Teleportation of *N*-dimensional states

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We present a general solution to the construction of such an entanglement in an N-dimensional Hilbert space that effects state teleportation. For $N>2$ the construction is nonunique and can be chosen to transfer the state to a preselected basis. We discuss possible applications to state preparation in various physical systems. [S1050-2947(98)02012-5]

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I. INTRODUCTION

The teleportation process suggested by Benett *et al.* [1] is an ingenious application of the nonlocal properties of entangled states. Such states play an important role in many of the recent discussions of quantum correlated systems [2]. Now we also have experimental methods to prepare the necessary entangled states in some simple situations [3,4]. Correlated photons preserve quantum coherence best and hence early experimental realizations have utilized these [5,6]. It is now realistic to assume that even more involved experimental realizations of teleportation ideas may become possible. Hence we undertake to explore the technique to transfer a given state to another one, which may be of different physical character or located in a more propitious place.

II. TELEPORTATION OF *N*-DIMENSIONAL STATES

We discuss three quantum systems described by their state vectors, which we assume to have the same dimensions N. We construct explicit schemes to apply the mechanism proposed by Bennett *et al.* to systems of arbitrary dimensionality.

We consider the initial state of system 1 to be

$$|\psi_0\rangle_1=\sum_k c_k|k\rangle_1, \tag{1}$$

where $\{|k\rangle_1\}$ is an arbitrary orthonormal and complete basis in the state space of system 1. We assume that we can prepare two other systems, 2 and 3, in the fully entangled state

$$|\psi_0\rangle_{23}=\frac{1}{\sqrt{N}}\sum_j |j\rangle_2|j\rangle_3. \tag{2}$$

If each basis state $|j\rangle_2$ in system 2 is entangled to a unique basis state in 3, we can always write the form (2) by relabeling the basis states in system 3. The full quantum state of the combined systems 1–3 now contains products of the type $|k\rangle_1|j\rangle_2$. We want to reexpress these in terms of entangled probe states, which can be subjected to single measurements. In order to achieve this we introduce two transformations: a unitary transformation $\{\mathcal{U}_{ij}\}$ and a doubly unitary transformation $\{\Gamma_{abc}\}$ such that, fixing any one of its indices, we obtain a unitary transformation in the other two. In fact, this condition can be slightly relaxed, but we will return to this issue in another context.

We now define the probe states in the combined systems 1 and 2 by writing

$$|\psi_{\alpha\beta}\rangle_{12}=\sum_l \mathcal{U}^*_{\alpha l}|l\rangle_1\sum_j \Gamma^*_{\beta lj}|j\rangle_2. \tag{3}$$

A simple check shows that this basis is complete and orthonormal in the combined systems 1 and 2. Inverting this and inserting into the initial state of the combined systems, we now obtain

$$|\psi_0\rangle_{123}\equiv|\psi_0\rangle_1|\psi_0\rangle_{23}=\sum_{\alpha,\beta}|\psi_{\alpha\beta}\rangle_{12}\sum_k c_k|e_k^{(\alpha\beta)}\rangle_3, \tag{4}$$

where the new basis vectors are

$$|e_k^{(\alpha\beta)}\rangle_3=\sqrt{N}\mathcal{U}_{\alpha k}\sum_j \Gamma_{\beta kj}|j\rangle_3. \tag{5}$$

If we now can make an observation that projects the state (4) on the state $|\psi_{\alpha\beta}\rangle_{12}$, we have prepared system 3 in the original quantum state (1), but in the basis $\{|e_k^{(\alpha\beta)}\rangle_3\}$. This is the original teleportation idea; we only need to tell the keeper of system 3 the outcome of our measurement $\{\alpha\beta\}$ and one knows the basis in which the state is received. Alternatively, one can perform a known unitary transformation from this basis to the one originally chosen in system 3; cf. Eq. (15) below. In either case one has received a quantum copy of the state (1) without either the keeper or the sender having to know what it is.

