x},t)$ is the noise part:

$$\hat{\psi}_b^{(s)}(\vec{x},t)=\int U(\vec{x},t;\vec{x}',0)\hat{\psi}_b(\vec{x}',0)d^3x', \tag{18}$$

$$\hat{\psi}_b^{(n)}(\vec{x},t)=\frac{g\sqrt{N}}{\kappa}\int_0^t dt'\hat{f}^\dagger(t')e^{-i\omega_c t'}\times\int d^3x' U(\vec{x},t-t';\vec{x}',0)\eta(\vec{x}'). \tag{19}$$

Such a separation allows us to distinguish the contribution of noise which does not depend on the input $\hat{\psi}_b(\vec{x},0)$.

Let us now consider that the input wave is initially in the form of a wave packet. We write

$$\hat{\psi}_b(\vec{x},0)=\hat{b}_s w(\vec{x},0)+\hat{\xi}(\vec{x},0), \tag{20}$$

where $w(\vec{x},0)$ is the initial wave-packet wave function, $\hat{b}_s$ is the corresponding annihilation operator, and $\hat{\xi}(\vec{x},0)$ describes the contributions of modes that are orthogonal to $w(\vec{x},0)$. Since input atoms are already assumed to be in the wave packet, we have $\langle\hat{\xi}^\dagger(\vec{x},0)\hat{\xi}(\vec{x},0)\rangle=0$ and $\langle b_s^\dagger\hat{\xi}(\vec{x},0)\rangle=0$. Hence $\hat{\xi}(\vec{x},0)$ does not make contributions for normally ordered observables. To show the amplifying action, we look at the number of atoms in the signal part:

$$n^{(s)}(t)\equiv\int d^3x\langle\hat{\psi}_b^{(s)\dagger}(\vec{x},t)\hat{\psi}_b^{(s)}(\vec{x},t)\rangle. \tag{21}$$

It can be shown that the rate of change of $n^{(s)}$ is always non-negative,

$$\frac{dn^{(s)}}{dt}=\frac{2|g|^2Nn_b^{(s)}(0)}{\kappa}\left|\int \eta^*(\vec{x})w(\vec{x},t)d^3x\right|^2\geqslant 0, \tag{22}$$

i.e., amplification occurs. Here $n^{(s)}(0)\equiv\langle\hat{b}_s^\dagger\hat{b}_s\rangle$ is the initial number of atoms and $w(\vec{x},t)$ is the wave packet at the time t defined by

$$w(\vec{x},t)=\int U(\vec{x},t;\vec{x}',0)w(\vec{x}',0)d^3x'. \tag{23}$$

The fact that $n^s(t)$ is proportional to $n_b^{(s)}(0)$ is indeed what we demand for the amplifier [see Eq. (1)]. The gain coefficient G is given by

$$G=1+\frac{2|g|^2N}{\kappa}\int dt\left|\int \eta^*(\vec{x})w(\vec{x},t)d^3x\right|^2. \tag{24}$$

According to Eq. (18), those atoms generated in the signal part propagate in the form of $w(\vec{x},t)$. The integral in Eq. (22) can be interpreted as a Frank-Condon factor because it measures the effective overlap between the wave packet and the condensate. Once the wave packet leaves the condensate, the integral vanishes and the rate $\dot{n}^{(s)}$ becomes zero. For the atoms created in the noise part [Eq. (19)], the number density is given by

$$\langle\hat{\psi}_b^{(n)\dagger}(\vec{x},t)\hat{\psi}_b^{(n)}(\vec{x},t)\rangle=\frac{2g^2N}{\kappa}\int_0^t dt'|\eta(\vec{x},t-t')|^2, \tag{25}$$

