Complement of the Hamiltonian

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The much-studied energy-time uncertainty relation has well-known difficulties that are exacerbated for a system with discrete energy levels. The difficulty in representing time in the abstract sense by an operator raises the related question of whether or not there is some other quantity that is complementary to the Hamiltonian of a quantum system. Such a quantity would have dimensions of time but would be a property of the system itself. We examine this question for a system with discrete energy eigenstates for which the ratios of the energy differences are rational. We find that such a quantity does exist and can be represented both by a probability-operator measure and by an Hermitian operator, but in a state space larger than the minimal space needed to include the states of the system. The uncertainty relation with the energy is slightly more complicated than the momentum-position uncertainty relation, but is readily interpretable. To describe such a quantity the name ''age'' is suggested. [S1050-2947(98)03012-1]

I. INTRODUCTION

One of the earliest issues in quantum mechanics has been the question of an energy-time uncertainty principle. Fundamental to this problem is that in quantum mechanics the energy is a dynamical variable of the quantum system represented by a Hermitian operator, but time in the normal sense is a parameter, as it is classically. (For a recent overview of this, see Ref. [1] and the references therein.) Another important difficulty is that, whereas the momentum-position uncertainty relation can be derived from the commutation relation

$$[\hat{p}_x, \hat{x}] = -i\hbar, \tag{1}$$

there is apparently no operator $\hat{\tau}$ that is canonically conjugate to the Hamiltonian operator $\hat{E}$ in the sense of Eq. (1) from which a corresponding uncertainty relation can be derived. Indeed, a relation of the form

$$[\hat{E}, \hat{\tau}] = -i\hbar \tag{2}$$

simply cannot be correct for quantum systems with discrete energy states. This is because if we find the expectation value of each side of Eq. (2) for an energy eigenstate, we obtain zero on the left-hand side and $-i\hbar$ on the right-hand side. This is exactly the difficulty involved with early attempts to find a phase operator canonically conjugate to the excitation number operator of an harmonic oscillator [2] (for recent reviews and bibliographies of the quantum phase problem, see Refs. [3] and [4]). Commutators of canonically conjugate operators in the form (1) and (2) can be regarded as special cases of a more general form of commutator [5] for conjugate, or complementary, operators. For momentum and position this more general commutator reduces to Eq. (1), but for phase and excitation number it contains an extra term that removes the difficulty described above. Of course this means that the number-phase uncertainty relation is more complicated than the momentum-position uncertainty relation. As Eq. (2) cannot be true, we might expect likewise

that for the uncertainty relation for the Hamiltonian with discrete eigenvalues and its conjugate, or complement, a similar complication should apply and Eq. (2) should be generalized. The question then arises as to whether or not there is some operator conjugate or complementary to the Hamiltonian in this more general sense.

In this paper we wish to explore the possibility of the existence of a quantity that can be regarded as the complement of the Hamiltonian for a quantum system with discrete energy levels, recognizing that if this quantity is represented by an operator the appropriate commutator must be a generalized form of Eq. (2). Although this quantity will have dimensions of time, it will not be appropriate to refer to such an operator as a time operator. As an operator it would represent an observable of the quantum system and not time in the abstract or coordinate sense or as a reading on an external clock. Its eigenstates would represent a state of the quantum system and some measurement on the system, at least in principle, should tell us about the quantity involved. Although this quantity is not time, we would hope that for a particular ideal system its expectation value may change linearly with time, for example, so that a measurement of the quantity would also give a measure of time. In this case, we would be using the system as a clock. Before seeking an operator for the complement of the Hamiltonian we first represent this quantity by a nonorthogonal probability-operator measure, which is a more general concept [6].

II. DISCRETE ENERGY AND THE α QUANTITY

We consider a quantum system with $p+1$ energy states $|E_i\rangle$ with $i=0,1,\ldots,p$. To avoid unnecessary complications, for this paper we shall let the states be nondegenerate and we choose our zero of energy so that the lowest-energy eigenvalue $E_0=0$ with the other energies increasing with, but not necessarily proportional to, i. We assume that these include all the accessible, or essential, states of the system. As an example of accessible states, we note that an effective two-level atom can be prepared from a multilevel atom for

particular experiments. In this case not all the usual states of the atom are accessible from the initial state by means of the particular interaction applied to the atom. By allowing more interactions that couple more states, we can increase the number of accessible states until eventually we include all physically accessible, or physical, states of the system, that is, states that are accessible from each other by some series of physical interactions.

