Hydrodynamic modes and pulse propagation in a cylindrical Bose gas above the Bose-Einstein transition

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We study hydrodynamic oscillations of a cylindrical Bose gas above the Bose-Einstein transition temperature using the hydrodynamic equations derived by Griffin, Wu, and Stringari. This extends recent studies of a cylindrical Bose-condensed gas at $T=0$. Explicit normal mode solutions are obtained for non-propagating solutions. In the classical limit, the sound velocity is shown to be the same as a uniform classical gas. We use a variational formulation of the hydrodynamic equations to discuss the propagating modes in the degenerate Bose-gas limit and show there is little difference from the classical results. We discuss the propagation of sound pulses above and below T_{BEC}. [S1050-2947(98)02311-7]

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I. INTRODUCTION

Recently Andrews *et al.* [1] reported a measurement of a sound pulse propagation of Bose-condensed cloud in a highly asymmetric cigar-shaped trap. They have measured the sound velocity in a Bose condensate as a function of the density. Zaremba [2] gave a detailed analysis of the collective excitations in a cylindrical Bose gas starting from the $T=0$ quantum hydrodynamic equation of Stringari [3], which is based on the Thomas-Fermi approximation. The sound velocity of a condensate pulse obtained in Ref. [2] is in good agreement with the experimental observations [1]. More recently, several other theoretical studies have discussed the excitations and pulse propagation in a cigar-shaped trap at $T=0$ [4–6].

In this paper, we consider the analogous modes in a hydrodynamic regime (where collisions ensure local thermal equilibrium) for a cylindrical Bose gas above the Bose-Einstein transition temperature T_{BEC}. We use the results to discuss pulse propagation above T_{BEC}. Below T_{BEC}, one expects first and second sound pulses.

II. HYDRODYNAMIC NORMAL MODE EQUATIONS FOR A CYLINDRICAL BOSE GAS

The linearized hydrodynamic equation for the velocity fluctuations $\mathbf{v}(\mathbf{r},t)$ derived by Griffin, Wu, and Stringari is [7,8]

$$m\frac{\partial^2\mathbf{v}}{\partial t^2}=\frac{5P_0(\mathbf{r})}{3n_0(\mathbf{r})}\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{v})-\boldsymbol{\nabla}[\mathbf{v}\cdot\boldsymbol{\nabla}U_0(\mathbf{r})]-\frac{2}{3}(\nabla\cdot\mathbf{v})\boldsymbol{\nabla}U_0(\mathbf{r})-\frac{\partial}{\partial t}\boldsymbol{\nabla}\delta U(\mathbf{r},t), \tag{1}$$

where $U_0(\mathbf{r})$ is the static cylindrical trap potential and $\delta U(\mathbf{r},t)$ is a small time-dependent external perturbation. The equilibrium local density $n_0(\mathbf{r})$ and the equilibrium local kinetic pressure $P_0(\mathbf{r})$ in Eq. (1) are given by

$$n_0(\mathbf{r})=\frac{1}{\Lambda^3}g_{3/2}(z_0),\quad P_0(\mathbf{r})=\frac{k_BT}{\Lambda^3}g_{5/2}(z_0), \tag{2}$$

where $z_0(\mathbf{r})\equiv e^{\beta(\mu_0-U_0)}$ is the local equilibrium fugacity, $\Lambda\equiv(2\pi\hbar^2/mk_BT)^{1/2}$ is the thermal de Broglie wavelength, and $g_n(z)=\Sigma_{l=1}^{\infty}z^l/l^n$ are the well-known Bose-Einstein functions.

