Dynamical squeezing of photon-added coherent states

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We study the dynamical squeezing of the photon-added coherent state (PACS) due to a time dependence of the frequency of the electromagnetic field oscillator in a cavity or a vibrational frequency of an ion inside an electromagnetic trap. Explicit expressions for the time dependence of various functions characterizing the quantum state, such as the photon distribution, the Wigner function, the mean values and variances of the quadrature components and of the photon number, show that the dynamically squeezed PACS possesses a larger squeezing coefficient than the usual squeezed states. The dynamical squeezing is accompanied by a change of the sub-Poissonian photon statistics to the super-Poissonian one. [S1050-2947(98)07410-1]

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I. INTRODUCTION

Different types of nonclassical states in quantum optics were studied intensively during recent years (see, e.g. [1], and references therein). The most known examples include, in particular, the Titulaer-Glauber generalized coherent states [2], squeezed states [3], even and odd coherent states [4], displaced and squeezed number states [5], and binomial states [6]. Wide new families of nonclassical states were discussed recently in [7].

An interesting class of nonclassical states consists of the *photon-added* states

$$|\psi,m\rangle=\mathcal{N}_m\hat{a}^{\dagger m}|\psi\rangle, \tag{1}$$

where $|\psi\rangle$ may be an arbitrary quantum state, $\hat{a}^\dagger$ is the boson creation operator, m is a positive integer—the number of added quanta, and $\mathcal{N}_m$ is a normalization constant. For the first time these states were introduced by Agarwal and Tara [8] as the *photon-added coherent states* (PACS) $|\alpha,m\rangle$, i.e., for the initial Glauber's coherent state $|\alpha\rangle$ [9]. Taking the initial state $|\psi\rangle$ in the form of a squeezed vacuum state, one obtains *photon-added squeezed states* [10,11]. These states possess the features of the Schrödinger-cat-like states [11]. The *even-odd photon-added states* were studied in [12]. For a *mixed* quantum state described by means of a statistical operator $\hat{\rho}$, one can define the *mixed photon-added state* as $\hat{\rho}_m=\tilde{\mathcal{N}}_m\hat{a}^{\dagger m}\hat{\rho}\hat{a}^m$, where $\tilde{\mathcal{N}}_m$ is a normalization constant. A concrete example is the *photon-added thermal state* [13,14]. Replacing the creation operator $\hat{a}^\dagger$ in the definition (1) by the annihilation operator $\hat{a}$ one obtains *photon-subtracted states* studied in detail in [15–17]. It was shown in [8] that photon-added states can be produced in the processes of the field-atom interaction in a cavity. An effective method of generating photon-added and photon-subtracted states in a traveling light beam by means of conditional measurements on a beam splitter was proposed in [11,15–18].

Photon-added states possess many interesting properties. In particular, if the basic (pure) state $|\psi\rangle$ is a superposition of the Fock states,

$$|\psi\rangle=\sum_{n=0}^{\infty}c_n|n\rangle, \quad \hat{a}^\dagger\hat{a}|n\rangle=n|n\rangle, \tag{2}$$

then the photon-added state (1) can be written as

$$|\psi,m\rangle=\mathcal{N}_m\sum_{n=m}^{\infty}c_{n-m}\left[\frac{n!}{(n-m)!}\right]^{1/2}|n\rangle. \tag{3}$$

Consequently, the probability $p_n^{(m)}$ of detecting n quanta (photons) in the state $|\psi,m\rangle$ can be expressed in terms of the initial probabilities, $p_n^{(0)}\equiv|c_n|^2$, as

$$p_n^{(m)}=\mathcal{N}_m^2\frac{n!}{(n-m)!}p_{n-m}^{(0)}, \tag{4}$$

so that the probability of detecting n quanta is exactly zero for $n<m$, for all kinds of photon-added states. Besides, the PACS exhibits simultaneously a significant squeezing and the sub-Poissonian statistics of quanta [8]. The PACS can be considered also as an eigenstate of the *boson inverse operator* [19]. However, all the above mentioned properties hold in a "static" case, when the frequency of the field mode oscillator does not depend on time.

In the present paper we study the *dynamics* of the states, which originates from the PACS in the processes involving a time dependence of the oscillator frequency. We demonstrate a twofold influence of the frequency time dependence on the PACS. On the one hand, it transforms the sub-Poissonian photon statistics of PACS into a strongly super-Poissonian statistics, and leads to a nonzero probability of detecting $n<m$ photons; i.e., it deteriorates the properties of the pure PACS. On the other hand, the *dynamical squeezing* of PACS enables one, under certain conditions, to achieve a stronger

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degree of squeezing than for the usual squeezed states with the same energy of quantum fluctuations.

The paper is organized as follows. In Sec. II we introduce the time-dependent photon-added states and give explicit expressions for the quadrature distribution, the Wigner function, and the photon distribution function. The evolution of the squeezing factor and the photon number statistics is studied in Sec. III. The results of the paper are summarized in Sec. IV. The details of calculations are given in the Appendix.

II. TIME-DEPENDENT PHOTON-ADDED STATES

The photon-added coherent state can be defined according to Eq. (1), if the basis state $|\psi\rangle$ coincides with the coherent state $|\alpha\rangle$. The photon distribution in the PACS is a shifted Poisson distribution,

$$p_n^{(m)}=\mathcal{N}_m^2\exp(-|\alpha|^2)\frac{n!|\alpha|^{2(n-m)}}{[(n-m)!]^2}, \tag{5}$$

where the normalization constant is [8]

$$\mathcal{N}_m=[m!L_m(-|\alpha|^2)]^{-1/2}, \tag{6}$$

with $L_m(z)$ being the Laguerre polynomial [20].