For now we shall keep p finite, but it is possible to take a limit at a later stage. We let Ψ_p be the $(p+1)$-dimensional state space spanned by these orthogonal energy states. This is the minimal space required for a description of the quantum system. The Hamiltonian operator for the system is

$$\hat{E} = \sum_{i=0}^{p} E_i |E_i\rangle\langle E_i|, \tag{3}$$

where E_i are the energy eigenvalues. A general state of the system can be written as

$$|f\rangle = \sum_{i=0}^{p} f_i |E_i\rangle \tag{4}$$

and will evolve in time according to

$$\exp(-i\hbar^{-1}\hat{E}t)|f\rangle = \sum_{i=0}^{p} f_i \exp(-i\hbar^{-1}E_i t)|E_i\rangle. \tag{5}$$

We are interested in a quantity α of the system, which we shall refer to simply as the α quantity, which will have dimensions of time and will be conjugate to the Hamiltonian in the sense that $\hat{E}$ is the generator of shifts in α quantity. Thus we seek a state $|\alpha\rangle$ for which

$$\exp(-i\hbar^{-1}\hat{E}\delta\alpha)|\alpha\rangle = |\alpha + \delta\alpha\rangle. \tag{6}$$

Writing

$$|\alpha\rangle = \sum_{i=0}^{p} c_i(\alpha)|E_i\rangle, \tag{7}$$

we find that we can ensure that Eq. (6) is true by setting

$$c_i(\alpha) \propto \exp(-i\hbar^{-1}E_i\alpha) \tag{8}$$

and thus, with the appropriate normalization factor included, we have

$$|\alpha\rangle = \frac{1}{\sqrt{p+1}} \sum_{i=0}^{p} \exp(-i\hbar^{-1}E_i\alpha)|E_i\rangle. \tag{9}$$

If we replace $\delta\alpha$ on the left-hand side of (6) by δt, we obtain a time translation expression, that is, the state $|\alpha + \delta\alpha\rangle$ is the state to which $|\alpha\rangle$ would evolve in a time $\delta t = \delta\alpha$. Thus, although the quantity represented by α is not the time because it is a property of the system, it bears some relation to time.

The states $|\alpha\rangle$ are not orthogonal and their number exceeds the dimensions of the space Ψ_p spanned by the $p+1$ energy states $|E_i\rangle$, so these cannot be eigenstates of a Hermitian operator on Ψ_p with which we might hope to repre-

sent the α quantity. Further, for unequally spaced energy levels, we cannot even pick out a subset of $p+1$ states $|\alpha\rangle$ that are orthogonal. We can, however, make progress for unequally spaced levels if the ratios E_i/E_1 are all rational numbers or can be sufficiently closely approximated by rational numbers. For E_i/E_1 rational

$$\frac{E_i}{E_1} = \frac{C_i}{B_i}, \tag{10}$$

where C_i and B_i are integers with no common factors. We write the lowest common multiple of the values of B_i with $i>1$ as r_1 and define $r_i = r_1 C_i/B_i$ for $i>1$ and as zero for $i=0$. Then r_i is an integer for all $i \geq 0$ with $r_0 = 0$. From Eq. (10) we can write

$$E_i = r_i \frac{2\pi\hbar}{T}, \tag{11}$$

where

$$T = 2\pi\hbar r_1/E_1. \tag{12}$$

Let us now select $s+1$ states $|\alpha\rangle$ for which the values of α are uniformly spread over the range T. That is, we choose values of α denoted by

$$\alpha_m = \alpha_0 + m \frac{T}{s+1}, \tag{13}$$

with $m = 0,1,...,s$. We find then that the states $|\alpha_m\rangle$ have the interesting property that

$$\sum_{m=0}^{s} |\alpha_m\rangle\langle\alpha_m| = \frac{1}{p+1} \left\{ \sum_{m=0}^{s} \sum_{i} |E_i\rangle\langle E_i| \right.$$

$$+ \sum_{i \neq k} \sum_{m=0}^{s} \exp[i(r_k - r_i)\alpha_m 2\pi/T]$$

$$\left. \times |E_i\rangle\langle E_k| \right\} \tag{14}$$

and thus

$$\frac{p+1}{s+1} \sum_{m=0}^{s} |\alpha_m\rangle\langle\alpha_m| = \hat{1}, \tag{15}$$

where $\hat{1}$ is the unit operator on the space Ψ_p, provided the second term on the right-hand side of Eq. (14) is zero or at least negligible compared with the first term. For E_i/E_1 rational and thus $r_k - r_i$ an integer, this term will be zero and Eq. (15) true provided $r_k - r_i$ is not a multiple of $s+1$. This follows from substituting Eq. (13) into Eq. (14) and summing the geometric progression involved. We can ensure $r_k - r_i$ is not a multiple of $s+1$ by choosing $s+1 > r_p$, which is the largest value of r_i.