Throughout this paper, we shall limit our discussion to a purely cylindrical harmonic trap potential

$$U_0(\mathbf{r})=\tfrac{1}{2}m\omega_0^2(x^2+y^2). \tag{3}$$

Inserting this into Eq. (1), we obtain coupled equations for the radial $\mathbf{v}_\perp$ and axial v_z velocity fluctuations

$$m\frac{\partial^2\mathbf{v}_\perp}{\partial t^2}=\frac{5P_0}{3n_0}\boldsymbol{\nabla}_\perp(\boldsymbol{\nabla}_\perp\cdot\mathbf{v}_\perp)-m\omega_0^2\boldsymbol{\nabla}_\perp(\mathbf{r}_\perp\cdot\mathbf{v}_\perp)-\frac{2}{3}m\omega_0^2(\boldsymbol{\nabla}_\perp\cdot\mathbf{v}_\perp)\mathbf{r}_\perp+\left(\frac{5P_0}{3n_0}\boldsymbol{\nabla}_\perp-\frac{2}{3}m\omega_0^2\mathbf{r}_\perp\right)\frac{\partial v_z}{\partial z}, \tag{4a}$$

$$m\frac{\partial^2 v_z}{\partial t^2}=\frac{5P_0}{3n_0}\frac{\partial^2 v_z}{\partial z^2}+\left(\frac{5P_0}{3n_0}\boldsymbol{\nabla}_\perp-m\omega_0^2\mathbf{r}_\perp\right)\cdot\frac{\partial\mathbf{v}_\perp}{\partial z}, \tag{4b}$$

where we have set $\delta U(\mathbf{r},t)=0$ since we are interested in normal mode solutions (driven solutions will be discussed in Sec. VI). We use the convention $\mathbf{v}(\mathbf{r},t)=\mathbf{v}_\omega(\mathbf{r})e^{-i\omega t}$. Finally, we shall look for solutions of the kind

$$\mathbf{v}_\omega(\mathbf{r})=(xf(r_\perp),yf(r_\perp),h(r_\perp))e^{ikz}, \tag{5}$$

where $r_\perp=\sqrt{x^2+y^2}$. That is, we assume that the functions f and h do not depend on the radial azimuthal angle.

Using Eq. (5) in Eq. (4), one finds, after some algebra, a coupled set of equations for the radial function f and the axial function h:

$$\omega^2 f = -c_0^2(r_\perp)\left(\frac{\partial^2 f}{\partial r_\perp^2} + \frac{3}{r_\perp}\frac{\partial f}{\partial r_\perp}\right) + \omega_0^2\left(\frac{10}{3}f + \frac{5}{3}r_\perp\frac{\partial f}{\partial r_\perp}\right) - ik\left[c_0^2(r_\perp)\frac{1}{r_\perp}\frac{\partial h}{\partial r_\perp} - \frac{2}{3}\omega_0^2 h\right], \tag{6a}$$

$$\omega^2 h = c_0^2(r_\perp)k^2 h - ik\left[c_0^2(r_\perp)\left(2f + r_\perp\frac{\partial f}{\partial r_\perp}\right) - \omega_0^2 r_\perp^2 f\right]. \tag{6b}$$

The position-dependent local ''sound velocity'' $c_0(r_\perp)$ is defined by

$$c_0^2(r_\perp) \equiv \frac{5P_0(r_\perp)}{3mn_0(r_\perp)} = \frac{5k_B T}{3m}B(z_0), \quad B(z_0) \equiv \frac{g_{5/2}(z_0)}{g_{3/2}(z_0)}. \tag{7}$$

One can show that the normal mode solutions of Eqs. (6a) and (6b) satisfy the following orthogonality:

$$\int d\mathbf{r}\ n_0(\mathbf{r})\mathbf{v}_{\omega'}^*(\mathbf{r})\cdot\mathbf{v}_\omega(\mathbf{r}) = 0, \quad \text{if } \omega_{n'} \neq \omega_n, \tag{8a}$$

or, more explicitly,

$$\int_0^\infty dr_\perp r_\perp n_0(r_\perp)[r_\perp^2 f_{n'}^*(r_\perp)f_n(r_\perp) + h_{n'}^*(r_\perp)h_n(r_\perp)] = 0, \quad \text{if } n' \neq n. \tag{8b}$$

Here the label n specifies the different normal mode solutions. We remark that, while it is not obvious, if we set the trap frequency ω_0 to zero, Eqs. (6a) and (6b) have solutions involving Bessel functions $J(k_\perp r_\perp)$ with the expected phonon dispersion relation $\omega^2 = c_0^2(k^2 + k_\perp^2)$.

