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Casimir force between a dielectric sphere and a wall: A model for amplification of vacuum fluctuations

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The interaction between a polarizable particle and a reflecting wall is examined. A macroscopic approach is adopted in which the averaged force is computed from the Maxwell stress tensor. The particular case of a perfectly reflecting wall and a sphere with a dielectric function given by the Drude model is examined in detail. It is found that the force can be expressed as the sum of a monotonically decaying function of position and of an oscillatory piece. At large separations, the oscillatory piece is the dominant contribution and is much larger than the Casimir-Polder interaction that arises in the limit that the sphere is a perfect conductor. It is argued that this enhancement of the force can be interpreted in terms of the frequency spectrum of vacuum fluctuations. In the limit of a perfectly conducting sphere, there are cancellations between different parts of the spectrum that no longer occur as completely in the case of a sphere with frequency-dependent polarizability. Estimates of the magnitude of the oscillatory component of the force suggest that it may be large enough to be observable. [S1050-2947(98)03412-X]

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I. INTRODUCTION

It was noted some time ago that if one wishes to assign a frequency spectrum to the Casimir force between reflecting planar boundaries, the result is a wildly oscillating function of frequency [1,2]. The integral of this function over all frequencies can only be performed with the aid of a suitable convergence factor. The net Casimir energy is much smaller than the contribution of each individual oscillation peak. The effect of integration over all frequencies is almost, but not quite completely, to cancel the various frequency regions against one another. This leads to the speculation [3] that one might be able to upset this cancellation in some way and thereby greatly amplify the magnitude of the Casimir force and possibly change its sign.

In the case of parallel plane boundaries, no natural way to do this has been demonstrated. However, the Casimir-Polder interaction between a polarizable particle and a reflecting plane offers similar possibilities. Casimir and Polder [4] originally derived the potential between an atom in its ground state and a perfectly reflecting wall. In the large distance limit, their result takes the particularly simple form[1]

$$V_{CP} \sim -\frac{3\alpha_0}{8\pi z^4}, \tag{1}$$

where z is the distance to the wall and α_0 is the static polarizability of the atom. This asymptotic potential may be derived from the interaction Hamiltonian

$$H_{int} = -\frac{1}{2}\alpha_0 \mathbf{E}^2, \tag{2}$$

where $\mathbf{E}$ is the quantized electric-field operator. If one expands this operator in terms of a complete set of the Maxwell equations in the presence of the boundary, the asymptotic Casimir-Polder may be written as

$$\langle H_{int}\rangle = \frac{\alpha_0}{4\pi z^3}\int_0^\infty d\omega\, \sigma(\omega), \tag{3}$$

where

$$\sigma(\omega) = [(2\omega^2 z^2 - 1)\sin 2\omega z + 2\omega z \cos 2\omega z]. \tag{4}$$

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[1] Gaussian units with $c = \hbar = 1$ will be used in this paper.

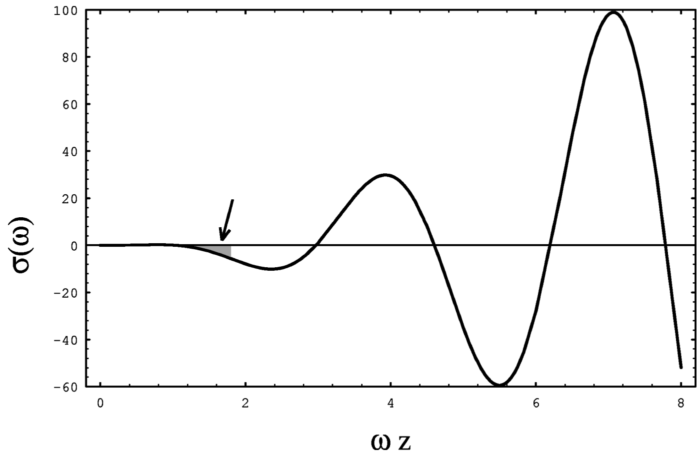

FIG. 1. Frequency spectrum $\sigma(\omega)$ for the Casimir-Polder potential. The oscillations almost exactly cancel, leaving a net area under the curve equal to that of the shaded region indicated by the arrow.

The integrand $\sigma(\omega)$ is an oscillatory function whose amplitude *increases* with increasing frequency. Nonetheless, the integral can be performed using a convergence factor (e.g., insert a factor of $e^{-\beta\omega}$ and then let $\beta \rightarrow 0$ after integration). The result is the right-hand side of Eq. (1). It is clear that massive cancellations have occurred (see Fig. 1) and that the area under an oscillation peak can be much greater in magnitude than the final result. This again raises the possibility of tampering with this delicate cancellation and dramatically altering the magnitude and sign of the force.

