Ionization of Rydberg atoms by Coriolis forces

T. Uzer,[1] Ernestine Lee,[2] and David Farrelly[2]
[1]School of Physics, Georgia Institute of Technology, Atlanta, Georgia 30332-0430
[2]Department of Chemistry and Biochemistry, Utah State University, Logan, Utah 84322-0300
(Received 3 March 1998)

When viewed from a rotating frame the ionization of a hydrogen atom in a circularly polarized microwave field is largely governed by Coriolis forces. Although simulations show that the ionization mechanism is chaotic, we find that ionization is governed by a *linear* resonance. Because the electron is fed into the resonance by chaotic scattering from the core the overall ionization process, when it occurs, is chaotic. Suppression of ionization by an applied magnetic field is shown to be possible: it is due to detuning of the linear resonance by the field. [S1050-2947(98)05112-9]

PACS number(s): 32.80.Rm, 03.20.+i, 05.45.+b, 32.60.+i

I. INTRODUCTION

This paper addresses two basic questions concerning the ionization of a hydrogen atom interacting with combinations of external electric and magnetic fields: when can the atom ionize and, if it can, what is the mechanism? On the surface the answer to the first question might seem, in principle, obvious. As we will show, however, the issue of how a hydrogen atom will ionize when subjected to certain combinations of electric and magnetic fields is quite subtle due to the presence of velocity-dependent Coriolis (or Coriolis-like) forces. The second question is also complicated and involves two aspects; (i) the mechanism by which an electron enters the ionization channel—in general this might involve a transition to chaos—and (ii) the process that eventually sweeps the electron out of the atom. While a vast amount of experimental and theoretical effort has been directed to studying the nonlinear dynamics of order-chaos transitions in Rydberg atoms [1,2] the ultimate act of ionization itself has been neglected.

Experiments as well as simulations of the hydrogen atom (i) in crossed static electric and magnetic fields (the ''crossed-fields problem'') and (ii) in a circularly polarized microwave (CPM) field show clearly that ionization can be induced by the fields [3]. The mechanism is significantly different from ionization in a linearly polarized microwave field [4] and involves a chaotic sequence of collisions between the electron and either the core or a centrifugal barrier that surrounds the core [5]. These findings differ from those of Zakrzewski *et al.* [6] who concluded that ionization does not have to proceed through interactions with the core or nucleus. In Ref. [5] we showed that this difference can be traced to issues of state preparation. We also find that in the CPM problem the addition of a magnetic field perpendicular to the plane of polarization completely suppresses ionization in the plane of the electric field. This result is quite unexpected because the addition of a magnetic field to the static Stark problem (i.e., the crossed-fields problem) destroys the integrability of the system.

We will show that in these systems an important ionization channel is one in which the electron ionizes through a *linear* resonance. Quite spectacularly, detuning from this resonance can lead to completely bound motion *along the electric field direction*. This happens for a hydrogen atom in a circularly polarized microwave field when a magnetic field is added perpendicular to the plane of polarization. Ionization, when it occurs, is exclusively along the *magnetic* field direction, i.e., it is orthogonal to the plane containing the electric field vector.

While our results are presented in the context of the ionization of Rydberg atoms, the Hamiltonian and the resulting dynamics are very similar to those for dust particles inhabiting planetary rings such as the ethereal rings of Jupiter and Neptune [7,8]. There, in addition to gravitational forces, dust grains are subjected to solar radiation pressure and magnetic fields and, after moving to a rotating frame, also to Coriolis forces. In fact, the CPM Hamiltonian is actually identical to one used by Mignard and later Deprit [9] to study the dynamics of orbiting dust subject to solar radiation pressure. Thus our findings might prove helpful in understanding the escape of dust particles from planetary rings.

The Hamiltonians for the crossed-fields problem [10] and the hydrogen atom in a CPM field in a rotating frame are similar because they both contain a nonconserved velocity-dependent term: in the former case this term is a paramagnetic term which arises from the presence of the magnetic field while in the latter it is explicitly a Coriolis term due to being in a rotating frame. The Coriolis term destroys any qualitative similarity with the more familiar static Stark effect. Although the CPM problem has been the subject of significant theoretical and experimental efforts, its ionization dynamics are only partially understood [3]. This reflects the complexity of the electronic motions involved, and, consequently the objective of most recent studies of these systems has been to relate ionization to transitions to chaos *within* the atom [10]. We take an opposite, though complementary view, that is motivated by scattering theory: if one thinks of ionization as a ''half collision,'' then it is natural to consider the asymptotic behavior of the electron. This is the starting point of our study.