For this scenario to work, however, we need to verify some properties of the transformations introduced above. The basis vectors (5) have to be orthonormal and a simple calculation shows that

$${}_3\langle e_k^{(\alpha\beta)}|e_{k'}^{(\alpha\beta)}\rangle_3=N|\mathcal{U}_{\alpha k}|^2\delta_{kk'}=\delta_{kk'}. \tag{6}$$

This is possible only if the unitary transformation $\mathcal{U}_{\alpha k}$ is of the type we called a Zeilinger matrix in our earlier work [7], i.e., all the elements of the unitary matrix have to be of the same absolute magnitude. With this condition imposed, the completeness follows

$$\sum_k |e_k^{(\alpha\beta)}\rangle_{3\,3}\langle e_k^{(\alpha\beta)}| = \sum_j |j\rangle_{3\,3}\langle j| = 1. \tag{7}$$

There are no further conditions on the arrays Γ_{abc}. In Ref. [1] these properties were satisfied: The unitary transformation was the discrete Fourier transform discussed in Ref. [7] and the mixing of the basis vectors in Eq. (3) was just a cyclic permutation that gives the transformation

$$\Gamma_{abc} = \sum_{n=-\infty}^{\infty} \delta(c|a+b+nN), \tag{8}$$

where the Kronecker delta is denoted by $\delta(a|b) \equiv \delta_{ab}$. It is straightforward to check that the transformation (8) satisfies the conditions imposed on Γ_{abc}.

The solutions chosen in Ref. [7], however, do suggest alternative ways to achieve the conditions explained above. There are other possibilities to choose the Zeilinger matrices; as we have shown, we can make a very simple and attractive construction in spaces of dimensions $N=2^n$. Then we start from the realization ($n=1$)

$$\mathcal{U}^{(1)} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{9}$$

and define recursively

$$\mathcal{U}^{(n+1)} = \mathcal{U}^{(n)} \otimes \mathcal{U}^{(1)} = \frac{1}{\sqrt{2}}\begin{bmatrix} \mathcal{U}^{(n)} & \mathcal{U}^{(n)} \\ \mathcal{U}^{(n)} & -\mathcal{U}^{(n)} \end{bmatrix}. \tag{10}$$

This can be shown to be unitary for any n and it is of the Zeilinger type by construction. These matrices have, in addition, the good property that

$$\mathcal{U}_{1k}^{(n)} = \mathcal{U}_{l1}^{(n)} = \frac{1}{2^{n/2}} = \frac{1}{\sqrt{N}} \tag{11}$$

for all k and l.

The cyclic permutation chosen in Ref. [1] suggests an interesting construction of the transformation Γ_{abc}. Choose any complete, orthonormal basis set in system 3: $\{|u_i\rangle_3\}$. Now define an $N\times N$ matrix, where the first row consists of these basis vectors and each following one a permutation $\{|u(P_i^\alpha)\rangle_3\}$, where the column position is indicated by the index i and the row by the permutation α; for $\alpha=1$, the unit permutation is intended. These permutations have to be chosen in such a way that each row or column selected gives a complete orthonormal basis in the space 3; a simple consideration shows that this can always be achieved in a nonunique way. The set $\{|u(P_i^\alpha)\rangle_3\}$ is consequently a complete orthonormal basis whether we fix α and vary i or vice versa.

We now set the transformation Γ to be

$$\Gamma_{abc} = {}_3\langle c|u(P_a^b)\rangle_3, \tag{12}$$

where $\{|c\rangle_3\}$ is the original basis set. It is easy to check explicitly that this choice satisfies the conditions imposed on Γ_{abc}. This is not the most general representation of our requirements, but considering the fact that the fixing of the original basis $\{|u_i\rangle_3\}$ is arbitrary and that there are several ways to arrange the permutations, the method offers many possibilities.