It is convenient to introduce a dimensionless radial variable

$$s \equiv \frac{r_\perp^2}{R^2}, \quad R \equiv \left(\frac{2k_B T}{m\omega_0^2}\right)^{1/2}. \tag{9}$$

We also introduce a dimensionless frequency and wave vector

$$\bar{\omega} \equiv \frac{\omega}{\omega_0}, \quad \bar{k} \equiv kR. \tag{10}$$

We observe from Eq. (2) that R denotes the ''size'' of the radial density profile produced by the harmonic potential trap. In these units, we note $z_0 = e^{\beta\mu_0}e^{-s}$ and the classical density profile $n_0(r_\perp) \sim e^{-s}$. In terms of these new dimensionless variables, it is useful to introduce the new functions:

$$\bar{f}(s) \equiv -iRf(r_\perp), \quad \bar{h}(s) \equiv h(r_\perp). \tag{11}$$

Using Eq. (6), the coupled equations for $\bar{f}$ and $\bar{h}$ are given by

$$\bar{\omega}^2\bar{f} = \hat{L}[\bar{f}] - \bar{k}\left(\frac{5}{3}B(z_0)\frac{d\bar{h}}{ds} - \frac{2}{3}\bar{h}\right), \tag{12a}$$

$$\bar{\omega}^2\bar{h} = \frac{5}{6}B(z_0)\bar{k}^2\bar{h} - \bar{k}\left[\frac{5}{3}B(z_0)s\frac{d\bar{f}}{ds} + \left(\frac{5}{3}B(z_0) - s\right)\bar{f}\right], \tag{12b}$$

where we have introduced the operator $\hat{L}$:

$$\hat{L}[\bar{f}] \equiv -\frac{10}{3}\left[B(z_0)s\frac{d^2\bar{f}}{ds^2} + [2B(z_0) - s]\frac{d\bar{f}}{ds} - \bar{f}\right]. \tag{13}$$

The rest of this paper is based on the equations in Eqs. (12a) and (12b), which determine the normal mode velocity fluctuations using Eq. (11) and (5). The associated density fluctuations $\delta n(\mathbf{r},t)$ can be found by using the number conservation law

$$\frac{\partial\delta n}{\partial t} = -\nabla\cdot(n_0\mathbf{v}). \tag{14}$$

III. NONPROPAGATING MODES

We first examine nonpropagating solutions ($k=0$) of Eqs. (12a) and (12b), in which case they reduce to the two independent equations

$$\bar{\omega}^2\bar{f} = \hat{L}[\bar{f}], \quad \bar{\omega}^2\bar{h} = 0. \tag{15}$$

There is a trivial zero frequency solution $\bar{f}=0$, $\bar{h}\neq 0$, corresponding to $\mathbf{v}_\perp = 0$, $v_z \neq 0$. In this case, the dependence of $\bar{h}$ on s cannot be determined uniquely. Using Eq. (14), one finds that $\delta n = 0$ for this mode. The interesting solutions of (15) with $\bar{\omega}\neq 0$ are given by

$$\bar{h} = 0, \quad \bar{\omega}^2\bar{f} = \hat{L}[\bar{f}]. \tag{16}$$

These correspond to oscillations only in the radial direction, which we shall now discuss.