The purpose of this paper is to explore this question in the context of a specific model. The force between a dielectric sphere and a perfectly conducting plane will be examined. The polarizability of the sphere will be taken to be a function of frequency, thereby introducing the possibility of modifying the contributions of different parts of the spectrum. This or similar problems have been discussed before by several authors. However, it will be examined here from a different viewpoint. The force may be calculated from the Maxwell stress tensor. In Sec. II A, a formula for the force on a small sphere in an arbitrary applied electromagnetic field will be derived in an electric-dipole approximation and discussed. In Sec. III, this formula will be applied to the calculation of the force on a dielectric particle near an interface in terms of the Fresnel coefficients of the interface. This result will be applied to the case of a dielectric sphere and a perfectly reflecting boundary in Sec. IV. It will be shown that the force has a component that is an oscillatory function of position and that it is possible for the sphere to be in stable equilibrium at a finite distance from the boundary. The results are summarized and discussed in Sec. V.

II. FORCE ON A SMALL PARTICLE

A. Electric-dipole approximation

In this section we will discuss the force that an applied electromagnetic field exerts on a small dielectric sphere. The applied electric field will be taken to be $\mathbf{E}_a(\mathbf{x},t)$ and the corresponding magnetic field to be $\mathbf{B}_a(\mathbf{x},t)$. We assume that the induced (scattered) field is that of electric-dipole radiation from a time-varying dipole moment $\mathbf{p}$. Later $\mathbf{p}$ will be taken to be linearly related to $\mathbf{E}_a$, but for now it is unspecified. We further assume that the particle is small compared to the characteristic spatial scale over which $\mathbf{E}_a(\mathbf{x},t)$ and $\mathbf{B}_a(\mathbf{x},t)$ vary. The latter assumption is not really independent of the electric-dipole approximation: If the size of the sphere is not small then one would in general have to include the contributions of higher multipoles. Just outside the particle, the electric and magnetic fields due to the dipole take the near-zone forms

$$\mathbf{E}_d \approx \frac{3\hat{\mathbf{n}}(\hat{\mathbf{n}}\cdot\mathbf{p}-\mathbf{p})}{r^3}, \quad \mathbf{B}_d \approx -\frac{\hat{\mathbf{n}}\times\dot{\mathbf{p}}}{r^2}. \tag{5}$$

Here r is the radial distance from the dipole and $\hat{\mathbf{n}}$ is the outward directed unit normal vector.

The net force acting upon the particle can be calculated by integrating the Maxwell stress tensor over a spherical surface just outside the particle,

$$F^i = \oint da_j T^{ij}, \tag{6}$$

where

$$T^{ij} = \frac{1}{4\pi}\left[E^i E^j + B^i B^j - \tfrac{1}{2}\delta^{ij}(\mathbf{E}^2+\mathbf{B}^2)\right]. \tag{7}$$

If we insert the net fields $\mathbf{E}_a+\mathbf{E}_d$ and $\mathbf{B}_a+\mathbf{B}_d$ into this expression, there will be three types of terms: those involving only the applied fields, those involving only the dipole fields, and the cross terms. However, the pure dipole terms yield no net contribution. Furthermore, any force due to the pure applied field terms is independent of the polarizability and hence not of interest. Thus we consider only the cross terms in T^{ij} between the applied and dipole fields:

$$\mathbf{F} = \frac{1}{4\pi}\oint da[(\hat{\mathbf{n}}\cdot\mathbf{E}_a)\mathbf{E}_d + (\hat{\mathbf{n}}\cdot\mathbf{E}_d)\mathbf{E}_a + (\hat{\mathbf{n}}\cdot\mathbf{B}_a)\mathbf{B}_d \\ -\hat{\mathbf{n}}(\mathbf{E}_a\cdot\mathbf{E}_d + \mathbf{B}_a\cdot\mathbf{B}_d)]. \tag{8}$$

Note that $\hat{\mathbf{n}}\cdot\mathbf{B}_d = 0$.

Because the particle is assumed to be small, we may expand $\mathbf{E}_a$ and $\mathbf{B}_a$ in a Taylor series around $\mathbf{x}=\mathbf{x}_0$, the location of the particle. The leading nonzero contributions to the force come from the zeroth-order term in $\mathbf{B}_a$ and the first-order term in $\mathbf{E}_a$:

$$\mathbf{B}_a(\mathbf{x},t) \approx \mathbf{B}_0,$$

$$E_a^i(\mathbf{x},t) \approx E_0^i + r\hat{\mathbf{n}}\cdot\nabla E_0^i + \cdots. \tag{9}$$

We now insert these expansions and Eq. (5) into Eq. (8) and perform the angular integration, using the relation

$$\oint da\, n^i n^j = \frac{4\pi r^2}{3}\delta^{ij} \tag{10}$$

to find

$$F^i = \frac{2}{3}p^j \partial_j E_0^i + \frac{1}{3}p_j \partial^i E_0^j + \frac{2}{3}(\dot{\mathbf{p}}\times\mathbf{B}_0)^i. \tag{11}$$

It is of interest to check the static limit of this expression. In this limit, $\dot{\mathbf{p}}=\mathbf{0}$ and $\nabla\times\mathbf{E}_0=0$. If we use these relations and set $\mathbf{p}=\alpha_0\mathbf{E}_0$, where α_0 is the static polarizability of the particle, the result is

$$F^i = \alpha_0 p_j \partial^i E_0^j = \frac{1}{2}\alpha_0 \partial^i \mathbf{E}_0^2. \tag{12}$$