For background we mention previous work related to the problems at hand: Raithel *et al.* [11] have done much of the experimental work in the crossed-fields problem and have identified a class of quasi-Landau (QL) resonances in the spectra of rubidium Rydberg atoms. Similar to the original QL resonances observed by Garton and Tomkins [12], this set of resonances is associated with a rather small number of *planar* periodic orbits [13,14]. In the same system, Main and Wunner [15] have demonstrated that the problem exhibits the hallmarks of a chaotic scattering system during ionization. Similar conclusions apply to the CPM problem [5]. The interaction of Rydberg states with circularly and elliptically polarized fields has also become an active area of research: Gallagher and co-workers [16] and Koch and co-workers [3] have studied the dependence of ionization thresholds on the polarization of the microwave field. In turn, these experiments have stimulated a number of theoretical studies [17,18,6,5]. In a different context interest has resurfaced in this problem since it has been shown that it is possible to create nonstationary, nondispersive electronic wave packets in Rydberg atoms using CPM fields [19]. We have demonstrated that an applied magnetic field [20] can dramatically enhance the stabilization of these wave packets [8,21,22] To this end there have, in fact, been a number of previous attempts to create outer potential minima in Rydberg atoms in crossed electric and magnetic ($\mathbf{E}\times\mathbf{B}$) fields [23] but in most cases these suggestions have been complicated by the existence of a velocity-dependent paramagnetic term in the Hamiltonian. One approach has been to neglect the paramagnetic term altogether in the one-particle Hamiltonian in the symmetric gauge. This is clearly incorrect because the procedure is gauge dependent and does not reproduce the $\mathbf{E}\times\mathbf{B}$ drift of the electron (one exception is the one-dimensional model of an electron bound at a liquid helium surface for which there is a real outer potential minimum in crossed electric and magnetic fields [23] in the direction perpendicular to the He surface). For highly excited atomic systems there exists another possibility considered by Gorkov and Dzyaloshinskii [24]: The two-body atomic problem can be ''pseudoseparated'' and the Coulomb potential is centered far from the origin at which point there is, instead, an oscillator potential. The paramagnetic term can be neglected if the wave function is concentrated in the oscillator potential and does not overlap the Coulomb potential from which it is displaced by a large distance. This situation corresponds to two oppositely charged particles performing $\mathbf{E}\times\mathbf{B}$ drift without mutual interaction. Thus the potential model applies for ''ionized'' atoms in crossed fields for which it is natural that the relative motion takes place in an oscillatory potential, the extension of which is determined by the heavier particle. Incidentally, these considerations are apropos to two oppositely charged particles in a pure magnetic field when the particles each execute cyclotron motions, provided that the centers of the cyclotron motions are displaced from each other by more than the cyclotron radius of the heavier particle. This model fails, however, when the particles are strongly interacting, since in this case the paramagnetic term is no longer conserved, i.e., the model is only appropriate for ionized atoms in crossed fields. This paper is a member of a sequence of papers that have studied in detail the use of the zero-velocity surface in treating Rydberg electrons subjected to Coriolis-like forces [18,20,25–27,22,5,8].

The paper is organized as follows. Section II introduces the classical Hamiltonian for a hydrogen atom subjected to various combinations of electric and magnetic fields. The important concepts of the *surface of zero velocity* and the *rotational potential* are also introduced in this section. All of our analysis relies on the concept of the rotational potential which is better known in celestial mechanics than in atomic physics. Thus we devote some attention to explaining this key concept. The stability of the Hamiltonians introduced in Sec. II can conveniently be studied using the cranking model of nuclear physics which is discussed in Sec. III. Section IV connects ionization to the presence of a linear resonance and conclusions are in Sec. V.

II. CLASSICAL HAMILTONIAN

A good description of the procedure for obtaining the Hamiltonian for either neutral or charged *N*-body systems in electric and magnetic fields is given in a thorough review by Johnson *et al.* [28]. For neutral systems the center-of-mass dynamics may be affected by the relative motion (RM) but the converse is not possible. In this paper we concern ourselves, therefore, only with the RM for a neutral two-body system consisting of an electron (mass m_1) and a positively charged particle (mass m_2) subjected simultaneously to a static electric or CPM field (field strength F and frequency ω_f) and a perpendicular homogeneous magnetic field $\mathbf{B}=B\mathbf{e}_z$. For the vector potential we choose the symmetric gauge $\mathbf{A}=\frac{1}{2}\mathbf{B}\times\mathbf{r}$. The Hamiltonian (in appropriate atomic units [28]) is given by

$$H^{(a)}=\frac{1}{2}(\mathbf{p}-q\mathbf{A})^2+V(\mathbf{r},t)=\frac{p_x^2+p_y^2+p_z^2}{2}-\frac{1}{r}-\Delta\frac{\omega_c}{2}(x\,p_y-y\,p_x)+\frac{\omega_c^2}{8}(x^2+y^2)+F(x\cos\omega_f t+y\sin\omega_f t), \tag{1}$$

where ω_c is the cyclotron frequency (the choice of sign is determined by the direction of the magnetic field) and the parameter $\Delta=(1-\delta)/(1+\delta)$ where $\delta=m_1/m_2$. With $\omega_f=0$ this reduces to the Hamiltonian for the atomic crossed-fields system and also can be adapted to describe excitons in crossed electric and magnetic fields [29,30]. In the CPM problem the time dependence in Eq. (1) can be eliminated by going to a frame that rotates at the constant angular velocity ω_f. This leads to the Hamiltonian

$$K^{(a)}=\frac{p_x^2+p_y^2+p_z^2}{2}-\frac{1}{r}-\left(\omega_f\pm\Delta\frac{\omega_c}{2}\right)(x\,p_y-y\,p_x)+F\,x+\frac{\omega_c^2}{8}(x^2+y^2). \tag{2}$$

where $K^{(a)}$ is called the Jacobi constant [18]. To avoid a proliferation of symbols, Hamiltonians in a rotating frame will always be assigned the letter K but no special notation will be attached to coordinate systems. Further, it is also useful to scale the coordinates and momenta as follows: $\mathbf{r}'=\omega^{2/3}\mathbf{r},\mathbf{p}'=\omega^{-1/3}\mathbf{p}$ [5] where ω is the coefficient of the term $(x\,p_y-y\,p_x)$.

