Nonlocality of the Einstein-Podolsky-Rosen state in the Wigner representation

Konrad Banaszek and Krzysztof Wódkiewicz*
Instytut Fizyki Teoretycznej, Uniwersytet Warszawski, Hoża 69, PL-00-681 Warszawa, Poland
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We demonstrate that the Wigner function of the Einstein-Podolsky-Rosen state, though positive definite, provides direct evidence of the nonlocal character of this state. The proof is based on an observation that the Wigner function describes correlations in the joint measurement of the phase-space displaced parity operator. [S1050-2947(98)06712-2]

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Einstein, Podolsky, and Rosen (EPR) in their argument about the completeness of quantum mechanics used the following wave function for a system composed of two particles [1]:

$$\Psi(x_1,x_2)=\int_{-\infty}^{\infty} e^{(2\pi i/h)(x_1-x_2+x_0)p}\,dp. \tag{1}$$

Despite its obvious simplicity, this wave function has not been explicitly used in arguments relating the nonlocality of quantum mechanics with the Bell inequalities. Following Bohm [2] the EPR correlations have been analyzed with the help of a singlet state of two spin-1/2 particles. For this state the nonlocality of quantum correlations has been demonstrated [3].

Quantum correlations for position-momentum variables can be analyzed in phase space using the Wigner distribution function. Using this phase-space approach to the EPR correlations Bell argued [4] that the original EPR wave function (1) will not exhibit nonlocal effects because its joint Wigner distribution function $W(x_1,p_1;x_2,p_2)$ is positive everywhere and as such will allow for a local hidden variable description of position sign correlations.

In local hidden variable theories these and analogous correlations can be written in a form of a statistical ensemble of two local realities $\sigma(\mathbf{a},\lambda_1)=\pm 1$ and $\sigma(\mathbf{b},\lambda_2)=\pm 1$, for two spatially separated detectors with certain settings labeled by **a** and **b**:

$$E(\mathbf{a};\mathbf{b})=\int d\lambda_1\int d\lambda_2\,\sigma(\mathbf{a},\lambda_1)\,\sigma(\mathbf{b},\lambda_2)\,W(\lambda_1;\lambda_2). \tag{2}$$

In this relation $W(\lambda_1;\lambda_2)$ is a local, positive, and normalized distribution of hidden variables λ_1 and λ_2. In the Wigner representation, these variables can be associated, respectively, with the phase-space realities (x_1,p_1) and (x_2,p_2). Bell's argument against the nonlocality of the EPR wave function (1) goes as follows. If the Wigner function of the system is positive everywhere it can be used to construct a local hidden variable correlation in a form given by Eq. (2) and accordingly the Bell inequality is never violated. In order to emphasize this point Bell used a nonpositive Wigner function to show that the position sign correlation function will violate local realism. These examples indicated a relation between the locality and the positivity of the phase-space Wigner function.

The relation between the EPR correlations and the Wigner distribution function has been addressed in several papers [5–9]. Although the singular character of the wave function (1) and the corresponding unnormalized Wigner function has been criticized, the main point of the Bell argument relating the positivity of the Wigner function to the lack of nonlocality of such a state has not been questioned. It has been argued that the problem of normalization can be simply solved by a ''smoothing'' procedure of the original wave function (1).

An example of such a ''smoothing'' procedure, with a clear application to quantum optics, has been the use of a two-mode squeezed vacuum state produced in a process of nondegenerate optical parametric amplification (NOPA) [10]. The NOPA state has been generated experimentally [6] and applied to discuss the implications of the positivity of the phase-space Wigner function on the Bell inequality [7].

These discussions have led to rather ambiguous results. On one hand, it has been argued that the quantum description for the system of the NOPA as well as for the system originally discussed by EPR is consistent with deterministic realism [6]. From this remark one can conclude that the EPR wave function (1) cannot be used to test direct violations of the Bell inequality. This rather vexing conclusion indicates that tests of quantum nonlocality have to rely not on the original EPR wave function but on Bohm's spin-1/2 system or on exotic states described by negative Wigner functions. On the other hand, attempts have been made to design an experiment that would reveal the nonlocality of the EPR state [5,9].