This is equivalent to the familiar result that the interaction energy of an induced dipole with a static electric field is

$$V = -\frac{1}{2}\alpha_0 \mathbf{E}_0^2. \tag{13}$$

B. Interaction with a single plane wave

Here we apply the result, (11) to compute the force that a single, linearly polarized plane wave exerts on the particle. The electric and magnetic fields of this wave are given by

$$\mathbf{E}_0 = \mathrm{Re}(\hat{\boldsymbol{\epsilon}} A e^{i(\mathbf{k}\cdot\mathbf{x}_0-\omega t)}) = \hat{\boldsymbol{\epsilon}} A \cos(\mathbf{k}\cdot\mathbf{x}_0-\omega t),$$

$$\mathbf{B}_0 = \hat{\mathbf{k}}\times\hat{\boldsymbol{\epsilon}} A \cos(\mathbf{k}\cdot\mathbf{x}_0-\omega t), \tag{14}$$

where A is the amplitude and $\hat{\boldsymbol{\epsilon}}$ the polarization vector. The dipole moment is given by

$$\mathbf{p} = \mathrm{Re}(\alpha\mathbf{E}_0) = \hat{\boldsymbol{\epsilon}} A|\alpha|\cos(\mathbf{k}\cdot\mathbf{x}_0-\omega t+\gamma), \tag{15}$$

where

$$\alpha = |\alpha| e^{i\gamma} = \alpha_1 + i\alpha_2. \tag{16}$$

We are interested in the time-averaged force, measured over time scales long compared to $1/\omega$; so we henceforth understand F^i to be the time average of Eq. (11). In the present case this yields

$$\mathbf{F} = \frac{1}{2}\mathbf{k}A^2|\alpha|\sin\gamma = \frac{1}{2}\mathbf{k}A^2\alpha_2, \tag{17}$$

a force proportional to the imaginary part of the polarizability α_2. This result may be given a simple physical interpretation. The rate at which electromagnetic energy is dissipated is given by the usual Joule heating term

$$\dot{W} = \int \mathbf{J}\cdot\mathbf{E}\, d^3x, \tag{18}$$

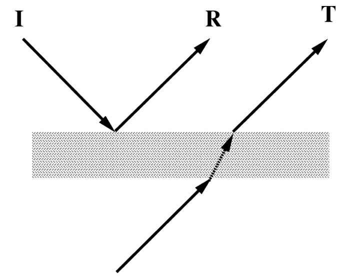

FIG. 2. Propagating modes above an interface consist of incident I and reflected R waves or transmitted T waves.

where $\mathbf{J}$ is the current density and the integration is taken over the volume of the particle. Because the electric field is approximately constant over this volume and because one may show [5] from the continuity equation that

$$\int \mathbf{J}\, d^3x = \dot{\mathbf{p}}, \tag{19}$$

we have that the time-averaged power absorbed by the particle is

$$\dot{W} = \frac{1}{2}\omega A^2|\alpha|\sin\gamma. \tag{20}$$

However, each photon carries energy ω and momentum $\mathbf{k}$, so the right-hand side of Eq. (17) is just the rate at which momentum is being absorbed by the particle due to the absorption of photons. There is of course also some momentum being transferred as a result of photon scattering. However, that effect is proportional to α^2 and is being neglected here.

III. FORCE ON A PARTICLE NEAR AN INTERFACE

In this section we will derive a formula for the Casimir force on a polarizable particle in the presence of a single plane interface. The interface will be assumed to have arbitrary reflectivity. We will, however, work in an approximation in which evanescent modes are neglected. The quantized electromagnetic field is to be expanded in a complete set of normalized solutions of the Maxwell equations. These solutions fall into three classes: (i) modes that are in the region above the interface and consist of an incident and a reflected part, as illustrated in Fig. 2; (ii) modes that originate on the far side of the interface and are outwardly propagating transmitted waves in the region above the interface; and (iii) evanescent modes that are propagating inside the material comprising the interface, but are exponentially decaying in the region above it. These last modes will be left out of the present discussion.

Let us focus first on the reflected modes in class (i). The net electric field is

$$\mathbf{E} = \mathbf{E}_I + \mathbf{E}_R, \tag{21}$$

where the incident wave is

$$\mathbf{E}_I = \hat{\boldsymbol{\epsilon}} A \cos(\mathbf{k}\cdot\mathbf{x}_0-\omega t) \tag{22}$$

and the reflected wave is

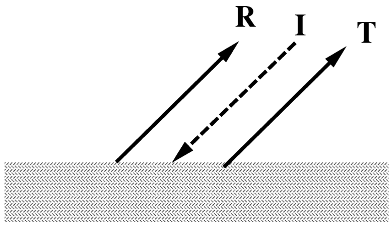

FIG. 3. Force due to an incident wave I, canceled by the sum of the forces due to a reflected wave R and a transmitted wave T.

$$\mathbf{E}_R = \hat{\boldsymbol{\epsilon}}' AR \cos(\mathbf{k}'\cdot\mathbf{x}_0 - \omega t + \delta). \tag{23}$$