The ionization dynamics of the CPM problem have been extensively studied. Reference [5] provides a comprehensive discussion of the literature as well as a detailed study of the classical dynamics of ionization. Be that as it may, the addition of a magnetic field perpendicular to the plane of polarization has a surprising result. Figure 1 is a typical Poincaré surface of section for the CPM/magnetic field system com-

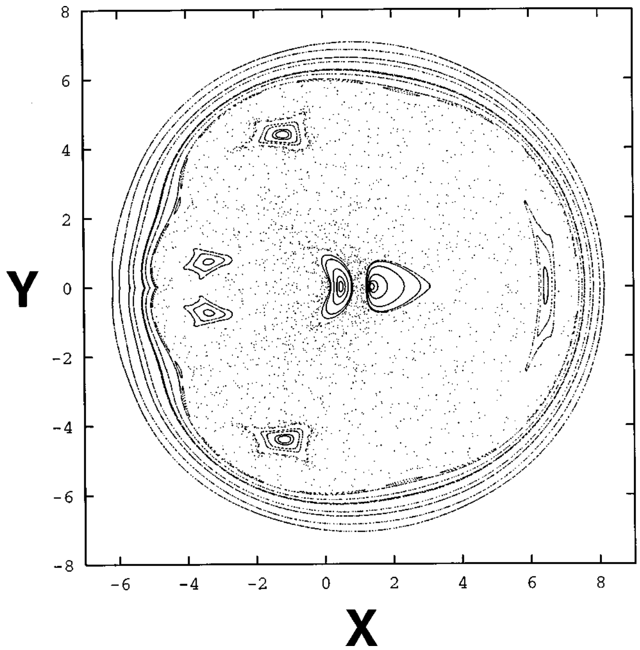

FIG. 1. Combined Poincaré surfaces of section for 50 trajectories in the x-y plane for a hydrogen atom in circularly polarized microwave and magnetic fields. Scaled atomic units are used: $\omega_s = \omega_c/\omega$ and $\varepsilon = F/\omega^{4/3}$, with $\varepsilon = 0.6$ and $\omega_s = 0.8$.

puted in the rotating frame for the planar limit ($z = p_z = 0$) of Hamiltonian (2). The chaotic region is generated by repeated collisions of the electron with the nucleus [5]. However, sets of Kolmogorov-Arnol'd-Moser (KAM) curves encircle the giant chaotic sea in the center of the figure, thereby preventing the escape of the electron to infinity. Note that the xy coordinate phase plane is being plotted and the KAM curves are, therefore, essentially the bounding surface of an effective potential. This is a slightly unorthodox but still legitimate way of defining a surface of section. This is done for the important insight it provides into the asymptotics of ionization. In the absence of a magnetic field the chaotic sea extends to infinity in the plane (many examples are provided in [5]).

At this point it is useful to introduce the concept of a surface of zero relative velocity to understand the motion of the electron in a rotating frame.

A. Rotating axes and the zero-velocity surface

A consequence of Newton's second law is that if a conservative force $\mathbf{P} = -\nabla V$ acts on a particle then motion with respect to axes that are rotating with constant angular velocity ω about the z axis will be determined by

$$\mathbf{P} = m[\ddot{\mathbf{r}} + \{2\omega\hat{\mathbf{z}}\times\dot{\mathbf{r}}\} + \{\omega^2\hat{\mathbf{z}}\times(\hat{\mathbf{z}}\times\mathbf{r})\}], \tag{3}$$

where the extra terms (as compared to Newton's second law in an inertial frame) in the first and second sets of curly braces are the Coriolis and centrifugal forces, respectively. The following relation has been used:

$$\frac{d\mathbf{r}}{dt} = \frac{\partial\mathbf{r}}{\partial t} + \omega\times\mathbf{r}, \tag{4}$$

which relates the rate of change of a vector $\mathbf{r}$ in a fixed frame of reference to that in a frame rotating with angular velocity ω. If $\mathbf{r}$ is decomposed into perpendicular and planar components

$$\mathbf{r} = z\hat{\mathbf{z}} + \boldsymbol{\rho} \tag{5}$$

then

$$\mathbf{P} = m[\ddot{\mathbf{r}} + 2\omega\hat{\mathbf{z}}\times\dot{\mathbf{r}} - \omega^2\boldsymbol{\rho}]. \tag{6}$$