Assuming that the state $|\alpha,m\rangle$ was created in some way in a cavity or in an ion trap at the moment of time $t=0$, let us suppose that the oscillator (the field mode) eigenfrequency begins to vary in time in accordance with a given law $\Omega(t)$ at $t>0$. In the cavity case, such a time dependence could arise either due to the variation of the dielectric function of the medium filling the cavity [21], or due to the motion of the cavity walls [22,23]. In the trap case one could vary the voltage applied between the electrodes of the trap. The evolution of the quantum state of the oscillator at $t>0$ is governed by the Schrödinger equation with the Hamiltonian

$$\hat{H}(t)=\tfrac{1}{2}[\hat{p}^2+\Omega^2(t)\hat{q}^2],$$

where the dimensionless units are chosen in such a way that $\Omega(0)=1$, and the quadrature components $\hat{q}$ (''coordinate'') and $\hat{p}$ (''momentum'') are defined by means of the usual relations,

$$\hat{a}=(\hat{q}+i\hat{p})/\sqrt{2},\quad \hat{a}^\dagger=(\hat{q}-i\hat{p})/\sqrt{2}.$$

The unitary evolution operator $\hat{U}(t)$ transforms the state $|\alpha,m\rangle$ into

$$\begin{aligned}|\alpha,m;t\rangle=\hat{U}(t)|\alpha,m\rangle&=\mathcal{N}_m[\hat{U}(t)\hat{a}^{\dagger m}\hat{U}^\dagger(t)]\hat{U}(t)|\alpha\rangle\\&=\mathcal{N}_m[\hat{A}^\dagger(t)]^m|\alpha;t\rangle,\end{aligned} \tag{7}$$

where $|\alpha;t\rangle\equiv\hat{U}(t)|\alpha\rangle$ is a time-dependent coherent state [24,25], and $\hat{A}^\dagger(t)=\hat{U}(t)\hat{a}^\dagger\hat{U}^\dagger(t)$ is an integral of motion of the parametric oscillator [26], satisfying the initial condition $\hat{A}^\dagger(0)=\hat{a}^\dagger$. The operators $\hat{A}^\dagger(t)$ and $\hat{A}(t)=\hat{U}(t)\hat{a}\hat{U}^\dagger(t)$ satisfy the same commutation relation as $\hat{a}$ and $\hat{a}^\dagger$, $[\hat{A}(t),\hat{A}^\dagger(t)]=\hat{1}$. Thus the algebra defined by the operators $\hat{A}(t)$, $\hat{A}^\dagger(t)$, $\hat{1}$, is the same Heisenberg-Weyl algebra as in the case of operators $\hat{a}$, $\hat{a}^\dagger$, $\hat{1}$. The knowledge of the time-dependent integrals of motion (invariant operators) enables one to find all the functions characterizing the quantum system: the propagator, mean values of various observables, transition amplitudes between the initial and final states, etc. For example, the state $|\alpha;t\rangle$ can be easily obtained from the eigenvalue equation

$$\hat{A}(t)|\alpha;t\rangle=\alpha|\alpha;t\rangle, \tag{8}$$

because the explicit form of the integral of motion $\hat{A}(t)$ can be found independently from the equation (we assume hereafter $\hbar\equiv 1$)

$$i\partial\hat{A}(t)/\partial t=[\hat{H}(t),\hat{A}(t)]. \tag{9}$$

The detailed exposition of the method of quantum invariants [26] was given in [25], for its applications to different physical problems see, e.g., [27–29].

The linear integral of motion of the quantum time-dependent oscillator reads [24]

$$\hat{A}(t)=\frac{i}{\sqrt{2}}[\varepsilon(t)\hat{p}-\dot{\varepsilon}(t)\hat{q}], \tag{10}$$

where the c-number function $\varepsilon(t)$ satisfies the equation

$$\ddot{\varepsilon}(t)+\Omega^2(t)\varepsilon(t)=0 \tag{11}$$

and the normalization condition

$$\dot{\varepsilon}\varepsilon^*-\varepsilon\dot{\varepsilon}^*=2i. \tag{12}$$

If the frequency $\Omega(t)$ assumes its initial value $\Omega=1$ upon some time T, then the solution to Eq. (11) for $t>T$ reads

$$\varepsilon(t)=C_+e^{it}+C_-e^{-it}, \tag{13}$$

where the constant coefficients

$$C_\pm=\tfrac{1}{2}e^{\mp iT}[\varepsilon(T)\mp i\dot{\varepsilon}(T)]$$

satisfy the constraint

$$|C_+|^2-|C_-|^2=1, \tag{14}$$

which is equivalent to Eq. (12). Due to Eq. (14), it is convenient to parametrize the coefficients $C_\pm$ as

$$C_+=\cosh(\tau)e^{i\phi_+},\quad C_-=-i\sinh(\tau)e^{i\phi_-}, \tag{15}$$

with real parameters τ and $\phi_\pm$. The physical meaning of such a parametrization becomes especially clear, if one considers a specific time dependence of the frequency corresponding to the parametrically excited oscillator,

$$\Omega(t)=1+2\gamma\cos(2t),\quad |\gamma|\ll 1. \tag{16}$$

Then we have the following expressions for τ and $\phi_\pm$ (up to small oscillating corrections of an order of γ) [21,22]:

$$\tau=\gamma T,\quad \phi_+=\phi_-=0, \tag{17}$$

so that τ can be interpreted as a ''slow time.'' We shall use the parametrization (15) with the conditions (17) to illustrate the general formulas, referring to it as to the ''resonance parametrization.''