In the *classical* gas limit, the operator $\hat{L}$ in Eq. (13) simplifies since $B(z_0)=1$. In this case, we can obtain the complete set of normal mode solutions of Eq. (16), namely,

$$\bar{f}_n^{(0)}(s) = \frac{1}{n!\sqrt{n}}\frac{d}{ds}L_n(s), \quad [\bar{\omega}_n^{(0)}]^2 = \frac{10n}{3}, \quad n = 1,2,3,\ldots . \tag{17}$$

Here $L_n(s)$ is the Laguerre polynomial defined by

$$L_n(s) \equiv e^s\frac{d^n}{ds^n}(s^n e^{-s}), \tag{18}$$

and the orthogonal functions $\bar{f}_n^{(0)}$ are normalized according to [see Eq. (8b)]

$$\int_0^\infty ds\ \ se^{-s}\bar{f}_n^{(0)}(s)\bar{f}_{n'}^{(0)}(s) = \delta_{nn'}. \tag{19}$$

In ordinary variables, the dispersion relation of these $k=0$ solutions corresponds to

$$\omega_n^2 = n\frac{10}{3}\omega_0^2, \quad n = 1,2,3,\ldots, \tag{20}$$

and the associated density fluctuation is given by

$$\delta n(\mathbf{r},t) \propto L_n(s=r_\perp^2/R^2)\exp(-r_\perp^2/R^2)e^{-i\omega_n t}. \quad (21)$$

In the lowest ($n=1$) mode with $\omega_1^2 = 10\omega_0^2/3$, $\overline{f_1^{(0)}}(s)$ is independent of s and $L_1(s)=1-s$. This is the two-dimensional radial breathing mode. We note that this particular mode corresponds to one of the coupled monopole-quadrupole modes found in Ref. [7] for an anisotropic trap in the limit that the axial trap frequency (ω_z) is set to zero. In this same limit ($\omega_z=0$), the other mode has zero frequency and a velocity fluctuation given by $\mathbf{v}_\omega = \alpha(x,y,-5z)$. This kind of solution is not described by the form (5), which we are dealing with in this paper.

For a degenerate Bose gas, in which $B(z_0)$ is now weakly dependent on s through $z_0 = e^{\beta\mu_0 - s}$, one cannot solve Eq. (16) analytically. We discuss variational solutions in Sec. V. However, one can check that $\overline{f}(s)=\text{const}$ is a solution, with frequency $\omega^2 = \frac{10}{3}\omega_0^2$. This is the analog of the $n=1$ normal mode in Eq. (17) for the classical gas (we recall that $\overline{f_1^{(0)}}$ is independent of s).

For comparison with Eq. (20), the analogous nonpropagating normal modes in a cigar trap at $T=0$ have a spectrum given by [2,9]

$$\omega_l^2 = 2l(l+1)\omega_0^2, \quad l=0,1,2,\ldots\,. \quad (22)$$

IV. PROPAGATING MODES

In this section, we discuss the more interesting propagating solution ($\overline{k}\neq 0$) of equations (12a) and (12b). In the classical limit [$B(z_0)=1$], one immediately finds a phonon mode solution

$$\overline{h} = \exp(2s/5), \quad \overline{f}=0, \quad \overline{\omega}^2 = \frac{5}{6}\overline{k}^2. \quad (23)$$

This is a longitudinal sound wave with the dispersion relation $\omega = c_0 k$, where the sound velocity $c_0^2 = 5k_B T/3m$ is the same as for a uniform classical gas. There is no radial oscillation (i.e., $\overline{f}=0$) associated with this phonon mode in the classical limit. Using Eq. (14), the associated density fluctuation is found to be

$$\delta n(\mathbf{r},t) \propto \exp(-3r_\perp^2/5R^2)\exp(ikz - ic_0 kt). \quad (24)$$

In the classical limit, one can show that the dispersion relation $\omega = c_0 k$ is valid for any cylindrical trap potential $U_0(r_\perp)$, i.e., it is not limited to parabolic potentials. In this more general case, the solution of the hydrodynamic equation corresponding to the phononlike mode in the classical limit is given by $h = \exp(2U_0/5k_B T)$ and $f=0$, with the associated density fluctuation $\delta n \propto \exp(-3U_0/5k_B T)$.