If E_i/E_1 is irrational but sufficiently well approximated by a rational number for the second term on the right-hand side of Eq. (14) to be negligible, then this rational number would in general have a large denominator, so r_1 will be very large. This implies from Eq. (12) that T will be very

much greater than the natural period $2\pi\hbar/E_i$ associated with the state with energy E_i. When E_i/E_1 are exact rational numbers T has a simple physical interpretation. From Eqs. (5) and (11) T is equal to the smallest time taken for the system to return to its initial state. Thus the state $|\alpha\rangle$ will be the same as the state $|\alpha+T\rangle$. It follows that restricting our selection of states $|\alpha\rangle$ to those for which the values of α are uniformly spread over the range T then prevents us from including the same state twice.

Expression (15) is a resolution of the identity. Thus, although the α quantity is not an observable represented by a Hermitian operator on the space Ψ_p of $p+1$ dimensions, it can be represented by a nonorthogonal probability-operator measure, which is a more general concept [6]. The $s+1$ nonorthogonal elements of this probability-operator measure are $(p+1)(s+1)^{-1}|\alpha_m\rangle\langle\alpha_m|$. Expression (15) can be used to expand the general state given by Eq. (4) as

$$|f\rangle = \frac{p+1}{s+1} \sum_{m=0}^{s} \langle\alpha_m|f\rangle|\alpha_m\rangle \quad (16)$$

and so, using $\langle f|f\rangle=1$, we obtain

$$\sum_{m=0}^{s} \frac{p+1}{s+1} |\langle f|\alpha_m\rangle|^2 = 1. \quad (17)$$

Thus each term in Eq. (17), all of which are positive, can represent a probability with the total probability correctly normalized. From quantum detection theory [6], this is the probability that the application of the probability-operator measure by means of a suitable measuring instrument yields the result α_m. Of course, this does not indicate immediately how to perform such a measurement, but that is not our concern here, where we are interested merely in identifying the quantity that can be regarded as the complement of the Hamiltonian and in studying some of its properties.

So far the choice of s is arbitrary apart from the requirement that $s \geq r_p$. In order for the α quantity to be applicable to all systems of the type considered here, however large the value of r_p, and to be independent of an arbitrary choice of s, we now define the α quantity as that represented by the above probability-operator measure in the limit as $s \to \infty$. In this limit, the difference between successive values of α_m tend to zero and the probability for a value of α in the small range between α and $\alpha + \delta\alpha$ is $P(\alpha)\delta(\alpha)$, where the probability density is given by

$$P(\alpha) = \frac{1}{T} |\langle f|\bar{\alpha}\rangle|^2, \quad (18)$$

where

$$|\bar{\alpha}\rangle = \sqrt{p+1}|\alpha\rangle = \sum_i \exp(-i\hbar^{-1}E_i\alpha)|E_i\rangle. \quad (19)$$

The resolution of the identity (15) then becomes simply

$$\int_{\alpha_0}^{\alpha_0+T} |\bar{\alpha}\rangle\langle\bar{\alpha}|d\alpha/T = \hat{I} \quad (20)$$

and the α quantity is represented by the probability-operator measure generated by the infinitesimal operators $|\bar{\alpha}\rangle\langle\bar{\alpha}|d\alpha/T$.

The probability density for the system in state $|f\rangle$ to be found with a value α for the α quantity is, from Eqs. (4) and (18),

$$P(\alpha) = \frac{1}{T} \left| \sum_{i=0}^{p} f_i^* \exp(-i\hbar^{-1}E_i\alpha) \right|^2. \quad (21)$$

This expression displays an essential feature of the complementary between energy and the α quantity: If the system is in an energy eigenstate then there is only one term, of modulus unity, in Eq. (21) and $P(\alpha)=T^{-1}$, that is, the α-quantity distribution is constant across the whole period T. Thus if the energy can be determined exactly the α quantity is completely random. The probability density (21) is all that is needed to calculate the statistical properties of the α quantity for any state $|f\rangle$ of our quantum system.