Solving Eq. (8) with operator $\hat{A}$ given by Eq. (10) we obtain the wave function of the time-dependent coherent state in the coordinate representation [24],

$$\langle q|\alpha;t\rangle=\pi^{-1/4}\varepsilon^{-1/2}\exp\left(\frac{i\dot{\varepsilon}q^2}{2\varepsilon}+\frac{\sqrt{2}\alpha q}{\varepsilon}-\frac{\alpha^2\varepsilon^*}{2\varepsilon}-\frac{|\alpha|^2}{2}\right). \tag{18}$$

Then Eq. (7) yields the following explicit expression for the *time-dependent photon-added coherent state* (TDPACS) in the coordinate representation,

$$\langle q|\alpha,m;t\rangle=\mathcal{N}_m\left(\frac{\varepsilon^*}{2\varepsilon}\right)^{m/2}H_m\left(\frac{q}{|\varepsilon|}-\sqrt{\frac{\varepsilon^*}{2\varepsilon}}\alpha\right)\langle q|\alpha;t\rangle, \tag{19}$$

where $H_m(z)$ is the Hermite polynomial [20], and the normalization factor is given by Eq. (6). The wave function of the ''static'' PACS [16] is obtained from Eq. (19) by means of the substitution the initial values $\varepsilon(0)=1$ and $\dot{\varepsilon}(0)=i$.

The photon distribution function (PDF) of the field in the state $|\alpha,m;t\rangle$ is defined as

$$p_n(\alpha,m;t)=|\langle n|\alpha,m;t\rangle|^2. \tag{20}$$

In the ''resonance parametrization'' the PDF can be written as (see Appendix)

$$\begin{aligned}p_n(\alpha,m;\tau)=&\frac{2^{-(m+n)}(\tanh\tau)^{m+n}}{\cosh\tau\ m!n!L_m(-|\alpha|^2)}\\&\times\exp[-|\alpha|^2+\mathrm{Im}(\alpha^2)\tanh\tau]\\&\times\left|\sum_{k=0}^{\min(m,n)}\left(\frac{2i}{\sinh\tau}\right)^k\frac{m!n!}{k!(m-k)!(n-k)!}\right.\\&\left.\times H_{n-k}\left(\frac{\alpha e^{-i\pi/4}}{\sqrt{\sinh(2\tau)}}\right)H_{m-k}\left(\frac{\alpha e^{i\pi/4}}{\sqrt{2\coth\tau}}\right)\right|^2.\end{aligned} \tag{21}$$

Formula (21) clearly shows that the probability of discovering $n<m$ (for instance, $n=0$) photons in TDPACS is not equal to zero when $\mathcal{E}>1$. Besides, it shows that the photon distribution actually does not depend on the ''fast time'' t. Considering Eq. (21) as the function of the phase ϕ of the complex number $\alpha=|\alpha|\exp(i\phi)$, one can check that it is periodic with the period π. Moreover, $p_n(\phi=0)=p_n(\phi=\pi/2)$. Nonetheless, the distribution is asymmetric with respect to the phase inversion $\phi\rightarrow-\phi$. In particular, we have quite different distributions p_n for $\phi=-\pi/4$ and for $\phi=\pi/4$ (all other parameters being the same), as shown in Figs. 1(a) and 1(b).

The phase space distribution can be described in terms of the *Wigner function*

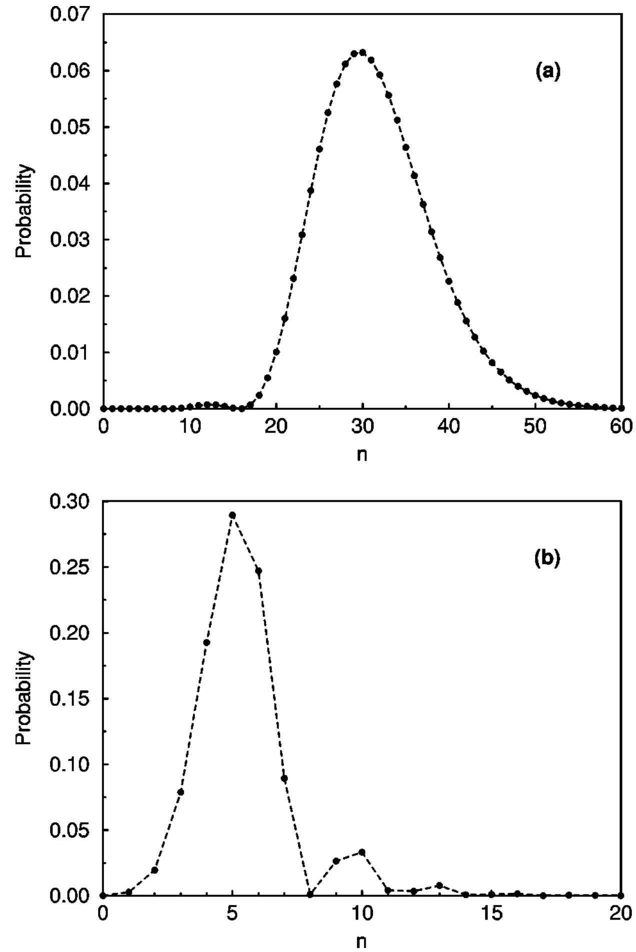

FIG. 1. The photon distributions in the ''resonance parametrization'' at $|\alpha|=2$, $m=5$, $\tau=0.5$, for extremal phases of the complex parameter α: (a) $\phi=-\pi/4$ and (b) $\phi=\pi/4$.