In order to find how the nonpropagating normal mode solutions in Eq. (17) are modified when $k\neq 0$, we expand $\overline{f}$ as follows:

$$\overline{f}(s) = \sum_n a_n \overline{f_n^{(0)}}(s). \quad (25)$$

This follows the approach of Zaremba [2] for a Bose-condensed gas at $T=0$ in a cigar-shaped trap. Substituting Eq. (25) into Eqs. (12a) and (12b), we obtain the coupled linear equations for the coefficients a_n:

$$\left(\overline{\omega}^2 - [\overline{\omega}_n^{(0)}]^2 - \frac{5}{6}\frac{[\overline{\omega}_n^{(0)}]^2}{\overline{\omega}^2 - \frac{5}{6}\overline{k}^2}\overline{k}^2\right)a_n + \frac{2}{3}\frac{\overline{k}^2}{\overline{\omega}^2 - \frac{5}{6}\overline{k}^2}\sum_{n'} M_{nn'}a_{n'} = 0, \quad (26)$$

where the matrix elements $M_{nn'}$ are defined by

$$M_{nn'} \equiv \int_0^\infty ds\ s^2 e^{-s}\overline{f_n^{(0)}}(s)\overline{f_{n'}^{(0)}}(s). \quad (27)$$

Using the identity for Laguerre polynomials

$$\int_0^\infty ds\ e^{-s}L_n(s)L_{n'}(s) = \delta_{nn'}(n!)^2, \quad (28)$$

we find

$$M_{nn'} = 2n\delta_{nn'} - \sqrt{nn'}\,\delta_{n',n\pm 1}. \quad (29)$$

To lowest order in $\overline{k}^2$, one finds the eigenvalue $\overline{\omega}^2$ of Eq. (26) is given by ($M_{nn}=2n$)

$$\overline{\omega}^2 \simeq \frac{10}{3}n + \left(\frac{5}{6} - \frac{M_{nn}}{5n}\right)\overline{k}^2 = \frac{10}{3}n + \frac{13}{30}\overline{k}^2. \quad (30)$$

In ordinary units, the excitation spectrum is given by

$$\omega_n^2(k) = n\frac{10}{3}\omega_0^2 + \frac{13}{15}\frac{k_B T}{m}k^2, \quad n=1,2,3,\ldots\,. \quad (31)$$

We note that the correction term in Eq. (31) is of order $(kR)^2$ relative to the first term, which is assumed to be large. Thus this spectrum for propagating modes for a classical gas in a cylindrical harmonic trap is only valid for $kR \ll 1$, where R is the radial size of the trapped gas density profile.

V. VARIATIONAL SOLUTIONS

For a Bose gas above T_{BEC}, one cannot easily solve the coupled equations in Eqs. (12a) and (12b) for $k\neq 0$. An alternative approach is to recast these equations into a variational form, following recent work [10] in solving the two-fluid hydrodynamic equations for a trapped Bose-condensed gas [11]. One finds that the functional

$$E[\bar{f},\bar{h}]\equiv\frac{\mathrm{Re}\int_0^\infty ds g_{3/2}(z_0)[s\bar{f}^*\hat{L}[\bar{f}]+\frac{5}{6}B(z_0)|\bar{h}|^2\bar{k}^2-2s\bar{f}^*(\frac{5}{3}B(z_0)d\bar{h}/ds-\frac{2}{3}\bar{h})\bar{k}]}{\int_0^\infty ds g_{3/2}(z_0)(s|\bar{f}|^2+|\bar{h}|^2)}. \tag{32}$$

has the property that conditions $\delta E/\delta\bar{f}=0$, $\delta E/\delta\bar{h}=0$ yields Eqs. (12a) and (12b). Thus the normal mode eigenvalues ω^2 are given by the stationary value of this functional $E[\bar{f},\bar{h}]$.