The associated magnetic fields are $\mathbf{B}_I = \hat{\mathbf{k}}\times\mathbf{E}_I$ and $\mathbf{B}_R = \hat{\mathbf{k}}'\times\mathbf{E}_R$, respectively. Here the complex reflection (Fresnel) coefficient is

$$\mathcal{R} = Re^{i\delta}, \tag{24}$$

where R is the magnitude of the reflection coefficient and δ is the phase shift. This mode induces a dipole moment $\mathbf{p} = \mathrm{Re}(\alpha\mathbf{E})$, where $\alpha = |\alpha| e^{i\gamma}$ is again the complex polarizability. The portions of $\mathbf{p}$ arising from the incident and reflected waves are, respectively,

$$\mathbf{p}_I = \hat{\boldsymbol{\epsilon}} A \cos(\mathbf{k}'\cdot\mathbf{x}_0 - \omega t + \gamma) \tag{25}$$

and

$$\mathbf{p}_R = \hat{\boldsymbol{\epsilon}} AR|\alpha| \cos(\mathbf{k}\cdot\mathbf{x}_0 - \omega t + \delta + \gamma). \tag{26}$$

The force that a particular mode exerts on the polarizable particle is obtained by inserting the above expressions for the fields and dipole moment into Eq. (11). The resulting expression should then be summed over all modes. However, it is simpler first to combine it with the corresponding expression arising from the transmitted waves of class (ii). In the region above the interface, the electric field of these modes is of the form

$$\mathbf{E}_T = \hat{\boldsymbol{\epsilon}} AT \cos(\mathbf{k}\cdot\mathbf{x}_0 - \omega t), \tag{27}$$

where T is a transmission coefficient. Here we may think of the interface as being a slab of finite thickness. Below the slab, these modes have the same form as the incident waves above the slab [Eq. (22)]. If the material in the slab is nonabsorptive, then the transmission and reflection coefficients satisfy

$$T^2 + R^2 = 1. \tag{28}$$

The force due to the modes of class (i) can be expressed as a sum of three types of terms: those involving only the incident wave, those involving only the reflected wave, and cross terms between the two. (See Fig. 3.) The first two types of contributions are of the form discussed in Sec. II for a single plane wave, as are the contributions due to the class (ii) transmitted waves. As a consequence of the relation (28), these three sets of contributions cancel one another, leaving only the incident-reflected-wave cross terms. The resulting force for a single mode is

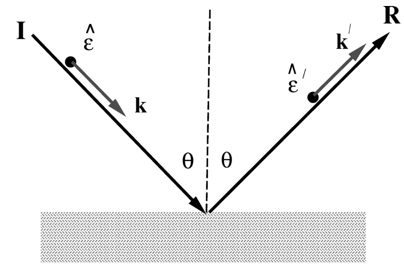

FIG. 4. Wave vectors $\mathbf{k}$ and $\mathbf{k}'$ and polarization vectors $\hat{\epsilon}$ and $\hat{\epsilon}'$ for the incident and reflected parts of an S-polarized wave.

$$F^i = \frac{2}{3}(p_I^j\partial_j E_R^i + p_R^j \partial_j E_I^i) + \frac{1}{3}(p_{Ij}\partial^i E_R^j + p_{Rj}\partial^i E_I^j) + \frac{2}{3}[(\dot{\mathbf{p}}_I\times\mathbf{B}_R)^i + (\dot{\mathbf{p}}_R\times\mathbf{B}_I)^i]. \tag{29}$$

We next insert the explicit forms for the fields and dipole moment and then average the resulting expression over time. The result is

$$\begin{aligned} F^i = \frac{1}{6}A^2 R\{ &\alpha_1[(k^i - k'^i)(\hat{\boldsymbol{\epsilon}}\cdot\hat{\boldsymbol{\epsilon}}') + 2\hat{\epsilon}^i(\mathbf{k}\cdot\hat{\boldsymbol{\epsilon}}') - 2\hat{\epsilon}'^i(\mathbf{k}'\cdot\hat{\boldsymbol{\epsilon}}) \\ &+ 2\omega\hat{\boldsymbol{\epsilon}}'\times(\mathbf{k}\times\hat{\boldsymbol{\epsilon}}) - 2\omega\hat{\boldsymbol{\epsilon}}\times(\mathbf{k}'\times\hat{\boldsymbol{\epsilon}}')]\sin\Delta \\ &+ \alpha_2[(k^i + k'^i)(\hat{\boldsymbol{\epsilon}}\cdot\hat{\boldsymbol{\epsilon}}') + 2\hat{\epsilon}^i(\mathbf{k}\cdot\hat{\boldsymbol{\epsilon}}') + 2\hat{\epsilon}'^i(\mathbf{k}'\cdot\hat{\boldsymbol{\epsilon}}) \\ &+ 2\omega\hat{\boldsymbol{\epsilon}}'\times(\mathbf{k}\times\hat{\boldsymbol{\epsilon}}) + 2\omega\hat{\boldsymbol{\epsilon}}\times(\mathbf{k}'\times\hat{\boldsymbol{\epsilon}}')]\cos\Delta\}. \end{aligned} \tag{30}$$

Here $\Delta = (\mathbf{k}' - \mathbf{k})\cdot\mathbf{x}_0 + \delta$ is the phase difference between the incident and reflected waves at the location of the particle.