$$\begin{aligned}W(q,p;\alpha,m;t)=&\int_{-\infty}^{\infty}\langle q+r/2|\alpha,m;t\rangle\langle\alpha,m;t|q-r/2\rangle\\&\times\exp(-ipr)dr.\end{aligned} \tag{22}$$

In the case involved, the integrand is the product of two Hermite polynomials of the same order, multiplied by the Gaussian weight function. This integral can be expressed in terms of the Laguerre polynomial, and finally we obtain

$$W(q,p;\alpha,m;t)=2(-1)^m\frac{L_m(|2\zeta-\alpha|^2)}{L_m(-|\alpha|^2)}\exp(-2|\zeta-\alpha|^2), \tag{23}$$

where

$$\zeta(q,p,t)=i(\varepsilon p-\dot{\varepsilon}q)/\sqrt{2}.$$

In the ''resonance parametrization,'' the actual argument of the Wigner function is $\rho=(q+ip)e^{it}$, which is related to the variable ζ as

$$\zeta=(\rho\cosh\tau+i\rho^*\sinh\tau)/\sqrt{2}.$$

Typical plots of the Wigner function are given in Figs. 2(a) and 2(b).

III. SQUEEZING AND STATISTICS OF QUANTA

To study the squeezing properties and the photon statistics of TDPACS we must calculate the average values of quadratures $\hat{q},\hat{p}$, the photon number operator $\hat{N}=\hat{a}^\dagger\hat{a}$, and their powers and products. This problem is reduced to calculating the average values of various products of the operators $\hat{A}$ and $\hat{A}^\dagger$ in the state $|\alpha,m;t\rangle$, due to the relations [which are immediate consequences of Eqs. (10) and (12)],

$$\hat{p}=\frac{1}{\sqrt{2}}[\hat{A}(t)\dot{\varepsilon}^*(t)+\hat{A}^\dagger(t)\dot{\varepsilon}(t)], \tag{24}$$

$$\hat{q}=\frac{1}{\sqrt{2}}[\hat{A}(t)\varepsilon^*(t)+\hat{A}^\dagger(t)\varepsilon(t)], \tag{25}$$

$$\hat{a}=\tfrac{1}{2}[\hat{A}(\varepsilon^*+i\dot{\varepsilon}^*)+\hat{A}^\dagger(\varepsilon+i\dot{\varepsilon})], \tag{26}$$

$$\hat{a}^\dagger\hat{a}=\mathcal{E}\hat{A}\hat{A}^\dagger-\tfrac{1}{2}(\mathcal{E}+1)+\tfrac{1}{4}[\hat{A}^{\dagger 2}(\varepsilon^2+\dot{\varepsilon}^2)+\text{H.c.}]. \tag{27}$$

In particular, we obtain the following formulas for the variances $\sigma_{ab}\equiv\frac{1}{2}\langle\hat{a}\hat{b}+\hat{b}\hat{a}\rangle-\langle\hat{a}\rangle\langle\hat{b}\rangle$ (see Appendix for the details of calculations):

$$\sigma_{qq}=\tfrac{1}{2}(|\varepsilon|^2\mathcal{L}_1-f^2\mathcal{L}_2), \tag{28}$$

$$\sigma_{pp}=\tfrac{1}{2}(|\dot{\varepsilon}|^2\mathcal{L}_1-\dot{f}^2\mathcal{L}_2), \tag{29}$$

$$\sigma_{pq}=\tfrac{1}{2}[\text{Re}(\varepsilon\dot{\varepsilon}^*)\mathcal{L}_1-f\dot{f}\mathcal{L}_2], \tag{30}$$

where $f(t)\equiv\varepsilon^*(t)\alpha+\varepsilon(t)\alpha^*$, whereas the non-negative coefficients $\mathcal{L}_1$ and $\mathcal{L}_2$ are expressed in terms of the associated Laguerre polynomials $L_m^{(b)}(x)$:

$$\mathcal{L}_1\equiv 2\frac{L_m^{(1)}(-|\alpha|^2)}{L_m(-|\alpha|^2)}-1, \tag{31}$$

$$\mathcal{L}_2\equiv\left[\frac{L_m^{(1)}(-|\alpha|^2)}{L_m(-|\alpha|^2)}\right]^2-\frac{L_m^{(2)}(-|\alpha|^2)}{L_m(-|\alpha|^2)}. \tag{32}$$

One can check that the combination

$$\mathcal{D}\equiv\sigma_{pp}\sigma_{qq}-\sigma_{pq}^2 \tag{33}$$

depends neither on time nor on the concrete value of function $\varepsilon(t)$,

$$\mathcal{D}=\tfrac{1}{4}\mathcal{L}_1^2-|\alpha|^2\mathcal{L}_1\mathcal{L}_2. \tag{34}$$

This is an example of the *quantum universal invariants* introduced in [30] (see also [25]). The specific feature of this invariant in the case of TDPACS is its independence of the phase of the complex number α. For any state, $\mathcal{D}\geqslant 1/4$ (if $\hbar=1$), which is the generalized uncertainty relation [25].