where

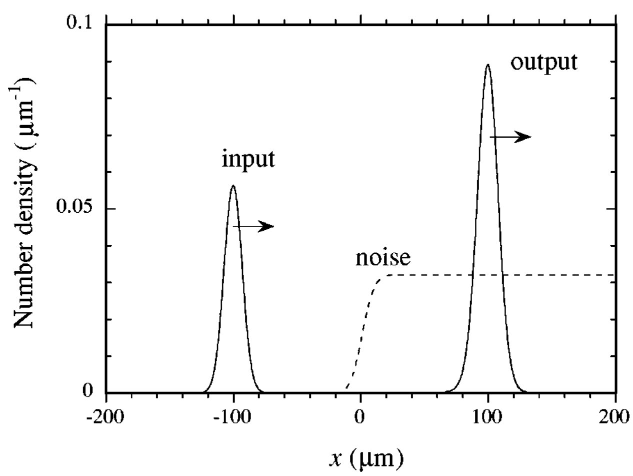

FIG. 2. An illustration of the amplification of wave packets in a one-dimensional system. The condensate is located at $x=0$, which has a width of 10 μm. The input atomic wave packet is initially located at $x=-100$ μm, with a width 10 μm. We show the shape of the wave packet when it reaches $x=+100$ μm. The dotted line is the number density due to the noise part. The average velocity of the input atom is 1.6 cm s^{-1}, and we use sodium atomic mass for this calculation.

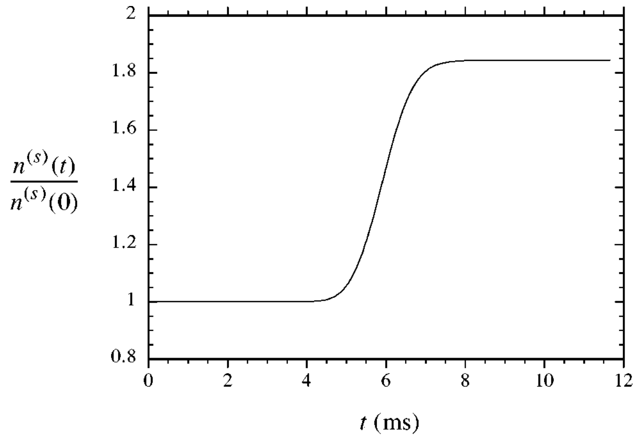

FIG. 3. The number of signal atoms as a function of time. Same parameters and initial conditions as in Fig. 2.

$$\eta(\vec{x},t) \equiv \int U(\vec{x},t;\vec{x}',0)\,\eta(\vec{x}')\,d^3x' \tag{26}$$

represents a wave function propagated from the effective mode $\eta(\vec{x}')$ through the propagator $U(\vec{x},t;\vec{x}',0)$.

The positive increasing rate [Eq. (22)] can be understood by the fact that the number of photons in the optical cavity is always very small. It is unlikely that an input atom can be absorbed by the condensate, because that requires an absorption of a cavity photon. On the other hand, the condensate can always convert its atoms into the output mode because photons are emitted (instead of being absorbed) during the atomic transition. This explains why we have a positive gain, and also why a cavity with a large leaking rate is useful here. We remark that the similar gain effect should also be expected in the free space. However, the use of an optical cavity provides control of the mode function, the instrinsic atom-field coupling strength, and the leaking rate. For example, a strong atom-field coupling can be achieved as in cavity QED systems [11]. In addition, since photons are emitted into the cavity mode, one can efficiently monitor the presence of input atoms by detecting the transmitted photons.