Let us further evaluate this expression. Let the z direction be perpendicular to the interface and let θ be the angle of incidence. Then

$$k_z' = -k_z = \omega c, \tag{31}$$

where $c = \cos\theta$. Furthermore,

$$\Delta = 2k_z' z + \delta = 2\omega z c + \delta. \tag{32}$$

We must now specify the polarization state. We adopt a linear polarization basis, using the usual S ($\hat{\boldsymbol{\epsilon}}$ perpendicular to the plane of incidence) and P ($\hat{\boldsymbol{\epsilon}}$ parallel to the plane of incidence) states. For S polarization (Fig. 4) we have

$$\hat{\boldsymbol{\epsilon}}' = \hat{\boldsymbol{\epsilon}} \tag{33}$$

and

$$\hat{\boldsymbol{\epsilon}}\times(\mathbf{k}\times\hat{\boldsymbol{\epsilon}}) = \mathbf{k}. \tag{34}$$

Only the z component of the force will remain after summation over all modes; so we need only consider that component. For S polarization we find

$$F_S^z = -A^2 R_S \alpha_1 c \sin\Delta. \tag{35}$$

For P polarization (Fig. 5) we have

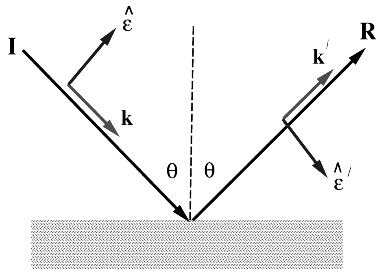

FIG. 5. Wave vectors and polarization vectors for a P-polarized wave.

$$\hat{\boldsymbol{\epsilon}}\cdot\hat{\boldsymbol{\epsilon}}'=\cos 2\theta, \tag{36}$$

$$\hat{\boldsymbol{\epsilon}}\cdot\hat{\mathbf{k}}'=\hat{\boldsymbol{\epsilon}}'\cdot\hat{\mathbf{k}}=\sin 2\theta, \tag{37}$$

$$\hat{\boldsymbol{\epsilon}}\times(\mathbf{k}'\times\hat{\boldsymbol{\epsilon}}')=-\mathbf{k}, \tag{38}$$

$$\hat{\boldsymbol{\epsilon}}'\times(\mathbf{k}\times\hat{\boldsymbol{\epsilon}})=-\mathbf{k}', \tag{39}$$

and

$$\epsilon_z=-\epsilon_z'=\sin\theta. \tag{40}$$

With the aid of these relations, Eq. (30) can be written for the case of P polarization as

$$F_P^z=A^2R_P\alpha_1c(1-2c^2)\sin\Delta. \tag{41}$$

Note that the force produced by the interference of incident and reflected waves depends upon α_1, the real part of the polarizability, rather than on the imaginary part as in Eq. (17).

The net force is obtained by integration of $F_S^z+F_P^z$ over all modes for which $k_z\leqslant 0$:

$$F=\int d^3k(F_S^z+F_P^z)=2\pi\int_0^\infty d\omega\,\omega^2\int_0^1 dc(F_S^z+F_P^z). \tag{42}$$

The modes are correctly normalized if we set

$$A^2=\frac{4\pi\omega}{(2\pi)^3}. \tag{43}$$

This leads to our final result for the force in the direction away from the interface:

$$F=\frac{1}{\pi}\int_0^\infty d\omega\,\omega^4\alpha_1(\omega)\int_0^1 dc\,c \times[-R_S\sin(2\omega zc+\delta_S)+R_P(1-2c^2)\sin(2\omega zc+\delta_P)]. \tag{44}$$

It is of interest to note that this result may also be derived from an effective interaction Hamiltonian of the form of Eq. (2), except with the static polarizability α_0 replaced by the real part of the dynamic polarizability $\alpha_1(\omega)$. The interaction potential is given by first-order perturbation theory [6,7] to be

$$V=\langle H_{int}\rangle=\frac{1}{2\pi}\int_0^\infty d\omega\,\omega^3\alpha_1(\omega)\times\int_0^1 dc[-R_S\cos(2\omega zc+\delta_S)+R_P(1-2c^2)\cos(2\omega zc+\delta_P)], \tag{45}$$

so that

$$F=-\nabla V. \tag{46}$$

IV. THE FORCE BETWEEN A DIELECTRIC SPHERE AND A PERFECTLY CONDUCTING PLANE

Let us consider the limit of Eq. (44) in which the interface is a perfect conductor. In this limit we have

$$R_S=R_P=1 \tag{47}$$

and

$$\delta_S=\delta_P=\pi. \tag{48}$$

This leads to

$$\begin{aligned}F&=\frac{2}{\pi}\int_0^\infty d\omega\,\omega^4\alpha_1(\omega)\int_0^1 dc\,c^3\sin(2\omega zc)\\&=-\frac{1}{4\pi z^4}\int_0^\infty d\omega\,\alpha_1(\omega)[3\sin 2\omega z-6z\omega\cos 2\omega z\\&\quad-6z^2\omega^2\sin 2\omega z+4z^3\omega^3\cos 2\omega z].\end{aligned} \tag{49}$$

Note that there are no evanescent modes in this case, so the previous approximation of ignoring such modes is not needed here.