Using the relation $\boldsymbol{\rho}\cdot\dot{\mathbf{r}} = \boldsymbol{\rho}\cdot\dot{\boldsymbol{\rho}}$ and forming the quantity $\mathbf{P}\cdot\dot{\mathbf{r}}$ we can calculate the work done in going from A to B,

$$W_{AB} = \int_A^B \mathbf{P}\cdot d\mathbf{r} = \frac{m}{2}(v_B^2 - v_A^2) - \frac{m\omega^2}{2}(\rho_B^2 - \rho_A^2), \tag{7}$$

where v_A and v_B are the mechanical velocities. For a conservative field $W_{AB} = V(A) - V(B)$ and so we obtain the result

$$K = \frac{1}{2}m\dot{r}^2 + V - \frac{1}{2}m\omega^2\rho^2, \tag{8}$$

where the last term is called the rotational potential [31]. It is apparent that the motion in the rotating frame is governed by the *modified* potential energy function

$$\Omega(x,y,z) = V - \frac{1}{2}m\omega^2(x^2+y^2), \tag{9}$$

which for fixed Ω is the locus of the surfaces of zero velocity [31]. In celestial mechanics the surface defined by $\Omega(x,y,z)$ is often termed the *surface of zero relative velocity* or simply the *zero-velocity surface* (ZVS). It is important to realize that Ω is not a potential even though it may share properties with a regular potential energy surface. Extensive discussion is provided in Refs. [18,20,25–27,22,5,8]. This analysis shows that even when the potential goes to infinity asymptotically (in the plane) the ZVS need not do so because of the presence of the rotational potential. It is clear that the presence of the rotational potential will modify the dynamics considerably and this is the subject of the rest of the paper.

B. Dynamics in the asymptotic limit $\rho\to\infty$

At this point we examine the effect of the rotational potential on the ionization dynamics and so consider the limit $\rho\to\infty$. For now we restrict our attention to the x-y plane: it is easy to see using Hamilton's equations that if the electron starts out in the plane with no component of momentum in the z direction then it must forever remain in that plane. Of course, in an experiment, an ensemble of initial conditions will not, in general, be restricted to this plane. However, by understanding this limit it is possible to glean considerable insight into ionization in the full three-dimensional problem. A critical observation is that while the Coulomb potential is extremely long ranged, in the presence of strong perturbations, it may easily be dominated even for relatively modest r values. Simulations support this expectation rather well.

As discussed in the preceding section, the question of whether the motion is bound or not in the ρ direction is somewhat complicated: the argument that the diamagnetic part of the potential rises to infinity asymptotically, thereby precluding ionization, is countered by the fact that in principle the nonconserved paramagnetic or Coriolis term can be negative. As noted above, the role of the rotational potential cannot be ignored. Depending on the particulars, we will show how the magnetic field, by tuning the rotational potential, can allow or prevent ionization in the plane.

To understand these findings we consider the dynamics of the electron in the plane and far from the nucleus: at some point one might imagine that an ionizing electron penetrates a boundary ''of no return'' beyond which the $1/r$ term in Eq. (2) can be neglected in comparison to the other fields. The resulting (planar) Hamiltonian in this limit is given by

$$K^{(b)}=\frac{p_x^2+p_y^2}{2}-\omega(x\,p_y-y\,p_x)+F\,x+\frac{\omega_c^2}{8}(x^2+y^2), \tag{10}$$

with $\omega=\omega_f\pm\Delta\omega_c/2$. Hamiltonian $K^{(b)}$ is similar to the cranked harmonic oscillator (CHO) model that has been used extensively in models of collective rotations in nuclear physics [32]. Our analysis proceeds by studying the stability of this oscillator which governs the asymptotic planar dynamics.

III. STABILITY OF THE CRANKED HARMONIC OSCILLATOR

The first step is to remove the linear term in x in the CHO by a canonical transformation

$$x=\xi+x_0,\quad p_x=p_\xi,\quad y=\eta,\quad p_y=p_\eta+p_0, \tag{11}$$

with $x_0=4F/(4\omega^2-\omega_c^2), p_0=4F\omega/(4\omega^2-\omega_c^2)$. Note that this transformation becomes singular when $\omega=\omega_c/2$ which occurs, e.g., when $\omega_f=0,\Delta=1$, i.e., the crossed-fields problem. We will return to this singular case shortly but, for now, merely note that this limit is exceptional. Further we restrict our study temporarily to $\Delta=1$. The Hamiltonian $K^{(b)}$ is taken by the transformation (11) into the more usual form of a cranked (isotropic) oscillator

$$\tilde{K}^{(b)}=\frac{p_\xi^2+p_\eta^2}{2}+\frac{\omega^2}{2}(a\,\xi^2+b\,\eta^2)-\omega(\xi p_\eta-\eta p_\xi), \tag{12}$$

with $a=b=\omega_c^2/4\omega^2$. Because the oscillator frequencies are equal the angular momentum $L_\zeta=(\xi p_\eta-\eta p_\xi)$ is conserved and one could, in principle, construct effective potentials at fixed values of L_ζ. For present purposes the dynamical stability of the system can best be understood by diagonalizing the problem using the canonical transformation