Since the quadrature variances σ_{qq} and σ_{pp} are periodical functions of time for $t>T$, it is reasonable to introduce the time-independent (in the stationary case) *squeezing coefficient*, defined as the minimal value of each variance (during the period) normalized by the oscillator ground-state variance 1/2. The explicit formula for this coefficient was found in [31],

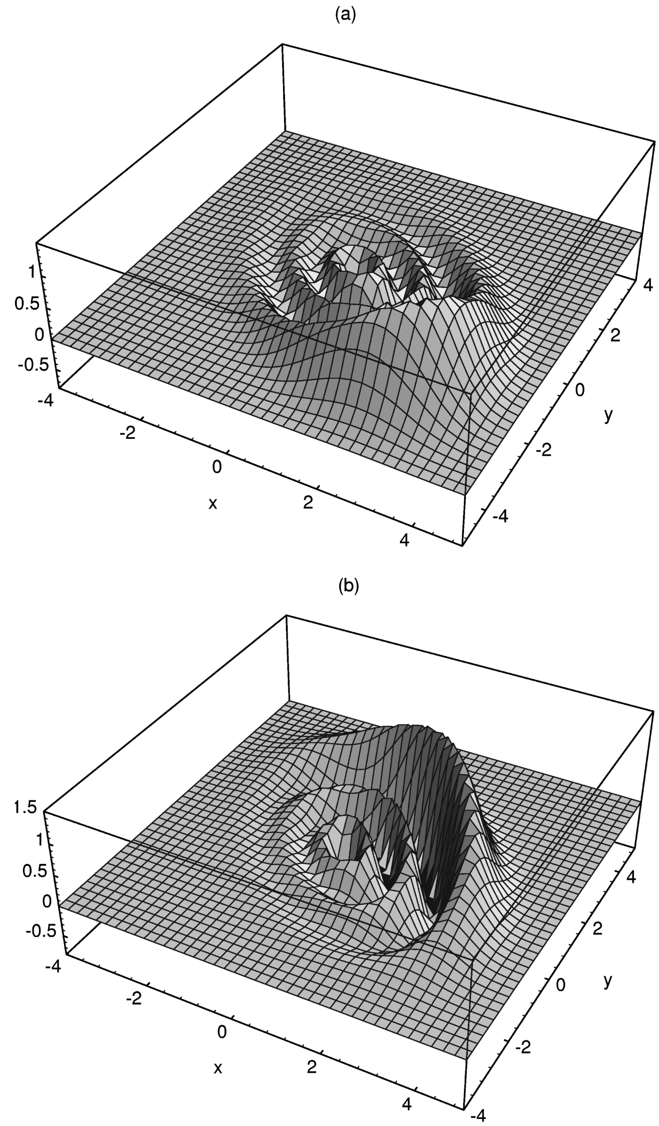

FIG. 2. The Wigner function (23) in the ''resonance parametrization'' at $|\alpha|=0.5$, $m=5$, $\tau=0.25$, for extremal phases of the complex parameter α: (a) $\phi=-\pi/4$ and (b) $\phi=\pi/4$. The axes labels are $x\equiv$ Re(ρ) and $y\equiv$ Im(ρ).

$$s=\sigma_{qq}+\sigma_{pp}-[(\sigma_{qq}+\sigma_{pp})^2-4\mathcal{D}]^{1/2}. \tag{35}$$

Using Eqs. (28), (29), and (34) we can write Eq. (35) as follows:

$$s=\mathcal{E}\mathcal{L}_1-2|\alpha|^2(\mathcal{F}+\mathcal{E})\mathcal{L}_2-\{[\mathcal{E}\mathcal{L}_1-2|\alpha|^2(\mathcal{F}+\mathcal{E})\mathcal{L}_2]^2-\mathcal{L}_1(\mathcal{L}_1-4|\alpha|^2\mathcal{L}_2)\}^{1/2}, \tag{36}$$

where $\mathcal{E}$ and $\mathcal{F}$ are the integrals of motion of Eq. (11) for $\Omega(t)\equiv 1$,

$$\mathcal{E}=\tfrac{1}{2}(|\varepsilon|^2+|\dot{\varepsilon}|^2)=\cosh(2\tau), \tag{37}$$

$$\mathcal{F}=\tfrac{1}{2}\,\text{Re}[e^{-2i\phi}(\dot{\varepsilon}^2+\varepsilon^2)]=-\sin(2\phi)\sinh(2\tau). \tag{38}$$

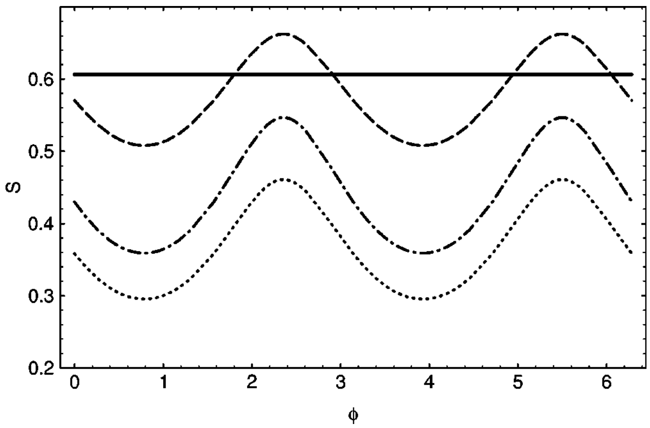

FIG. 3. The squeezing factor in the ''resonance parametrization'' vs the phase ϕ for $\tau=0.25$, $|\alpha|=2$, and $m=0$ (the solid line), $m=1$ (the dashed curve), $m=5$ (the dashed-dotted curve), $m=10$ (the dotted curve).