Now consider a sphere of radius a composed of a uniform material with dielectric function $\varepsilon(\omega)$. The complex polarizability is given by

$$\alpha(\omega)=a^3\frac{\varepsilon(\omega)-1}{\varepsilon(\omega)+2}. \tag{50}$$

We will take the dielectric function to be that of the Drude model,

$$\varepsilon(\omega)=1-\frac{\omega_p^2}{\omega(\omega+i\gamma)}, \tag{51}$$

where ω_p is the plasma frequency and γ is the damping parameter. From Eqs. (50) and (51) we find that the real part of the polarizability is given by

$$\alpha_1=a^3\omega_p^2\frac{\omega_p^2-3\omega^2}{(3\omega^2-\omega_p^2)^2+9\omega^2\gamma^2}. \tag{52}$$

Note that although $\alpha(\omega)$ has poles only in the lower half ω plane, its real part $\alpha_1(\omega)$ has poles in both the upper and lower half planes.

If we insert Eq. (52) into Eq. (49), we must evaluate the set of integrals

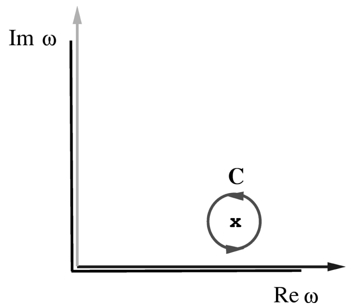

FIG. 6. Contours of integration for integrals of the form of Eq. (53). The integral on real ω can be expressed as a sum of an integral on imaginary ω plus a contribution C coming from the pole at $\omega=\Omega+\frac{1}{2}i\gamma$.

$$I_1=\int_0^\infty d\omega\ \alpha_1(\omega)\sin(2\omega z)=\mathrm{Im}\int_0^\infty d\omega\ \alpha_1(\omega)e^{2i\omega z}, \tag{53}$$

$$I_2=\frac{1}{2}\frac{dI_1}{dz}=\int_0^\infty d\omega\ \alpha_1(\omega)\omega\cos 2\omega z, \tag{54}$$

$$I_3=-\frac{1}{2}\frac{dI_2}{dz}=\int_0^\infty d\omega\ \alpha_1(\omega)\omega^2\sin 2\omega z, \tag{55}$$

and

$$I_4=\frac{1}{2}\frac{dI_3}{dz}=\int_0^\infty d\omega\ \alpha_1(\omega)\omega^3\cos 2\omega z. \tag{56}$$

In terms of these integrals, the force between the sphere and the plate is

$$F=-\frac{1}{4\pi z^4}(3I_1-6zI_2-6z^2I_3+4z^3I_4). \tag{57}$$

The second integral in Eq. (53) may be evaluated by rotating the contour of integration to the positive imaginary axis (Fig. 6). However, in this process we will also acquire a contribution from the residue of the pole of $\alpha_1(\omega)$ at $\omega=\Omega+\frac{1}{2}i\gamma$, where

$$\Omega=\frac{1}{6}\sqrt{12\omega_p^2-9\gamma^2}. \tag{58}$$

The result may be written as

$$I_1=J_1+P_1. \tag{59}$$

Here integrating over imaginary frequency yields

$$J_1=\int_0^\infty d\xi\ \alpha_1(i\xi)e^{-2\xi z}$$

$$=a^3\omega_p^2\int_0^\infty d\xi\frac{3\xi^2+\omega_p^2}{(3\xi^2+\omega_p^2)^2-9\xi^2\gamma^2}e^{-2z\xi}, \tag{60}$$

and the residue of the pole is

$$P_1=-\frac{\pi a^3\omega_p^2}{6\Omega}e^{-\gamma z}\cos 2\Omega z. \tag{61}$$

These results may be combined to obtain our final expression for the force between the sphere and the plate, which may be written as

$$F=J+P, \tag{62}$$

where J is the net contribution from integrals along the imaginary axis and P is that from the pole at $\omega=\Omega+\frac{1}{2}i\gamma$. The explicit forms of these two contributions are

$$J=-\frac{a^3\omega_p^2}{4\pi z^4}\int_0^\infty d\xi\frac{(3\xi^2+\omega_p^2)(4z^3\xi^3+6z^2\xi^2+6z\xi+3)}{(3\xi^2+\omega_p^2)^2-9\xi^2\gamma^2}\times e^{-2z\xi} \tag{63}$$

and

$$P=-\frac{a^3\omega_p^2}{48\Omega z^4}e^{-\gamma z}[2\Omega z(4\Omega^2z^2-3\gamma^2z^2-6\gamma z-6)\sin 2\Omega z+(12\gamma\Omega^2z^3-\gamma^3z^3+12\Omega^2z^2-3\gamma^2z^2-6\gamma z-6)\cos 2\Omega z]. \tag{64}$$

(Here and at other points in this paper the calculations were performed with the aid of the symbolic algebra program MACSYMA.)