The purpose of this paper is to demonstrate that the positive definite Wigner function of the EPR state provides direct evidence of the nonlocality exhibited by this state. We shall show that the positivity or the negativity of the Wigner function has a rather weak relation to the locality or the nonlocality of quantum correlations. In fact, we shall show that the NOPA wave function violates the Bell inequality and that the original EPR wave function (1) exhibits strong nonlocality, but one should be careful with the singular limit of strong squeezing (in this limit the NOPA state reduces to the EPR state). The NOPA phase space will be parametrized by

*Also at the Center of Advanced Studies and Department of Physics, University of New Mexico, Albuquerque, NM 87131.

two complex coherent states amplitudes α and β corresponding, respectively, to (x_1,p_1) and (x_2,p_2).

The starting point of our proof is an observation that the two-mode Wigner function $W(\alpha;\beta)$ can be expressed as

$$W(\alpha;\beta)=\frac{4}{\pi^2}\Pi(\alpha;\beta), \tag{3}$$

where $\Pi(\alpha,\beta)$ is a quantum expectation value of a product of displaced parity operators:

$$\hat{\Pi}(\alpha;\beta)=\hat{D}_1(\alpha)(-1)^{\hat{n}_1}\hat{D}_1^\dagger(\alpha)\otimes\hat{D}_2(\beta)(-1)^{\hat{n}_2}\hat{D}_2^\dagger(\beta). \tag{4}$$

The connection of the parity operator $(-1)^{\hat{n}}$ with the Wigner function provides an equivalent definition of the latter [11], as well as a feasible quantum optical measurement scheme [12]. In the above formula, $\hat{D}_1(\alpha)$ and $\hat{D}_2(\beta)$ denote the unitary phase-space displacement operators for the subsystems 1 and 2.

As the measurement of the parity operator yields only one of two values: $+1$ or -1, there exists an apparent analogy between the measurement of the parity operator and of the spin-1/2 projectors. The solid angle defining the direction of the spin measurement is now replaced by the coherent displacement describing the shift in phase space. Consequently, all types of Bell's inequalities derived for a correlated pair of spin-1/2 particles can be immediately used to test the nonlocality of the NOPA wave function. The two NOPA field modes are equivalent to an entangled state of two harmonic oscillators. As Eq. (4) clearly demonstrates, the correlation functions measured in such experiments are given, up to a multiplicative constant, by the joint Wigner function of the system. As a consequence we have the fundamental relation

$$E(\mathbf{a};\mathbf{b})\equiv\Pi(\alpha;\beta). \tag{5}$$

The original EPR state (1) is an unnormalizable δ function. In order to avoid problems arising from this singularity, we will consider a normalizable state that can be generated in a NOPA. Such a state is characterized by the dimensionless effective interaction time r (the squeezing parameter). The Wigner function of this NOPA state is well known [6,7] and is given by

$$\Pi(\alpha;\beta)=\exp[-2\cosh 2r(|\alpha|^2+|\beta|^2)+2\sinh 2r(\alpha\beta+\alpha^*\beta^*)]. \tag{6}$$

The Wigner function of the original EPR state (1) is obtained in the limit $r\to\infty$.

The correlation function is measured for any of four combinations of $\alpha=0,\sqrt{\mathcal{J}}$ and $\beta=0,-\sqrt{\mathcal{J}}$, where $\mathcal{J}$ is a positive constant characterizing the magnitude of the displacement. From these quantities we construct the combination [13]

$$\mathcal{B}=\Pi(0;0)+\Pi(\sqrt{\mathcal{J}};0)+\Pi(0;-\sqrt{\mathcal{J}})-\Pi(\sqrt{\mathcal{J}};-\sqrt{\mathcal{J}})$$
$$=1+2\exp(-2\mathcal{J}\cosh 2r)-\exp(-4\mathcal{J}e^{2r}), \tag{7}$$

which for local theories satisfies the inequality $-2\leqslant\mathcal{B}\leqslant 2$. Let us note that one of the components of the above combination describes perfect correlations: $\Pi(0,0)=1$, obtained for a direct measurement of the parity operator with no displacements applied. This is a manifestation of the fact that in the parametric process photons are always generated in pairs.