Note that $\mathcal{E}(t)$ is nothing but twice the energy of the quantum oscillator, which was initially in the ground state [25] [in units $\hbar=\Omega(0)=1$]. Due to Eq. (14) we have the inequality $|\mathcal{F}|\leqslant 2|C_+C_-|=\sqrt{\mathcal{E}^2-1}$. The transition to the case of the stationary PACS is achieved by putting $\mathcal{E}=1$ and $\mathcal{F}=0$.

The squeezing coefficient (36) in the stationary PACS reads

$$s_*=\mathcal{L}_1-4|\alpha|^2\mathcal{L}_2. \tag{39}$$

In the ''resonance parametrization,'' the squeezing factor is a periodic function of ϕ with the period π, which has a maximum at $\phi=-\pi/4$ and a minimum at $\phi=\pi/4$. Figure 3 shows the dependence $s(\phi)$ for fixed values of other parameters. The horizontal line $(m=0)$ corresponds to the usual squeezed state originating from the initial coherent state $|\alpha\rangle$ due to the parametric excitation. For this state, the squeezing coefficient does not depend on α,

$$s_0=[\mathcal{E}+\sqrt{\mathcal{E}^2-1}]^{-1}=e^{-2\tau}. \tag{40}$$

For $\tau\gg 1$, the squeezing factor is confined in the interval

$$s_*e^{-2\tau}\leqslant s\leqslant \mathcal{L}_1 e^{-2\tau}, \tag{41}$$

and it can be both greater and less than s_0, depending on the phase of α.

The explicit expression for the average number of photons in TDPACS reads (hereafter $x\equiv -|\alpha|^2$),

$$\langle \hat{N}\rangle=(m+1)\mathcal{E}\frac{L_{m+1}(x)}{L_m(x)}-\frac{1}{2}(\mathcal{E}+1)+|\alpha|^2\mathcal{F}\frac{L_m^{(2)}(x)}{L_m(x)}. \tag{42}$$

The variance of the photon number, $\sigma_N\equiv\langle(\hat{a}^\dagger\hat{a})^2\rangle-\langle\hat{a}^\dagger\hat{a}\rangle^2$, can be represented as

$$\begin{aligned}\sigma_N=&|\alpha|^4\mathcal{F}^2\left[\frac{L_m^{(4)}(x)}{L_m(x)}-\left(\frac{L_m^{(2)}(x)}{L_m(x)}\right)^2\right]+2|\alpha|^2\mathcal{E}F\\&\times\left\{(m+1)\left[\frac{L_{m+1}^{(2)}(x)}{L_m(x)}-\frac{L_{m+1}(x)L_m^{(2)}(x)}{[L_m(x)]^2}\right]-\frac{L_m^{(2)}(x)}{L_m(x)}\right\}\\&+(m+1)(m+2)(3\mathcal{E}^2-1)\frac{L_{m+2}(x)}{2L_m(x)}\\&-(m+1)(2\mathcal{E}^2-1)\frac{L_{m+1}(x)}{L_m(x)}\\&-(m+1)^2\mathcal{E}^2\left[\frac{L_{m+1}(x)}{L_m(x)}\right]^2+\frac{1}{2}(\mathcal{E}^2-1)\\&\times\left[1-x^2\frac{L_m^{(4)}(x)}{L_m(x)}\right].\end{aligned} \tag{43}$$

Putting $\mathcal{E}=1$ and $\mathcal{F}=0$ we recover the results of [8].

A distinctive feature of PACS is the sub-Poissonian photon statistics [8], i.e., the inequality $\sigma_N<\langle\hat{N}\rangle$. However, this inequality does not hold, in general, for TDPACS. This is clearly seen in Fig. 4(a), which shows the dependence of the Mandel parameter [32]

$$\mathcal{R}\equiv\sigma_N/\langle\hat{N}\rangle \tag{44}$$

on the ''slow time'' τ in the ''resonance parametrization.'' The initial sub-Poissonian statistics $(\mathcal{R}<1)$ is rapidly transformed into the super-Poissonian one $(\mathcal{R}>1)$. Figure 4(b) shows the dependence of the Mandel parameter on the phase of α. Although the positions of the minimum and maximum of the squeezing coefficient, $\phi=\pm\pi/4$, correspond to the *local* extrema of $\mathcal{R}(\phi)$, there is no correlation (or anticorrelation) between the absolute extrema of the functions $\mathcal{R}(\phi)$ and $s(\phi)$ in a generic case.

IV. CONCLUSION

Let us summarize the main results of the paper. We have obtained explicit analytical expressions for various functions characterizing the evolution of the photon-added coherent states in the case of a time-dependent frequency of the oscillator. The formulas are written in terms of some universal parameters, which have a clear physical meaning in the case of a parametric resonance. It appears that under certain conditions the time-dependent photon-added coherent states yield more strong squeezing than the usual squeezed states. In the process of the evolution, the initial sub-Poissonian statistics of quanta is transformed into a highly super-Poissonian one. One could use the results obtained to evaluate deteriorating effects due to time variations of the system parameters during an experiment. Or, on the contrary, one could deliberately introduce some time dependence of the parameters in order to achieve optimal values of certain quantities characterizing the quantum system, e.g., the squeezing coefficient.