III. RATE OF CHANGE WITH TIME

To find the way in which the properties of the α quantity change with time we first calculate the rate of change of $P(\alpha)$ in (18) from Schrödinger's equation as

$$\frac{dP(\alpha)}{dt} = \frac{1}{T} \left(\frac{d\langle f|}{dt} |\bar{\alpha}\rangle\langle\bar{\alpha}|f\rangle + \langle f|\bar{\alpha}\rangle\langle\bar{\alpha}| \frac{d|f\rangle}{dt} \right)$$

$$= \frac{i}{\hbar T} \langle f|[\bar{E},|\bar{\alpha}\rangle\langle\bar{\alpha}|]|f\rangle. \quad (22)$$

The rate of change of the mean of the α quantity for a state $|f\rangle$ is then

$$\frac{d\langle\alpha\rangle}{dt} = \int_{\alpha_0}^{\alpha_0+T} \alpha \frac{dP(\alpha)}{dt} d\alpha = \frac{i}{\hbar} \langle f|[\hat{E},\hat{A}]|f\rangle, \quad (23)$$

where

$$\hat{A} = \frac{1}{T} \int_{\alpha_0}^{\alpha_0+T} \alpha|\bar{\alpha}\rangle\langle\bar{\alpha}|d\alpha. \quad (24)$$

We can express Eq. (24), from Eqs. (11) and (19), as

$$\hat{A} = \frac{1}{T} \sum_{i,j} \int_{\alpha_0}^{\alpha_0+T} \alpha \exp[i2\pi(r_j-r_i)\alpha/T]d\alpha|E_i\rangle\langle E_j|$$

$$= \alpha_0 + T/2 + i\hbar \sum_{i\neq j} \frac{\exp[-i\hbar^{-1}(E_i-E_j)\alpha_0]}{E_i-E_j} |E_i\rangle\langle E_j|. \quad (25)$$

Thus

$$[\hat{E},\hat{A}] = i\hbar \sum_{i \neq j} \exp(-i\hbar^{-1}E_i\alpha_0)|E_i\rangle\langle E_j|\exp(iE_j\alpha_0)$$

$$= i\hbar\left(\sum_i \exp(-i\hbar^{-1}E_i\alpha_0)|E_i\rangle\right.$$

$$\left.\times \sum_j \langle E_j|\exp(iE_j\alpha_0) - \hat{1}\right), \tag{26}$$

which can be written from Eq. (19) simply as

$$[\hat{E},\hat{A}] = i\hbar(|\bar{\alpha}_0\rangle\langle\bar{\alpha}_0| - \hat{1}). \tag{27}$$

Thus, from Eq. (23)

$$\frac{d\langle\alpha\rangle}{dt} = 1 - \langle f|\bar{\alpha}_0\rangle\langle\bar{\alpha}_0|f\rangle. \tag{28}$$

From Eq. (18) the last term of Eq. (28) is just $TP(\alpha_0)$; thus, if the probability distribution $P(\alpha)$ for the state $|f\rangle$ of the system is sufficiently narrow for this last term to be negligible, the expectation value of the α quantity changes exactly as the time parameter. If the system is in an eigenstate of energy, however, it is clear from Eqs. (19) and (28) that the rate of change of $\langle\alpha\rangle$ is zero. This must be true, of course, as an energy eigenstate is a stationary state. Since no observable quantity of a stationary state varies with time its α quantity should not vary. This is a good example displaying the difference between the α quantity and the time parameter. It also shows the need for the second term in Eq. (28) and the inadequacy of a commutator similar to Eq. (2) that would predict $d\langle\alpha\rangle/dt = 1$ for all states. A particular example of a system for which $d\langle\alpha\rangle/dt \approx 1$ is a harmonic oscillator in a truncated phase state [7] where the truncation is at a very large excitation number. For such a state $r_i = i$ and p is very large. The mean α quantity of this system changes directly as the time for most of the time and so the harmonic oscillator makes a useful quantum clock. We note that even for this system, this direct variation with time cannot always apply. When the state eventually evolves to have a large overlap with the state $|\alpha_0\rangle$, $d\langle\alpha\rangle/dt$ suddenly becomes very large and negative and $\langle\alpha\rangle$ quickly reverts to the value it had one period of the oscillator earlier. This is just equivalent to the seconds reading on a digital clock jumping from 59 back to 00. On the other hand, $\langle\alpha\rangle$ of a hydrogen atom in a superposition of some of its lower energy states will not be so directly related to the time during one period and the variation of $\langle\alpha\rangle$ with time will be more complicated.