To illustrate the dynamics we solve Eq. (15) numerically for the number density $\langle \hat{\psi}_b^\dagger(x,t)\hat{\psi}_b(x,t)\rangle$. Since the propagation of the input and output atomic waves is essentially one dimensional, we consider a one-dimensional system in which the function $\eta(x)$ is modeled by $\eta(x) = \pi^{-1/4}\sigma^{-1/2}e^{-(x^2/2\sigma^2)}\sin k_c x$, where k_c is wave number associated with the standing-wave cavity mode. We choose the width $\sigma = 10$ μm, $g = 1$ KHz, $\kappa = 100$ MHz, and the number of atoms in the condensate is 2×10^4. The incident signal is a single atom Gaussian wave packet with $\hbar k_0 = \hbar k_l + \hbar k_c$ which matches the momentum conservation. In Fig. 2, we show the wave packet before and after passing through the BEC. The amplification is apparent because of the larger size of the output wavepacket. The number density of the noise part is also shown (dotted line). We see that the wave packet can be significantly higher than the noise level. If there are no input atoms, the noise part is the only contribution. The total number of atoms in the signal part is plotted in Fig. 3 as a function of time. In this example the final number of atoms is about 1.8, which is about twice the initial number. We point out that the amplifier is not limited to single-atom input, coherent sources such as atom lasers can be used. In fact a much better signal-to-noise ratio can be achieved when the number of atoms in the input; wave packet is more than one.

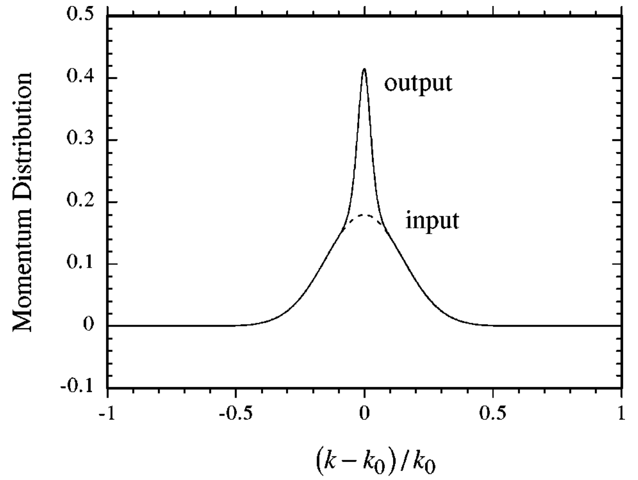

FIG. 4. Momentum distribution of the output wave packet (solid line) and input wave packet (dotted line). The parameters are the same as in Fig. 2, except that the input wave packet has a width of 2 μm. The areas under the curves represent the numbers of atoms.

As indicated in Eq. (22), the gain of atoms is governed by a Frank-Condon factor. Therefore, amplification can only be significant for those input waves that match the momentum conservation. Since the condensate considered here has a finite size, gain can be expected even if there are uncertainties in the momentum of input wave packet. This is more precisely measured by the Fourier spectrum of $\eta(\vec{x})$:

$$\tilde{\eta}(\vec{p}) \equiv \int \eta(\vec{x})e^{i\vec{p}\cdot\vec{x}/\hbar}d^3x. \tag{27}$$

The width of $\tilde{\eta}(\vec{p})$ determines the range of input momentum that can be amplified efficiently. We illustrate this by looking at the case when the input wave packet consists of a broad-band momentum spectrum (i.e., a narrow wave packet in position space). In Fig. 4 we show the momentum distributions of the input and output wave packets. We see that two distributions are basically the same except for a finite range of momentum where the output shows a sharp peak significantly above the input distribution. We note that the width of the peak is the same as the width of $\tilde{\eta}(\vec{p})$, which is not surprising given that the areas under the curves represent the numbers of atoms. In other words, Fig. 3 demonstrates that only those input momenta within the width of $\tilde{\eta}(\vec{p})$ (which can be interpreted as the bandwidth of the amplifier) interact with the condensate and are amplified.

In conclusion, we have described a matter wave amplifier which makes use of cavity-assisted Raman interaction with a trapped Bose-Einstein condensate. We have derived a wave equation in the bad cavity limit, and discussed the basic rate equations. In particular, we have considered the case of an input wave packet, and have shown that the input and output numbers follow a linear gain relationship. The question about the coherence properties of the output field is an important subject for further investigations. This work applies cavity QED technique to BEC systems, and more intriguing ideas should be expected in this interesting area [12,13].

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