In the case that $\gamma=0$, the integral for J may be evaluated in terms of sine and cosine integral functions. In the limit of small separations, one finds for this case that

$$J\sim a^3\omega_p\left(-\frac{\sqrt{3}}{8z^4}+\frac{\omega_p}{6\pi z^3}+O(z^{-1})\right) \tag{65}$$

and

$$P\sim a^3\omega_p\left(\frac{\sqrt{3}}{8z^4}+O(z^0)\right). \tag{66}$$

Thus the leading terms cancel and we find a repulsive force in this limit:

$$F\sim\frac{a^3\omega_p^2}{6\pi z^3}+O(z^{-1}),\quad a\ll z\ll\omega_p^{-1}. \tag{67}$$

It is of particular interest that P contributes an oscillatory term to the force. In the large separation limit $z\gg 1/\omega_p$ we have that

$$J\sim-\frac{3a^3}{2\pi z^5}. \tag{68}$$

This is just the attractive force due to the asymptotic Casimir-Polder potential (1), where $\alpha_0=a^3$ is the static polarizability of the sphere. The oscillatory term becomes, in the large distance limit,

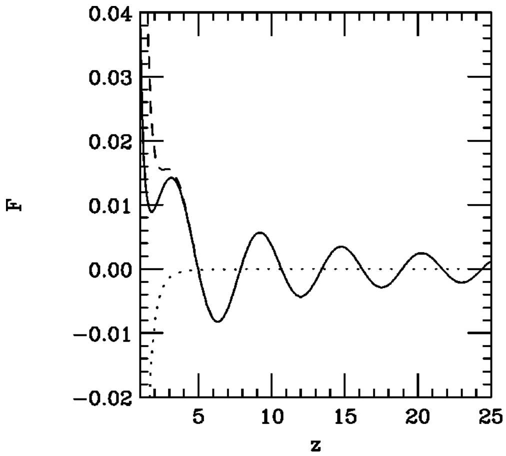

FIG. 7. Force F between a sphere and a perfectly reflecting wall for the case where $\gamma = 0.005\omega_p$. (Here F is in units of $\omega_p^5 a^3$ and z in units of ω_p^{-1}.) The stable equilibrium points are the zeros of F where the slope is negative. Here $F > 0$ corresponds to repulsion. The dotted line is the contribution of J, the imaginary frequency integral (63), and the dashed line is that of P, the pole contribution (64).

$$P \sim -\frac{\Omega \omega_p^2 a^3}{12z} e^{-\gamma z}(2\Omega \sin 2\Omega z + 3\gamma \cos 2\Omega z). \quad (69)$$

Although this term is exponentially decaying, it is possible for it still to be significant in the asymptotic region if, as is typically the case, $\gamma \ll \omega_p$. In this case, the oscillatory term P will dominate the Casimir-Polder term J and lead to a series of stable equilibrium points at finite distance from the boundary, separated from one another by a distance of approximately $\ell = \pi/\Omega$. A plot of the force at various separations is given in Fig. 7.

One might imagine trying to levitate the spheres in the Earth's gravitational field by this means. This will occur if $F_{max} \geqslant F_g$, where F_{max} is F evaluated at a peak value and F_g is the force of gravity. The ratio of these two forces may be expressed as

$$\frac{F_{max}}{F_g} \approx 27\left(\frac{\omega_p}{1\ \text{eV}}\right)^4 \left(\frac{1\ \mu\text{m}}{z}\right)\left(\frac{1\ \text{g/cm}^3}{\rho}\right) e^{-5(\gamma/1\ \text{eV})(z/1\ \mu\text{m})}, \quad (70)$$

where ρ is the mass density of the sphere. We have assumed that $\gamma \ll \omega_p$, so $\Omega \approx \sqrt{3}\omega_p/3$. Let $z = z_c$ be the distance at which this ratio of forces is unity and hence the maximum distance above the interface at which levitation can occur. In Table I values of z_c for various alkali metals are given, along with appropriate input parameters.

TABLE I. Parameters for some alkali metals. The Drude model parameters ω_p and γ, taken from Ref. [8], are in eV. The maximum levitation height z_c and the separation between equilibrium points ℓ are in μm.