IV. UNCERTAINTY RELATION

As the α quantity is represented by a probability-operator measure and its variance is found from the associated probability distribution, its uncertainty relation with energy cannot be written down immediately as is the case for observables represented by Hermitian operators. Using Eq. (20) we can, however, write the variance of energy as

$$\langle\Delta E^2\rangle = \int_{\alpha_0}^{\alpha_0+T} \langle f|(\hat{E}-\langle E\rangle)|\bar{\alpha}\rangle\langle\bar{\alpha}|(\hat{E}-\langle E\rangle)|f\rangle d\alpha/T. \tag{29}$$

Combining this with the variance of the α quantity obtained from Eq. (18) gives

$$\langle\Delta E^2\rangle\langle\Delta\alpha^2\rangle = \int_{\alpha_0}^{\alpha_0+T} |\langle f|(\hat{E}-\langle E\rangle)|\bar{\alpha}\rangle|^2 d\alpha$$

$$\times \int_{\alpha_0}^{\alpha_0+T} |\langle\bar{\alpha}|f\rangle(\alpha-\langle\alpha\rangle)|^2 d\alpha/T^2$$

$$\geq \left|\int_{\alpha_0}^{\alpha_0+T} \langle f|(\hat{E}-\langle E\rangle)|\bar{\alpha}\rangle\right.$$

$$\left.\times \langle\bar{\alpha}|f\rangle(\alpha-\langle\alpha\rangle)d\alpha/T\right|^2 \tag{30}$$

from Schwarz's inequality. Thus the product of the mean square deviations is

$$\Delta E\Delta\alpha \geq |\langle f|(\hat{E}-\langle E\rangle)(\hat{A}-\langle\alpha\rangle)|f\rangle|. \tag{31}$$

As α is real, operator $\hat{A}$ defined by Eq. (24) will be Hermitian and we can use the usual inequality [8] to obtain

$$\Delta E\Delta\alpha \geq \tfrac{1}{2}|\langle f|[\hat{E},\hat{A}]|f\rangle| = \frac{\hbar}{2}|1 - \langle f|\bar{\alpha}_0\rangle\langle\bar{\alpha}_0|f\rangle|. \tag{32}$$

When the system is in an energy eigenstate the uncertainty in energy must be zero, even for a finite $\Delta\alpha$. In this case the right-hand side of Eq. (32) vanishes, ensuring consistency. Thus the energy is exactly measurable in principle and does not suffer from the difficulty discussed in Ref. [1] associated with an uncertainty relation based on Eq. (2). As with Eq. (28), $\langle f|\bar{\alpha}_0\rangle\langle\bar{\alpha}_0|f\rangle$ is just $TP(\alpha_0)$ and so if $|f\rangle$ is orthogonal to $|\alpha_0\rangle$ or at least if the probability distribution $P(\alpha)$ for the state $|f\rangle$ of the system is sufficiently narrow for this last term to be negligible, the uncertainty product takes the more usual form. This is the same condition for the rate of change of $\langle\alpha\rangle$ with time to be unity as discussed earlier. Only under these special circumstances could a commutation relation as given by Eq. (2) be used.

V. THE SPACE Ψ_s

In this section we examine the possible existence of an Hermitian operator to represent the α quantity. In performing the preceding calculations we have found it convenient to define a Hermitian operator $\hat{A}$ by Eq. (24) with a more explicit form given by Eq. (25). It follows immediately from Eqs. (24) and (18) that

$$\langle\alpha\rangle = \langle f|\hat{A}|f\rangle \tag{33}$$

and the rate of change of $\langle f|\hat{A}|f\rangle$ will be the same as that of $\langle\alpha\rangle$. From Eq. (27) the uncertainty relation for the observable represented by $\hat{A}$ and the energy will be the same as Eq. (32). Furthermore, $\hat{A}$ is well defined by Eqs. (24) and (19) even for quantum systems whose energy eigenvalues E_i are not rational multiples of E_1. In view of these properties, why do we not simply take the observable represented by $\hat{A}$ as the complement of the Hamiltonian? The essential reason is that the eigenstates of $\hat{A}$ are not the states $|\alpha\rangle$ and the Hamil-

tonian does not generate shifts from one eigenstate of $\hat{A}$ to another in a manner similar to Eq. (6). We note that while Eq. (33) is true, in general $\langle f|\hat{A}^2|f\rangle \neq \langle \alpha^2 \rangle$ as the states $|\bar{\alpha}\rangle$ are not orthogonal and so, although the variance of $\hat{A}$ satisfies the same uncertainty inequality as $\Delta\alpha^2$, these two variances are not in general equal. Thus, although $\hat{A}$ is interesting in its own right, at best it can only be regarded as conjugate to $\hat{E}$ in a weak sense of satisfying the commutation relation (27). Its expectation value is equal to the expectation value of the α quantity, so it might provide a way of measuring this value, but it can be regarded only as the operator acting on Ψ_p that is nearest to an operator conjugate to $\hat{E}$.