When we insert the transformation (12) into the result (5) we find

$$|e_k^{(\alpha\beta)}\rangle_3 = \sqrt{N}\mathcal{U}_{\alpha k}\sum_j |j\rangle_{3\,3}\langle j|u(P_\beta^k)\rangle_3 = \sqrt{N}\mathcal{U}_{\alpha k}|u(P_\beta^k)\rangle_3. \tag{13}$$

Thus the basis to which we have transferred the original state is just one of those chosen in the definition of the transformation Γ. With $\mathcal{U}_{\alpha k}$ a Zeilinger matrix, it adds only an irrelevant phase to the basis set. In particular, if we have dimensionality 2^n, we can select the projection on to the probe state $\{\alpha,\beta\} \Rightarrow \{1,1\}$, in which case from Eq. (11) it follows that

$$|e_k^{(11)}\rangle_3 = |u_k\rangle_3, \tag{14}$$

that is, we have transferred the state to the basis introduced in order to construct the transformation Γ. This gives us the unique procedure to select the basis where the reconstruction is to take place: Choose the basis $\{|u_i\rangle_3\}$, construct the device to measure the states (3), and select the cases when the state $|\psi_{11}\rangle_{12}$ is obtained. Then the original state has been reconstructed in the selected basis. Of course, in one single measurement we cannot be certain to observe this outcome and hence we must be able to accept any result possible if we cannot repeat the experiment. However, any outcome offers the classical information telling in which basis the state has reemerged.

The unitary transform that maps the transferred state to the initial one in system 1 is equal to

$$\mathbb{U}_3^{(\alpha\beta)} = \sum_k |k\rangle_{3\,3}\langle e_k^{(\alpha\beta)}| \tag{15}$$

for an observed pair $\{\alpha,\beta\}$. If the recording of $\{\alpha,\beta\}$ is performed without destroying the state $|\psi_{\alpha\beta}\rangle_{12}$ we can find a further use for Eq. (15). We form the same unitary operator, but acting on the Hilbert space of system 1. Then we calculate

$$\begin{aligned}\mathbb{U}_1^{(\alpha\beta)\dagger}|\psi_{\alpha\beta}\rangle_{12} &= \sum_k |e_k^{\alpha\beta}\rangle_1 \mathcal{U}_{\alpha k}^* \sum_j \Gamma_{\beta kj}^* |j\rangle_2 \\ &= \sum_k \sqrt{N}|\mathcal{U}_{\alpha k}|^2 \sum_{j,j'} \Gamma_{\beta kj'}\Gamma_{\beta kj}^* |j'\rangle_1 |j\rangle_2 \\ &= \frac{1}{\sqrt{N}}\sum_j |j\rangle_1|j\rangle_2.\end{aligned} \tag{16}$$

We thus have regained a state of type (2), which can be used as the starting point for a second teleportation: We send system 2 to the intended target of the operation and take a state of the form (1) in a new system 4. The correlated probe measurement is now carried out on systems 1 and 4 and the state appears in system 2. Thus we only need one initially correlated state to teleport any number of states to desired locations. Because of the need for repeated transfer of each member of the entangled pairs, the procedure consumes a certain time for each step.

When we perform the operation (16), we are in some sense just recreating the original situation. We have only swapped the entanglement between systems 2 and 3 to systems 1 and 2. The original state of system 1 reappears in system 3. Thus our discussion exemplifies some of the arguments given by Nielsen and Caves [8].

III. TELEPORTATION AS A STATE PREPARATION METHOD

Many experiments on fundamental quantum properties require the preparation of a chosen initial state or a state suitably entangled between two systems of our choice. The problem to prepare an initial state of an experiment has thus been addressed in a variety of ways [9–20]; Ref. [21] covers the current state of the art in a comprehensive way. These methods, however, assume that we know the state we want to prepare and that we may apply an interaction to the system where we want to direct the state. In an experiment, the desired initial state may be the outcome of some earlier measurement and we cannot determine it fully or clone it from the one single copy we have. Yet we may want to copy its quantum information onto another quantum system to be used in the continuation of the experiment. Furthermore, it may not be desirable to couple an interaction to the system where we want the prepared state to reside; even the suggested adiabatic manipulations may add too much contact with an unfriendly environment for the sensitive quantum systems we may wish to address. We want to draw attention to the fact that teleportation provides the ideal tool to overcome both of these problems. As the method is based on nonlocal quantum correlations, the resulting state may be situated well away from all other parts of the experiment.