$$\begin{aligned}\xi'&=A\xi+Bp_\eta,\quad \eta'=A\eta+Bp_\xi,\\ p_{\xi'}&=p_\xi+C\eta,\quad p_{\eta'}=p_\eta+C\xi,\end{aligned} \tag{13}$$

with $A-BC=1$. The transformation coefficients A, B, and C turn out to be

$$A=\frac{[S+\omega^2(b-a)]}{2S},$$

$$B=\frac{2\omega}{S},$$

$$C=\frac{[\omega^2(b-a)-S]}{4\omega},$$

with $S=\mathrm{sgn}(\sqrt{b}-\sqrt{a})\,\omega^2\sqrt{(b-a)^2+8(b+a)}$.

In this way $\tilde{K}^{(b)}$ is reduced to the separable form

$$\tilde{K}^{(b')}=\frac{1}{2m_{\xi'}}p_{\xi'}^2+\frac{1}{2}m_{\xi'}\Omega_{\xi'}^2\xi'^2+\frac{1}{2m_{\eta'}}p_{\eta'}^2+\frac{1}{2}m_{\eta'}\Omega_{\eta'}^2\eta'^2, \tag{14}$$

where the new masses and frequencies are (taking the ''+'' sign in ω)

$$\begin{aligned}m_{\xi'}&=-\frac{\omega_c}{\omega_f},\quad m_{\eta'}=\frac{\omega_c}{(\omega_c+\omega_f)},\\ \Omega_{\xi'}&=\omega_f,\quad \Omega_{\eta'}=\omega_c+\omega_f.\end{aligned} \tag{15}$$

Taking the opposite sign in the definition of ω [Eq. (10)] interchanges the two frequencies. Provided that $\Omega_{\xi'}$ and $\Omega_{\eta'}$ are real, both the masses are finite, and at least one mass is positive, the motion will be stable [32]: this will be the case if $\omega_c\neq 0$, i.e., if a magnetic field is present the CHO will be stable. Asymptotically, this analysis is valid for the full Hamiltonian and means that the atom cannot ionize in the plane when a magnetic field is added to the CPM field. This is the origin of the bounding KAM curves in Fig. 1. Incidentally, an alternative way to establish stability is to find the eigenvalues of the matrix $\mathcal{A}$ where $\dot{\zeta}=\mathcal{A}\zeta$ are Hamilton's equations with the vector $\zeta=\{\xi',\eta',p_{\xi'},p_{\eta'}\}$. The four eigenvalues are found to be purely imaginary, confirming the result that a magnetic field prevents ionization in the plane [20,33].

IV. LINEAR RESONANCE

So far we have established that a magnetic field can prevent ionization in the plane. It remains to understand how ionization occurs in the absence of a magnetic field. Here we show that a linear resonance comes into play if the electron undergoes sufficiently large excursions from the nucleus in the ρ direction.

A. Hydrogen atom in a circularly polarized microwave field

Consider the pure CPM system (i.e., no magnetic field present) for which the problem cannot be reduced to a