We may, on the other hand, be able to identify the actual Hermitian operator representing the α quantity if we are guided by the work of Naimark [6], which indicates that a general resolution of the identity such as Eq. (15) can be extended to an orthogonal resolution of the identity in a *larger* space Ψ_s of which Ψ_p is a subspace. Returning to Eq. (15), we note that α_m has $s+1$ values, where $s+1 > r_p$, the largest of the integers r_i. We thus expect the state space Ψ_s that we seek to have $s+1$ dimensions spanned by $s+1$ eigenstates of the Hermitian operator of this space that represents the α quantity. The energy basis of Ψ_s will therefore also include extra states. As Ψ_p includes all the accessible states of the system, the extra states of Ψ_s will not be accessible, perhaps not even physically accessible. For example, if the matrix elements of any physical interaction Hamiltonian between the states of Ψ_p and the states of Ψ_s orthogonal to Ψ_p are all zero, the system can never evolve from the superposition (4) to include any of the extra states as components. Thus we still use the superposition (4) to describe a general physical state of the system.

We seek orthogonal states $|\theta_m\rangle$ in the space Ψ_s such that

$$\sqrt{\frac{p+1}{s+1}}\,|\alpha_m\rangle = \hat{1}|\theta_m\rangle, \tag{34}$$

where $\hat{1}$ is the unit operator for the space Ψ_p as used in Eq. (15), and such that

$$\sum_{m=0}^{s} |\theta_m\rangle\langle\theta_m| = \hat{I}, \tag{35}$$

where $\hat{I}$ is the unit operator for the space Ψ_s. Then operating on Eq. (35) from the left and from the right with $\hat{1}$ will yield Eq. (15). We can write the states we seek as

$$|\theta_m\rangle = \frac{1}{\sqrt{s+1}} \sum_{n=0}^{s} \exp(-i2\pi n\alpha_m/T)|n\rangle, \tag{36}$$

where $|n\rangle = |E_i\rangle$ for $n = r_i$. It can be shown that these states are orthogonal and satisfy Eq. (35) and, from Eqs. (9) and (11), also Eq. (34). To check Eq. (34) use can be made of the property that all states of Ψ_p are eigenstates of $\hat{1}$ with eigenvalue unity and states orthogonal to Ψ_p are eigenstates of $\hat{1}$ with eigenvalue zero. To check Eq. (35) it is useful to write the states as

$$|\theta_m\rangle = \frac{1}{\sqrt{s+1}} \sum_{n=0}^{s} \exp(-in\theta_m)|n\rangle, \tag{37}$$

where

$$\theta_m = 2\pi\alpha_m/T, \tag{38}$$

which lie in a 2π range between θ_0 and $\theta_0 + 2\pi$, with $\theta_m = \theta_0 + m2\pi/(s+1)$.

The states $|\theta_m\rangle$ form a complete orthonormal basis for the space Ψ_s. The space Ψ_s is mathematically equivalent to the finite $(s+1)$-dimensional space used to examine the harmonic oscillator, or quantized electromagnetic field mode, in Ref. [7] with angular frequency ω given by $2\pi/T$. Apart from an unimportant change in the sign of i, these states can be seen to be the same as the phase states of Ψ_s [7] and are the eigenstates of the operator

$$\hat{\phi}_\theta = \sum_{m=0}^{s} \theta_m |\theta_m\rangle\langle\theta_m|, \tag{39}$$

with eigenvalues θ_m, and the eigenstates of the associated Hermitian operator

$$\hat{\tau} = \hat{\phi}_\theta T/2\pi, \tag{40}$$

with eigenvalues α_m. With the values α_m given by Eq. (13), these are now seen to be the eigenvalues of $\hat{\tau}$, which acts on the space Ψ_s. Each term of Eq. (17), that is, $|\langle f|\alpha_m\rangle|^2(p+1)/(s+1)$, is seen to be the probability of projecting the state $|f\rangle$ onto the state $|\theta_m\rangle$, that is, it is the probability of obtaining the result α_m by a measurement of the Hermitian operator $\hat{\tau}$ given by Eq. (40). This is in accord with the previous interpretation we gave for the terms of Eq. (17), but we can now link the measurement to the Hermitian operator $\hat{\tau}$.