For $\bar{k}=0$ with $\bar{\omega}\neq 0$, we can use Eq. (32) to estimate the normal mode frequencies using the classical solutions of Eq. (17), $\bar{f}=\bar{f}_n^{(0)}$ and $\bar{h}=0$, as trial functions. The frequency so determined is given by

$$\bar{\omega}^2=\frac{\int_0^\infty ds\ \ g_{3/2}(z_0)s\bar{f}_n^{(0)}\hat{L}[\bar{f}_n^{(0)}]}{\int_0^\infty ds g_{3/2}(z_0)s(\bar{f}_n^{(0)})^2}=\frac{10}{3}n+\frac{10}{3}\frac{\int_0^\infty ds[g_{3/2}(z_0)-g_{5/2}(z_0)]\bar{f}_n^{(0)}[s^2(d^2\bar{f}_n^{(0)}/ds^2)+2s(d\bar{f}_n^{(0)}/ds)]}{\int_0^\infty ds g_{3/2}(z_0)s(\bar{f}_n^{(0)})^2}. \tag{33}$$

The second term in Eq. (33) gives the quantum correction to the classical limit.

For $k\neq 0$, the most interesting propagating mode is the phonon mode with $\omega\propto k$. For trial functions in Eq. (32), we take [see Eq. (23)]

$$\bar{f}=\bar{k}A_f,\quad \bar{h}=A_h\exp(2s/5), \tag{34}$$

where the constants A_f and A_h are real and independent of s. To first order in $\bar{k}$, we find a phononlike solution $\bar{\omega}=\overline{ck}$, with the (dimensionless) sound velocity $\bar{c}$ given by

$$\bar{c}^2=\frac{\left(\frac{5}{6}\int_0^\infty ds g_{5/2}(z_0)e^{4s/5}-\frac{2}{15}\left\{\int_0^\infty ds[g_{3/2}(z_0)-g_{5/2}(z_0)]se^{2s/5}\right\}^2\Big/\int_0^\infty ds g_{3/2}(z_0)s\right)}{\int_0^\infty ds g_{3/2}(z_0)e^{4s/5}}. \tag{35}$$

The amplitudes in Eq. (34) that are associated with this phonon mode have the ratio

$$\frac{A_f}{A_h}=-\frac{\int_0^\infty ds[g_{3/2}(z_0)-g_{5/2}(z_0)]se^{2s/5}}{5\int_0^\infty ds g_{3/2}(z_0)s}. \tag{36}$$

One can see that in a degenerate Bose gas, where $g_{3/2}(z_0)\neq g_{5/2}(z_0)$, the *radial* oscillations (A_f) are coupled to the *axial* oscillations (A_h).

The normal mode frequencies given by the variational expressions in Eqs. (33) and (35) can be numerically calculated. All the integrals can be evaluated analytically using the useful identity

$$\int_0^\infty ds\ \ g_n(z_0)s^m=m!g_{n+m+1}(\tilde{z}_0), \tag{37}$$

where $\tilde{z}_0\equiv e^{\beta\mu_0}$ and g_n is the Bose-Einstein function, as defined below Eq. (2). It is useful to plot the temperature dependence relative to the T_{BEC} for an ideal gas in a cigar-shaped trap described by Eq. (3). For a trap of length L with N atoms, we have [using Eq. (2)]

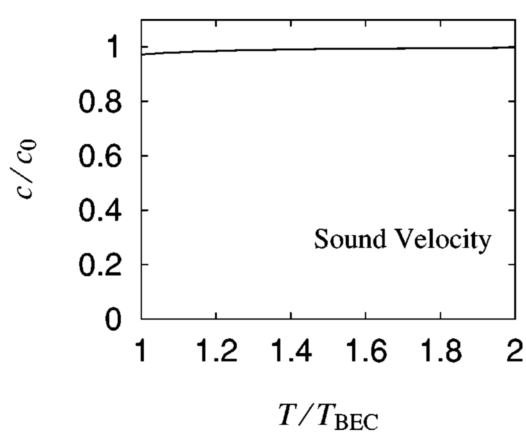

FIG. 1. The sound velocity c as a function of temperature relative to T_{BEC}. The values are normalized to the classical gas result $c_0=\sqrt{5k_BT/3m}$.