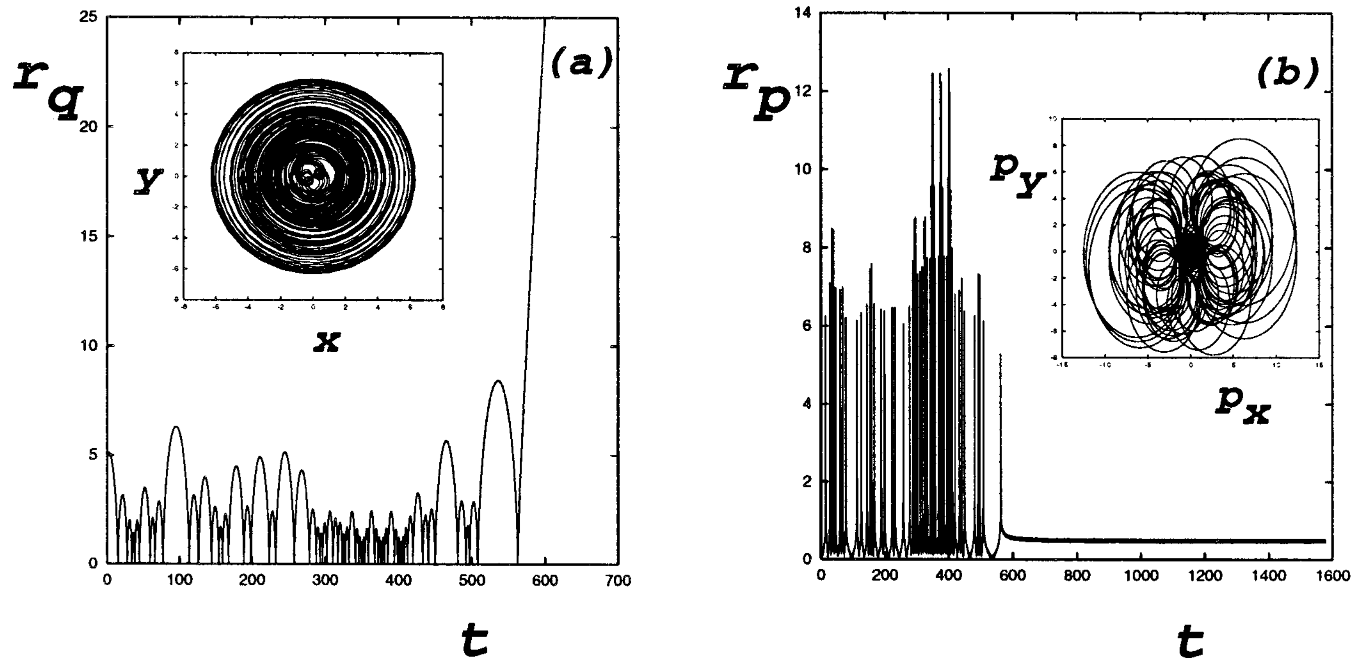

FIG. 2. Plots of (a) $r_q(t)=\sqrt{x(t)^2+y(t)^2}$ and (b) $r_p(t)=\sqrt{p_x(t)^2+p_y(t)^2}$ for an ionizing electron with $\omega_f=4$, $\varepsilon=0.15$ in similarly scaled units to Fig. 1. The insets show the x-y (prior to ionization) and p_x-p_y projections of the orbits. Note how, after the electron is scattered from the nucleus for the last time, r_p immediately settles down to a constant, while the amplitude r_q exhibits the characteristic behavior of a linear resonance, as predicted by Eq. (16). Scaled atomic units are used.

cranked oscillator because no oscillator potential is present [set $\omega_c=0$ in Eq. (12)]. It is easy to show that asymptotically the dynamics is governed by the set of equations

$$\ddot{\xi}+\omega_f^2\xi=2\omega_f p_\eta, \quad \ddot{\eta}+\omega_f^2\eta=-2\omega_f p_\xi,$$
$$\ddot{p}_\xi+\omega_f^2 p_\xi=0, \quad \ddot{p}_\eta+\omega_f^2 p_\eta=0. \tag{16}$$

Clearly the motion in the p_ξ-p_η, and, therefore also in the p_x-p_y phase plane is harmonic [$p_\xi(t),p_\eta(t)$ are periodic with frequency ω_f] and thus the coordinate motion is a linear oscillator driven at its own frequency ω_f, i.e., the classic example of resonance [34]. Ionization in the CPM problem occurs, therefore, because of an instability caused by a linear resonance that pumps the electron. Figures 2(a) and 2(b) plot the classical quantities $r_q=\sqrt{x^2+y^2}$ and $r_p=\sqrt{p_x^2+p_y^2}$ for an ionizing electron: as the electron ionizes the p_x-p_y motion settles down to a circle—this is the signature of ionization and the presence of a linear resonance [cf, Eq. (11)] [35]. Interestingly, as in the Kepler problem itself, the *hodograph* shown in the inset of Fig. 2 is asymptotically a circle (an interesting account of this phenomenon is given by Gutzwiller [2]). *This is the signature of a linear resonance and shows that such a resonance is responsible for the rapid exit of the electron*: i.e., it does not just slowly ''drift away'' but is swept away by the resonance. Of course, the mechanism by which the electron reaches the asymptotic region is *nonlinear* (i.e., chaotic) but it is a *linear* process that quickly leads to ionization. As shown clearly by Fig. 2(a) the electron ionizes after undergoing a sequence of chaotic collisions with the nucleus—for details see [5]. We emphasize that this behavior is generic to ionizing trajectories that originate in the chaotic sea. In fact, Fig. 2(a) is quite reminiscent of chaotic scattering of atoms from a corrugated surface [36]. The effect of an applied magnetic field is to detune the linear resonance in the system of equations (16) leading to total suppression of ionization in the plane even though a chaotic sea may still surround the core—see Fig. 1. In this case Eq. (16) now reads (here $\omega=\omega_f+\omega_c/2$)

$$\ddot{\xi}+\left(\omega^2+\frac{\omega_c^2}{4}\right)\xi=2\omega p_\eta,$$
$$\ddot{\eta}+\left(\omega^2+\frac{\omega_c^2}{4}\right)\eta=-2\omega p_\xi,$$
$$\ddot{p}_\xi+\left(\omega^2+\frac{\omega_c^2}{4}\right)p_\xi=-\frac{1}{2}\omega\omega_c^2\eta,$$
$$\ddot{p}_\eta+\left(\omega^2+\frac{\omega_c^2}{4}\right)p_\eta=\frac{1}{2}\omega\omega_c^2\xi. \tag{17}$$

Essentially the motion settles down to a system of normal modes defined by the CHO and shown previously by Eq. (14).