The conjugate of Eq. (39) is the number operator [5]

$$\hat{N} = \sum_{n=0}^{s} n|n\rangle\langle n|, \tag{41}$$

from which we can define an operator

$$\hat{H}_s = \hat{N}2\pi\hbar/T, \tag{42}$$

for which $\hat{1}\hat{H}_s\hat{1} = \hat{E}$. Ψ_s will also be spanned by the $s+1$ eigenstates $|n\rangle$ of $\hat{H}_s$ with $n = 0,1,...,s$, with a uniform energy difference of $2\pi\hbar/T$ between successive states. States $|n\rangle$ for which $n = r_i$ will be the same as the states $|E_i\rangle$ of Ψ_p. It is not difficult to show that $\hat{\tau}$ is the conjugate, or complement, of $\hat{H}_s$ in the strict sense of Ref. [5], that is, $\hat{H}_s$ is the generator of shifts in the values of α_m and $\hat{\tau}$ is the generator of energy shifts. The first property follows from Eq. (36),

$$\exp\{-i\hbar^{-1}\hat{H}_s[T/(s+1)]\}|\theta_m\rangle = |\theta_{m+1}\rangle, \tag{43}$$

and the second follows from the relation

$$\exp[i\hbar^{-1}\hat{\tau}(2\pi\hbar/T)] = \exp(i\hat{\phi}_\theta). \tag{44}$$

Because the right-hand side of Eq. (44) shifts a number state $|n\rangle$ to a neighboring number state [7], the left-hand side shifts an eigenstate of $\hat{H}$ to a neighboring eigenstate. In Eq. (43), $T/(s+1)$ is the difference in α between successive τ states and, from Eq. (42), $2\pi\hbar/T$ in Eq. (44) is the energy step size.

From Eqs. (42), (41), and (40) and with Eq. (36) substituted into Eq. (39), we find eventually

$$[\hat{\tau},\hat{H}_s]=\frac{2\pi\hbar}{s+1}$$
$$\times\sum_{n\neq n'}\frac{(n-n')\exp[i(n-n')2\pi\alpha_0/T]|n'\rangle\langle n|}{\exp[i(n-n')2\pi/(s+1)]-1},$$
(45)

which, apart from the sign of i, is just $\hbar$ times the expression for $[\hat{\varphi}_\theta,\hat{N}]$ given in [7].

For consistency with our previous definition of the α quantity, we must define $\hat{\tau}$ to represent the α quantity in the limiting sense of $s+1$ being extremely large, that is, $s+1 \gg r_p$. Then the difference between successive values of the α quantity becomes vanishingly small. As with the phase problem [7], we must be careful how we take the infinite-s limit however. We can ensure consistency with the probability distribution derived from Eq. (18) by taking the limit of expectation values of functions of the operator rather than the limit of the operator itself.

The uncertainty relation derived from the commutator (45) is

$$\Delta H_s \Delta\tau \geq \tfrac{1}{2}|\langle f|[\hat{\tau},\hat{H}_s]|f\rangle|,$$
(46)

where ΔH_s and $\Delta\tau$ are the uncertainties in the observables represented by $\hat{H}_s$ and $\hat{\tau}$. The rate of change of the expectation value is

$$\frac{d\langle\tau\rangle}{dt}=-\frac{i}{\hbar}\langle f|[\hat{\tau},\hat{H}_s]|f\rangle.$$
(47)

The state $|f\rangle$, given by Eq. (4), contains only the states $|E_i\rangle$ of Ψ_p so, when Eq. (45) is substituted, all terms on the right-hand sides of Eqs. (46) and (47) will be zero for $n,n' > r_p$. In our limiting case where $s+1 \gg r_p$, we find

$$\langle f|[\hat{\tau},\hat{H}_s]|f\rangle=-i\hbar\sum_{i\neq k}\exp[i(r_k-r_i)2\pi\alpha_0/T]$$
$$\times\langle f|E_i\rangle\langle E_k|f\rangle$$
$$=i\hbar-i\hbar|\langle f|\bar{\alpha}_0\rangle|^2$$
(48)