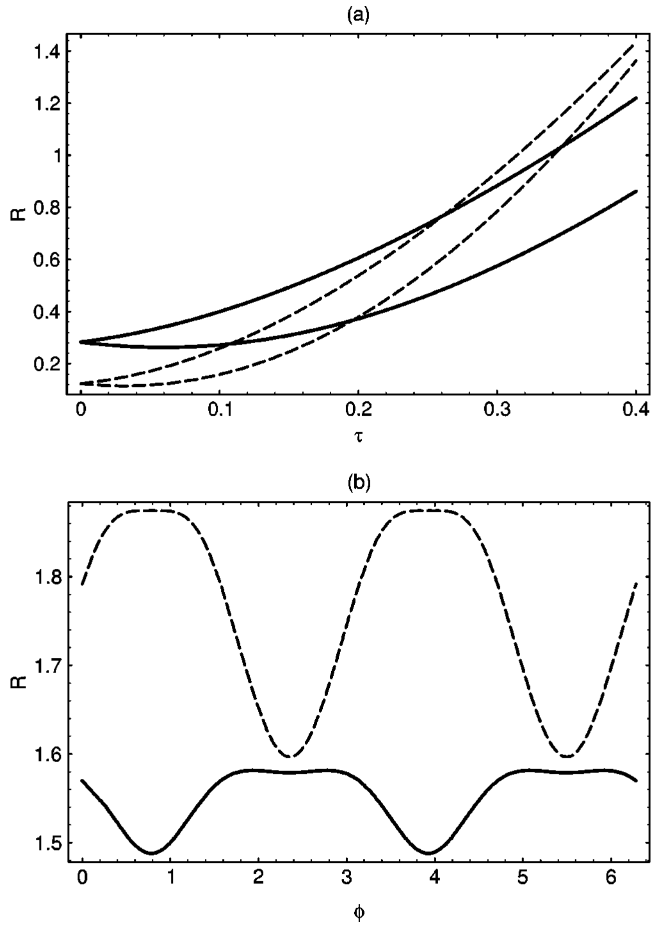

FIG. 4. Mandel's parameter in the "resonance parametrization." (a) $\mathcal{R}(\tau)$ for $|\alpha|=0.5$ and different values of m and ϕ: $m=1, \phi=\pi/4$ (lower solid curve), $m=1, \phi=-\pi/4$ (upper solid curve), $m=5, \phi=\pi/4$ (lower dashed curve), and $m=5, \phi=-\pi/4$ (upper dashed curve). (b) $\mathcal{R}(\phi)$ for $m=1$ (solid curve) and $m=5$ (dashed curve), at $\tau=0.25, |\alpha|=2$.

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APPENDIX A: DETAILS OF CALCULATIONS

To calculate the scalar product between the Fock state and the TDPACS in Eq. (20), we notice that the state $|\alpha,m;t\rangle$ is proportional to the coefficient at the term β^m in the Taylor expansion of the state $|\beta\rangle=\exp(\beta\hat{a}^\dagger)|\alpha;t\rangle$ [see Eq. (7)], whereas the state $|n\rangle$ coincides (up to a constant) with the coefficient at the term γ^n in the expansion of the state $|\gamma\rangle=\exp(\gamma\hat{a}^\dagger)|0\rangle$. Consequently, the calculation of the scalar product $\langle n|\alpha,m;t\rangle$ is equivalent to finding the coefficient at the term $\gamma^{*n}\beta^m$ in the power expansion of the function $\langle\gamma|\beta\rangle$. But this function is obviously an exponential of some quadratic form with respect to γ^* and β, so the coefficients of the expansion can be expressed in terms of the two-dimensional Hermite polynomials defined via the generating function (we have slightly changed the definition given in [20])

$$\exp\left(-\frac{1}{2}\mathbf{aRa}+\mathbf{az}\right)=\sum_{n,m=0}^{\infty}\frac{a_1^n a_2^m}{\sqrt{n!m!}}\tilde{H}_{nm}^{\{\mathbf{R}\}}(z_1,z_2). \quad \text{(A1)}$$

Here $\mathbf{R}$ is a 2×2 symmetrical matrix, whereas $\mathbf{z}=(z_1,z_2)$ and $\mathbf{a}=(a_1,a_2)$ are two-dimensional vectors. After some algebra we obtain the formula

$$p_n(\alpha,m;t)=\sqrt{\frac{2}{\mathcal{E}+1}}\frac{|\tilde{H}_{nm}^{\{\mathbf{R}\}}(z_1,z_2)|^2}{L_m(-|\alpha|^2)}\times\exp\left[-|\alpha|^2\left(1+\frac{\mathcal{F}}{\mathcal{E}+1}\right)\right], \quad \text{(A2)}$$

where the elements of the matrix $\mathbf{R}$ and vector $\mathbf{z}$ read

$$R_{11}=\frac{i\dot{\varepsilon}+\varepsilon}{i\dot{\varepsilon}-\varepsilon}=i\tanh\tau e^{-2it},$$