B. The crossed-fields problem in a rotating frame

We now turn to the crossed-fields problem itself for which $\omega_f=0$ in Eq. (1). First of all consider the case of a *finite* nuclear mass, i.e., $\Delta<1$ when it is easy to show that the Hamiltonian is equivalent to $K^{(b)}$ with $\omega=\Delta\omega_c/2$ and, therefore, ionization is impossible in the plane, a fact that has been noted before [30]. Explicitly the Hamiltonian is

$$H=\frac{p_x^2+p_y^2}{2}-\frac{1}{r}+\Delta\frac{\omega_c}{2}(x\,p_y-y\,p_x)+F\,x+\frac{\omega_c^2}{8}(x^2+y^2), \tag{18}$$

with the ZVS being given by

$$\Omega=-\frac{1}{r}+F\,x+\frac{(1-\Delta)^2\omega_c^2}{8}(x^2+y^2). \tag{19}$$

Note how simply the ZVS shows that, if $\Delta<1$, the motion in the plane is bounded by curves of zero velocity which prevent ionization. Such an example is seen in celestial mechanics where asteroids (or test particles) lying beyond Jupiter, despite having chaotic orbits, may be blocked from colliding with Jupiter by curves of zero velocity [37].

The situation is slightly more complicated if $\Delta=1$ because then the transformation (11) is singular. To handle this case we move to a new rotating frame so as to eliminate the Coriolis term in Eq. (1): the price paid is that we end up with an explicitly time-dependent Hamiltonian (after dropping the $1/r$ term)

$$K^{(d)}=\frac{p_x^2+p_y^2}{2}+\frac{\omega^2}{2}(x^2+y^2)+F(x\cos\omega t+y\sin\omega t), \tag{20}$$

where, now, $\omega=\omega_c/2$. The advantage is that the Newtonian equations of motion can be written in an elegant and physically transparent form,

$$\begin{aligned}\ddot{x}+\omega^2x&=-F\cos\omega t,\quad \ddot{p}_x+\omega^2p_x=F\omega\sin\omega t,\\ \ddot{y}+\omega^2y&=-F\sin\omega t,\quad \ddot{p}_y+\omega^2p_y=-F\omega\cos\omega t.\end{aligned} \tag{21}$$

These transformations make it obvious that the asymptotic dynamics consists of a set of uncoupled harmonic oscillators each of which is being driven at its own frequency, leading to instability and ionization: again by a linear resonance. The system of equations (21) generates a spiral [38] which corresponds to a trochoid in the nonrotating frame.

C. Ionization in three dimensions

Now we discuss the relevance of the preceding considerations to ionization in the full three-dimensional problem. First consider a case where ionization is possible: the CPM problem. Unless the electron is rigorously confined to the plane then, as it seeks its egress from the atom, it will move along the z direction to a greater or lesser extent. As it does this, the Coriolis forces (arising from being in a rotating frame) will deflect the electron, causing it to spiral around the z axis. If there is no confinement perpendicular to this direction then the electron will eventually escape. This occurs after a chaotic series of collisions with the core. The last such collision is the one that propels the particle into the linear resonance. A similar picture holds in the crossed-fields problem where the paramagnetic term replaces the Coriolis term.

If, on the other hand, the motion is bounded asymptotically in the plane, then ionization, if it occurs, will be along the z direction. This will be possible only if

$$\begin{aligned}K&>V(x,y;z\to\infty)-\frac{1}{2}\omega^2(x^2+y^2)\\ &=F\,x-\frac{\omega_f(\omega_f\pm\omega_c)}{8}(x^2+y^2).\end{aligned} \tag{22}$$

The ionization threshold is determined by finding the minimum of the right hand side of Eq. (22) which only exists if the minus sign is taken and $\omega_c>\omega_f$. We obtain

$$K_{\text{ion}}=\frac{F^2}{2\omega_f(\omega_f-\omega_c)}. \tag{23}$$

If $K<K_{\text{ion}}<0$ then the electron cannot ionize at all.

V. CONCLUSIONS

We have identified a linear resonance in the ionization of Rydberg electrons for several systems that are of considerable current interest in both quantum physics and nonlinear dynamics. In the absence of a magnetic field in the CPM system ionization occurs when the electron is scattered into a linear resonance by collisions with the nucleus. The addition of a magnetic field perpendicular to the polarization plane stops ionization in the plane. Similarly, in the crossed-fields problem a linear resonance was also shown to be responsible for ionization. It is rather surprising that the ionization dynamics of these strongly chaotic systems [5] occurs ultimately through the simplest physical example of resonance: a pumped *linear* oscillator.

ACKNOWLEDGMENT

Support of this research by the National Science Foundation is gratefully acknowledged.