Alkali metal	ρ	ω_p	γ	ℓ	z_c
Li	0.53	6.6	0.031	0.16	49
Na	0.97	5.6	0.028	0.19	46
K	0.86	3.8	0.021	0.28	47

The maximum elevation z_c at which a sphere could levitate is in the range of $46-49\ \mu$m. This is larger than the distance at which Casimir forces are usually expected to have a noticeable effect. Recall that all of the discussion in this paper is at zero temperature. Thermal effects at finite temperature can mask this vacuum energy effect. For example, for a sodium sphere of radius $a = 50$ nm near the maximum levitation height, the difference in potential energy between successive equilibrium points corresponds to a temperature of approximately 0.1 K, and would be observable only at low temperatures. On the other hand, the corresponding energy difference near the minimum levitation height is about 2000 K. Thus the first several equilibrium points should be observable at room temperature. The use of a perfectly reflecting wall should be a reasonable approximation as long as the plasma frequency of the material in the wall is large compared to that in the sphere. Thus a wall composed of aluminum ($\omega_p = 14.8$ eV) [9] is a good reflector at frequencies of the order of the plasma frequencies of the alkali metals.

V. DISCUSSION

In the preceding sections we have seen that a polarizable sphere with a dispersive polarizability in the vicinity of a perfectly reflecting boundary can experience a Casimir force that is much larger than would be experienced by a perfectly conducting sphere at the same separation. This can be understood in terms of the oscillatory frequency spectrum of vacuum energy effects. Cancellations between different parts of the spectrum that occur in the perfectly conducting limit seem to be upset by the dispersive properties of the sphere's material. A perfectly reflecting sphere would have a frequency-independent polarizability of $\alpha = \alpha_0 = a^3$ and the force exerted by the wall would be given by Eq. (68) at all separations. In addition to its amplification, the force now becomes an approximately oscillatory function of position, leading to the possibility of trapping the sphere in stable equilibrium.

Note that this type of oscillatory force does not arise in the case of a pair of half spaces of dielectric material separated by a gap. If the material in the half spaces is a homogeneous dielectric, whose dielectric function satisfies the Kramers-Kronig relations, then the Lifshitz theory [10] predicts a force of attraction that is always less than that in the case of two perfectly conducting planes. Apparently, the effect of the infinite spatial volume of the half spaces is to average over the spatial oscillations. A similar result was found recently by Lambrecht *et al.* [11] for the case of mirrors for a scalar field in one spatial dimension.

It is of interest to compare the macroscopic sphere problem discussed in this paper with the problem of an atom near a perfect mirror. The case where the atom is in the ground state was discussed in the original Casimir-Polder paper [4], where a monotonically decreasing expression was obtained that reduces to Eq. (1) in the large-z limit. This result is of the same form as the contribution J to the net force found in Sec. III coming from the integration over imaginary frequencies. Various authors [6,7,12] have treated the problem of a polarizable particle near an interface. However, these authors were primarily interested in the case where the polarizable particle is an atom in its ground state and hence included only the imaginary frequency contribution. The case of an atom in an excited state was treated by Barton [13] and other authors [14–16], who found that the potential now has an oscillatory component. Furthermore, this oscillatory term at large distances has a form similar to Eq. (69), with the magnitude of the oscillatory part decreasing as $1/z$. Thus at large separations, the net potential is dominated by this oscillatory term. In the case of the atom in an excited state, the oscillating potential can be given a classical interpretation. The atom behaves like a radiating antenna in the presence of a mirror. Such an antenna will experience an oscillatory back-reaction force whose sign depends upon whether the reflected wave interferes constructively or destructively with the original radiated wave. The oscillatory force found in the present problem does not seem to have such an interpretation because the dielectric sphere is not radiating. Nonetheless, it is plausible that there should be some parallels between an atom in an excited state and a macroscopic system such as the sphere with a continuum of quantum states just above the ground state.

The oscillatory force can be understood in this case as arising from a position dependence of the cancellation of the different parts of the frequency spectrum. One can see from Fig. 1 that a particle whose polarizability is nonzero only in a narrow band of frequency will experience an oscillatory force. (See Ref. [3] for further discussion of this point.) The delicate cancellation is perhaps one reason that it is difficult to predict the sign of a Casimir force in advance of an explicit calculation.

Finally, let us recall the assumptions that were employed in the analysis of this paper. The general formula for the force [Eq. (44)] was derived in Sec. III by assuming that the scattered wave is dipolar and there are no evanescent modes. The dipole approximation should be valid as long as the size of the particle is small compared to the wavelength of any modes that contribute significantly to Eq. (44). The assumption of no evanescent modes places some restrictions on the material of the interface. In particular, a perfectly conducting interface will have no evanescent modes. More generally, in frequency ranges in which the real part of the index of refraction is less than unity, there will be no such modes. This will be the case for all frequencies if the interface is composed of a metal for which the collisionless Drude model [Eq. (51) with $\gamma=0$] is a good approximation. In Sec. IV, we made some further approximations. These included the assumption that the particle is a small sphere whose dielectric function has the form given by the Drude model (51). Here the dipole approximation is expected to be valid when $a \ll \omega_p^{-1}$. A final approximation was made in assuming that the interface is perfectly conducting. This is expected to be valid when the interface is composed of a metal whose plasma frequency is large compared to that of the sphere. Then the dominant contributions to Eq. (44), those for which $\alpha_1 \neq 0$, come from modes for which Eqs. (47) and (48) are approximately valid. The extension of the results of this paper to the case where the interface is an imperfect reflector is currently under investigation.

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