In the experiments using the theory as a method to teleport an unknown state from a sender to a receiver, the physical character of systems 1–3 has usually been assumed to be the same. The experiment [6] offers an exception. When we use the method as a state preparation tool, we may choose to have totally different physical realizations play the roles of the various systems. A desired quantum state may be easier to achieve in one system than in the one on which the experiment is to be carried out. A suitable transfer may then create the desired state where we want it. It may also be easier to transport quantum coherence between some systems than others. Because the main coherence in the teleportation scheme is carried by the standard entangled state of systems 2 and 3, we may be able to construct a standard solution for this part of the experiment and then apply the preparation mechanism to the totally different system 1. Whenever we are able to devise a suitable measurement scheme, the projection based preparation of the desired state in system 3 is essentially interaction free. This may help in isolating it from unwanted influences from the environment during the preparation stage.

One possible application is the transfer of a quantum state from one cavity to another [22–24]. The state is prepared in cavity 1, and the state in cavity 3 is entangled with an atomic system. The atom is transferred to cavity 1 and the appropriate joint measurement is carried out there. As a consequence, the state in cavity 3 becomes an image of the disappearing state in cavity 1. Notice that this occurs without any action taking place at the position of the emerging state. We may well expect the coherent transportation of the atomic state from one cavity to the other to be simpler than the direct transfer of a cavity quantum state into the second cavity. It remains to be decided how the appropriate observations affecting the projections should be realized.

In discussions concerning the nonlocal properties of quantum mechanics, one often assumes that the entangled states are to be positioned at well separated spatial localizations. This emphasizes the apparent paradoxes of quantum theory and brings out the nonclassical features. However, there are many examples where the same physical object carries different quantum degrees of freedom. One example is the entanglement between internal spin states and the translational motion in an experiment of the Stern-Gerlach type. In connection with the state preparation arguments, we may think of a simple generalization of the standard entangled state (2) to describe correlations between the internal quantum level and the spatial wave function

$$\langle x|\psi_0\rangle_{23}=\frac{1}{\sqrt{N}}\sum_j |j\rangle_2\Phi_j^{(3)}(x). \qquad (17)$$

In another quantum system 1, we prepare the desired state, which may or may not be known exactly. After carrying out the correlated measurement procedure described in this paper, we have prepared system 3 in the translational state

$$\Psi^{(3)}(x)=\sum_k c_k\Phi_k^{(3)}(x). \qquad (18)$$

With a suitable choice of states $\{\Phi_j^{(3)}(x)\}$ and a large enough number N, we can use this to prepare system 3 in an initial state (18) of almost arbitrary shape. Which effects can be achieved depend on the possibilities to prepare the standard entangled state (17). If plane waves can be achieved, we have devised a Fourier synthesizer of spatial wave packets. It is of course obvious that the method can be used with momentum wave functions, if this is easier, or in any other representation found suitable from an experimental point of view.

Another possibility is to synthesize a desired motional state in an ion trap, assuming that the necessary entanglement can be prepared. The teleportation method then offers a tool to transfer a physical state prepared outside the trap to the ion motion, even if the state mentioned first is only transiently available. This is closely related to the synthesizing of ionic states suggested in Refs. [11,12].

The possibility to replace the interaction by state preparation utilizing teleportation could also be applied to realizations of quantum computation schemes [25]. One may also consider the generalization to infinite ranges of continuous labels instead of the discrete sums over N states [26], but there are mathematical complications to grapple with. As even the experimental realization of the discrete case for $N > 2$ may offer difficulties, we have chosen not to work through the continuous case in any detail.

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