[1] T. F. Gallagher, *Rydberg Atoms* (Cambridge University Press, Cambridge, England, 1994).
[2] M. Gutzwiller, *Chaos in Classical and Quantum Mechanics* (Springer-Verlag, New York, 1990).
[3] M. R. W. Bellerman, P. M. Koch, D. R. Mariani, and D. Richards, Phys. Rev. Lett. **76**, 892 (1996).
[4] P. M. Koch and K. A. H. van Leeuwen, Phys. Rep. **255**, 289 (1995).
[5] A. F. Brunello, T. Uzer, and D. Farrelly, Phys. Rev. A **55**, 3730 (1997).
[6] J. Zakrzewski, R. Gebarowski, and D. Delande, Phys. Rev. A **54**, 691 (1996); R. Gebarowski and J. Zakrzewski, *ibid.* **51**, 1508 (1995).
[7] J. A. Burns and L. Schaffer, Nature (London) **337**, 340 (1989).
[8] E. Lee, D. Farrelly, and T. Uzer, Opt. Express **1**, 221 (1997).
[9] F. Mignard, Icarus **49**, 347 (1982); A. Deprit, in *The Big Bang*

and George Lemaitre, edited by A. Berger (Reidel, Dordrecht, 1984), pp. 151–180.
[10] T. Uzer and D. Farrelly, Phys. Rev. A **52**, R2501 (1995).
[11] G. Raithel, M. Fauth, and H. Walther, Phys. Rev. A **44**, 1898 (1991).
[12] W. R. S. Garton and F. S. Tomkins, Astrophys. J. **158**, 839 (1969).
[13] J. von Milczewski, D. Farrelly, and T. Uzer, Phys. Rev. Lett. **78**, 2349 (1997).
[14] J. von Milczewski, D. Farrelly, and T. Uzer, Phys. Rev. A **56**, 657 (1997).
[15] J. Main and G. Wunner, Phys. Rev. Lett. **69**, 586 (1992).
[16] P. Fu, T. J. Scholz, J. M. Hettema, and T. F. Gallagher, Phys. Rev. Lett. **64**, 511 (1990).
[17] J. A. Griffiths and D. Farrelly, Phys. Rev. A **45**, R2678 (1992).
[18] D. Farrelly and T. Uzer, Phys. Rev. Lett. **74**, 1720 (1995).
[19] I. Bialynicki-Birula, M. Kaliński, and J. H. Eberly, Phys. Rev. Lett. **73**, 1777 (1994).
[20] D. Farrelly, E. Lee, and T. Uzer, Phys. Lett. A **204**, 359 (1995); E. Lee, A. F. Brunello, and D. Farrelly, Phys. Rev. Lett. **75**, 3641 (1995); A. F. Brunello, T. Uzer, and D. Farrelly, *ibid.* **76**, 2874 (1996).
[21] M. Raković, T. Uzer, and D. Farrelly, Phys. Rev. A **57**, 2814 (1998).
[22] C. Cerjan, E. Lee, D. Farrelly, and T. Uzer, Phys. Rev. A **55**, 2222 (1997).
[23] J. C. Gay, L. R. Pendrill, and B. Cagnac, Phys. Lett. **72A**, 315 (1979); L. A. Burkova, I. E. Dzyaloshinskii, G. F. Drukarev, and B. S. Monozon, Zh. Éksp. Teor. Fiz. **71**, 526 (1976) [Sov. Phys. JETP **44**, 276 (1976)]; S. K. Bhattacharya and A. R. P. Rau, Phys. Rev. A **26**, 2315 (1982).
[24] L. I. Gorkov and I. E. Dzyaloshinskii, Zh. Éksp. Teor. Fiz. **53**, 717 (1968) [Sov. Phys. JETP **26**, 449 (1968)].
[25] D. Farrelly, E. Lee, and T. Uzer, Phys. Rev. Lett. **80**, 3884 (1998).
[26] E. Lee, A. Puzder, M. Chou, T. Uzer, and D. Farrelly, Phys. Rev. B **57**, 12 281 (1995).
[27] E. Lee, A. F. Brunello, and D. Farrelly, Phys. Rev. A **55**, 2203 (1997).
[28] B. R. Johnson, J. O. Hirschfelder, and K. H. Yang, Rev. Mod. Phys. **55**, 109 (1988).
[29] D. Farrelly, Phys. Lett. A **191**, 265 (1994).
[30] P. Schmelcher, J. Phys. B **25**, 2697 (1992).
[31] J. M. A. Danby, *Fundamentals of Celestial Mechanics*, 2nd ed. (Willman-Bell, Richmond, VA, 1992), p. 47.
[32] D. Glas, U. Mosel, and P. G. Zint, Z. Phys. A **285**, 83 (1978).
[33] E. Lee, Ph.D. dissertation, Utah State University, Logan, UT, 1997.
[34] V. I. Arnold, *Ordinary Differential Equations* (MIT Press, Cambridge, MA, 1985), pp. 182–186.
[35] The shifted angular momentum term in Eq. (12) is conserved and, therefore, if $\omega_c = 0$, it is easy to see that $\dot{r}_p = 0$ since then $a = b = 0$. In fact, the exterior KAM curves in Fig. 1 are approximately level curves of the quantity $(\xi p_\eta - \eta p_\xi)$ which is *almost* conserved in the full problem at sufficiently large distances from the nucleus—see Eq. (11).
[36] F. Borondo, C. Jaffé, and S. Miret-Artes, Surf. Sci. **317**, 211 (1994).
[37] M. Lecar, Celest. Mech. Dyn. Astron. **64**, 163 (1996).
[38] A. D. Peters and J. D. Delos, Phys. Rev. A **47**, 3020 (1993).