$$R_{22}=\frac{i\dot{\varepsilon}^*-\varepsilon^*}{i\dot{\varepsilon}-\varepsilon}=i\tanh\tau,$$

$$R_{12}=\frac{2}{i\dot{\varepsilon}-\varepsilon}=-\frac{e^{-it}}{\cosh\tau},$$

$$z_1=-\alpha R_{12}, \quad z_2=-\alpha R_{22}.$$

(The expressions containing the parameter τ are given in the "resonance parametrization.") Using the relation between the two-dimensional Hermite polynomial and its one-dimensional counterpart [33,34] we can represent Eq. (A2) as a sum over products of the usual Hermite polynomials,

$$p_n(\alpha,m;t)=\sqrt{\frac{2}{\mathcal{E}+1}}\frac{2^{-(m+n)}}{m!n!L_m(-|\alpha|^2)}\left(\frac{\mathcal{E}-1}{\mathcal{E}+1}\right)^{(m+n)/2}\times\exp\left[-|\alpha|^2\left(1+\frac{\mathcal{F}}{\mathcal{E}+1}\right)\right]\times\left|\sum_{k=0}^{\min(m,n)}\left(-2i\sqrt{\frac{2}{\mathcal{E}-1}}\right)^k\times\frac{m!n!}{k!(n-k)!(m-k)!}\times H_{n-k}\left(i\sqrt{\frac{2}{\varepsilon^2+\dot{\varepsilon}^2}}\alpha\right)\times H_{m-k}\left(\sqrt{\frac{\mathcal{E}-1}{\varepsilon^2+\dot{\varepsilon}^2}}\alpha\right)\right|^2.$$

This formula is equivalent to Eq. (21).

To calculate average values of different operators, we need an explicit formula for the matrix elements

$$\langle\alpha,m;t|\hat{A}^k(t)\hat{A}^{\dagger l}(t)|\alpha,m;t\rangle\equiv\langle\alpha,m|\hat{a}^k\hat{a}^{\dagger l}|\alpha,m\rangle.$$

Firstly, we notice that due to the relation

$$|\alpha\rangle \equiv \exp(-|\alpha|^2/2)\exp(\alpha \hat{a}^\dagger)|0\rangle,$$

we may write the PACS $|\alpha,m\rangle$ as

$$|\alpha,m\rangle = \mathcal{N}_m \exp(-|\alpha|^2/2)\frac{\partial^m}{\partial\alpha^m}[\exp(|\alpha|^2/2)|\alpha\rangle]. \tag{A3}$$

Then the scalar product between different PACS can be written as

$$\begin{aligned}\langle\beta,n|\alpha,m\rangle &= \mathcal{N}_n(|\beta|)\mathcal{N}_m(|\alpha|)\exp[-\tfrac{1}{2}(|\alpha|^2+|\beta|^2)] \\ &\quad\times\frac{\partial^{n+m}}{\partial\beta^{*n}\partial\alpha^m}\left\{\exp\left[\frac{1}{2}(|\alpha|^2+|\beta|^2)\right]\langle\beta|\alpha\rangle\right\} \\ &= \mathcal{N}_n(|\beta|)\mathcal{N}_m(|\alpha|)\exp[-\tfrac{1}{2}(|\alpha|^2+|\beta|^2)] \\ &\quad\times\frac{\partial^{n+m}}{\partial\beta^{*n}\partial\alpha^m}e^{\beta^*\alpha}.\end{aligned}$$

The right-hand side of the last expression, being a multiple derivative of the exponential function of a quadratic form of β^* and α, can be written in terms of the Hermite polynomials of two variables [20]. However, in the specific case concerned (when the defining symmetric 2×2 matrix **R** has two zero diagonal elements) these polynomials are reduced to the associated Laguerre polynomials [25,34], so the final result reads

$$\frac{\langle\beta,n|\alpha,m\rangle\exp\left[\frac{1}{2}(|\alpha|^2+|\beta|^2)-\beta^*\alpha\right]}{\mathcal{N}_n(|\beta|)\mathcal{N}_m(|\alpha|)} = \begin{cases} m!\,\alpha^{n-m}L_m^{(n-m)}(-\beta^*\alpha), & n\geqslant m \\ n!\,\beta^{*m-n}L_n^{(m-n)}(-\beta^*\alpha), & m\geqslant n. \end{cases} \tag{A4}$$

In particular, taking $\alpha=\beta$ and $m=n$ we obtain the normalization factor (6), where $L_m(z)\equiv L_m^{(0)}(z)$. An immediate consequence of Eq. (A4) is the formula sought for:

$$\begin{aligned}\langle\alpha,m|\hat{a}^k\hat{a}^{\dagger l}|\alpha,m\rangle &= \frac{\mathcal{N}_m^2(|\alpha|)\langle\alpha,k+m|\alpha,l+m\rangle}{\mathcal{N}_{k+m}(|\alpha|)\mathcal{N}_{l+m}(|\alpha|)} \\ &= \begin{cases} \dfrac{(m+k)!}{m!}\dfrac{L_{m+k}^{(l-k)}(-|\alpha|^2)}{L_m(-|\alpha|^2)}\alpha^{*l-k}, & l\geqslant k \\ \dfrac{(m+l)!}{m!}\dfrac{L_{m+l}^{(k-l)}(-|\alpha|^2)}{L_m(-|\alpha|^2)}\alpha^{k-l}, & k\geqslant l. \end{cases}\end{aligned} \tag{A5}$$

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