from Eqs. (9), (13), (11), and (19). It is straightforward to show for the state $|f\rangle$ that $\Delta H_s = \Delta E$ and in the infinite-s limit that $\langle\tau\rangle=\langle\alpha\rangle$. By using the orthogonality of the states $|\theta_m\rangle$ to write from Eqs. (39) and (40)

$$\hat{\tau}^2=\sum_{m=0}^{s}\alpha_m^2|\theta_m\rangle\langle\theta_m|,$$
(49)

we can show that in the infinite-s limit

$$\langle f|\hat{\tau}^2|f\rangle=\langle\alpha^2\rangle,$$
(50)

where $\langle\alpha^2\rangle$ is the value derived from Eq. (18). It then follows that from Eqs. (46)–(48) we regain the previous expressions (28) and (32) for the rate of change and uncertainty relation for the α quantity. In general, expression (50) is true upon replacing the square with any power so the Hermitian operator $\hat{\tau}$ is indeed a good representation of the α quantity.

We can now understand more clearly the role of the Hermitian operator $\hat{A}$. From Eqs. (25), (40), and (39) we find that $\hat{A}$ is the operator acting on space Ψ_p that has the same matrix elements as $\hat{\tau}$ for the states of Ψ_p in the infinite-s limit. This is analogous to finding the weak limit of the phase operator on the space of physical states [9]. For example, the difference between the mean squares $\langle f|\hat{A}^2|f\rangle$ and $\langle\alpha^2\rangle$ noted earlier, and hence between $\langle f|\hat{A}^2|f\rangle$ and $\langle f|\hat{\tau}^2|f\rangle$, has a parallel in the case of phase arising because the weak limit of the square of the operator is not equal to the square of the weak limit of the operator.

VI. CONCLUSION

The question we have addressed here is basically that if time is not complementary to energy, particularly for a system with discrete energy levels, then what is? Does such a quantity exist? A sensible quantity that is complementary to the energy of a quantum system would have dimensions of time, but would be a property of the system. It would not represent time in the abstract [1] or universal sense [10]. We would expect in some special cases the expectation value of this quantity to vary directly with the time, but not in general. For stationary states the expectation value of this quantity should not vary with time at all.

We have seen that for a quantum system with discrete energy eigenvalues E_i such that E_i/E_1 is rational or can be approximated sufficiently closely by a rational number, the complement of the Hamiltonian is the α quantity that can be represented by the probability-operator measure generated by the operators $|\bar{\alpha}\rangle\langle\bar{\alpha}|d\alpha$. In the $(p+1)$-dimensional minimal state space Ψ_p, which would usually be used to describe the quantum system with $p+1$ accessible energy states, we cannot construct an operator that is the complement of the energy operator $\hat{E}$ in the strict sense. The α quantity can, however, be represented by a Hermitian operator in a larger space Ψ_s with $s+1$ dimensions in the infinite-s limit, provided care is taken in finding this limit in that the limit of expectation values and not the operator itself is found.

We have been dealing with a quantum system with a finite number $p+1$ of energy levels. We can make $p+1$ as large as we please, provided we are careful how we take limits to ensure that $s \gg r_p$ in expressions involving both of these quantities. We have already used the infinite-s limit to derive expression (21) which, of course, is independent of s so we can use Eq. (21) if we wish to examine the properties of the α quantity for, for example, the state of a system involving a continuum of energy values approximated by a very large number of uniformly spaced levels.

In this paper we have limited our considerations to a quantum system for which the ratios E_i/E_1 are rational or at least are represented by rational numbers to a sufficiently

good approximation. In this case T is the recurrence period. The general case for a system with irrational energy ratios is not as straightforward. For example, if we apply Eq. (21), we have the problem of identifying what value of T to use. We shall not explore this general case in this paper, but we note that for any nonzero $E_k - E_i$ the left-hand side of Eq. (20) will approach the unit operator for any value of T as long as T is made sufficiently large. This suggests that a limiting approach may be useful. A further modification will be necessary in the event of a degeneracy, but again this is outside the scope of this paper.

Finally, there is the question of a suitable name for the quantity that we have referred to so far as simply the α quantity. This quantity has dimensions of time, but is not time. Rather it is a property of the quantum system and depends on the state of the system. For particular states the rate of change of its mean value can approach unity, but for an energy eigenstate, its mean value does not change with time, precisely as expected for a stationary state. To describe such a quantity the name ''age'' suggests itself. We could then say, for example, that a system in a stationary state would not age as time goes on.

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