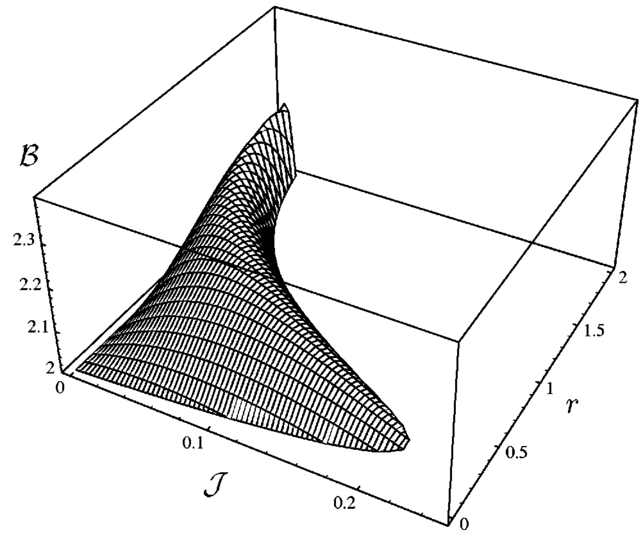

FIG. 1. Plot of the combination $\mathcal{B}$ defined in Eq. (7). Only values exceeding the bound imposed by local theories are shown.

As depicted in Fig. 1, the result (7) violates the upper bound imposed by local theories. With increased r, the violation of Bell's inequality is observed for smaller $\mathcal{J}$. We will therefore perform an asymptotic analysis for large r and $\mathcal{J}\ll 1$. In this regime we may approximate $\cosh 2r$ appearing in the argument of the first exponent in Eq. (7) just by $e^{2r}/2$. Then a straightforward calculation shows that the maximum value of $\mathcal{B}$ (for this particular selection of coherent displacements) is obtained for

$$\mathcal{J}e^{2r}=\frac{1}{3}\ln 2, \tag{8}$$

and equals $\mathcal{B}=1+3\times 2^{-4/3}\approx 2.19$. Thus, in the limit $r\to\infty$, when the original EPR state is recovered, a significant violation of Bell's inequality takes place. This result has been obtained without any serious attempt to find the maximum violation (for this purpose one should consider a general quadruplet of displacements). Let us note that in order to observe the nonlocality of the EPR state, very small displacements have to be applied, decreasing as $\mathcal{J}\propto e^{-2r}$. This shows the subtleties related to the original EPR state (1) and the need for considering its regularized version.

This discussion shows that despite conflicting claims, the original EPR wave function (1) exhibits strong nonlocality. The violation of the Bell inequality is achieved for a state that is described by a positive Wigner function. This example puts to rest various conjectures, relating the positivity or the negativity of the Wigner function to the violation of local realism. We have shown that in quantum mechanics, the correlation (2) can be a Wigner function itself. This is due to the fact that the Wigner function can be directly associated with the parity operator. This operator can be measured in a photon-photon coincidence experiment.

Apparently, the Wigner representation cannot serve as a model local hidden variable theory describing the joint parity measurement. A straightforward explanation of this fact is given by expressing the correlation function $\Pi(\alpha;\beta)$ in the form analogous to Eq. (2):

$$\Pi(\alpha;\beta)=\int d^2\lambda_1\int d^2\lambda_2\frac{\pi}{2}\delta^{(2)}(\alpha-\lambda_1)\times\frac{\pi}{2}\delta^{(2)}(\beta-\lambda_2)W(\lambda_1;\lambda_2), \tag{9}$$

where λ_1 and λ_2 are now complex phase-space look-alikes of hidden variables. Though the outcome of the parity measurement may be only $+1$ or -1, the analog of local realities appearing in the Wigner representation is described by unbounded δ functions

$$\sigma(\mathbf{a},\lambda_1)\equiv\frac{\pi}{2}\delta^{(2)}(\alpha-\lambda_1),\qquad \sigma(\mathbf{b},\lambda_2)\equiv\frac{\pi}{2}\delta^{(2)}(\beta-\lambda_2), \tag{10}$$

which makes the Bell inequality void.

A tempting aspect of the Wigner representation is the interpretation of quantum mechanics in classical-like terms in phase space. One well known difficulty with this approach is the negativity of the Wigner function [14]. The example discussed in this paper shows that quantum mechanics manifests its nature also in another, equally important way: the Wigner representations of quantum observables cannot be in general interpreted as phase-space distributions of possible experimental outcomes. In particular, the Wigner representation of the parity operator is not a bounded reality corresponding to the dichotomic result of the measurement. This enables violation of Bell's inequalities even for quantum states described by positive-definite Wigner functions.

The measurement described in this paper gains particular interest in the context of recent advances in quantum state reconstruction. Over the past several years it has become possible to obtain experimentally a complete picture of the quantum state of light in the Wigner representation [15,16]. This, along with the feasibility of generating the quantum optical NOPA state [6], provides an exciting possibility to test experimentally the nonlocal nature of the EPR state demonstrated in this work.

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