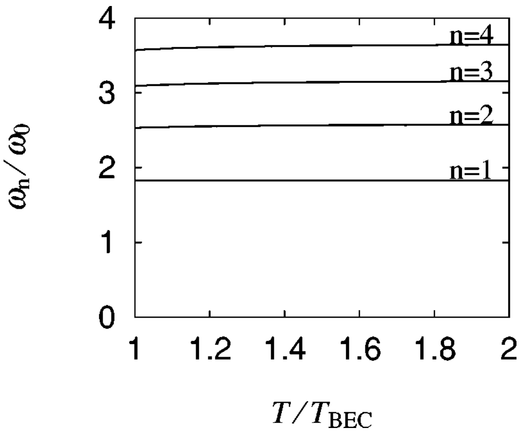

FIG. 2. The normal mode frequencies ω_n for $k=0$ as a function of temperature, as given by Eq. (33). The frequencies are normalized to the radial trap frequency ω_0.

$$N=\int d\mathbf{r}\ \ n_0(\mathbf{r})=\frac{2\pi L}{\Lambda^3}\int_0^\infty dr_\perp r_\perp g_{3/2}(z_0)$$
$$=\frac{\pi R^2 L}{\Lambda^3}\int_0^\infty ds g_{3/2}(\tilde{z}_0 e^{-s})$$
$$=L\left(\frac{m\omega_0}{2\pi\hbar}\right)^{1/2}\left(\frac{k_B T}{\hbar\omega_0}\right)^{5/2} g_{5/2}(\tilde{z}_0). \tag{38}$$

When $T=T_{BEC}$, we have $\mu_0=0$ and hence $\tilde{z}_0=1$. Thus Eq. (38) gives the Bose-Einstein transition temperature

$$k_B T_{BEC}=\hbar\omega_0\left[\frac{N}{L}\left(\frac{2\pi\hbar^2}{m\omega_0}\right)^{1/2}\frac{1}{\zeta(5/2)}\right]^{2/5} \tag{39}$$

for a cigar trap in the usual semiclassical approximation.

In Fig. 1, we show how the sound velocity given by Eq. (35) varies with temperature down to T_{BEC}, relative to the classical value $c_0=\sqrt{5k_B T/3m}$. In Fig. 2, we show the temperature-dependent results for the frequencies of the non-propagating modes based on Eq. (33). We recall that since $\bar{f}_1^{(0)}$ is constant, the correction term in Eq. (33) vanishes for the $n=1$ mode. The variational calculations shown in Figs. 1 and 2 indicate that there is little change in the normal mode frequencies given by the classical limit for temperature down to $T=T_{BEC}$. As noted at the end of Ref. [7], these results are to be expected since the only place where the Bose nature of the gas enters is in the first term of Eq. (1). This involves the ratio $B(z_0)=g_{5/2}(z_0)/g_{3/2}(z_0)$, which is remarkably close to the classical value of unity, even at the center of the trap.

VI. PROPAGATION OF SOUND PULSES

In this section, following the approach of Zaremba [2], we discuss the propagation of sound pulses induced by a small external perturbation $\delta U(\mathbf{r},t)$. We assume that $\delta U(\mathbf{r},t)$ has no radial dependence and is switched on at $t=0$, i.e., the external perturbation is of the form

$$\delta U(\mathbf{r},t)=\delta U(z)\theta(t). \tag{40}$$

The equation of motion (1) with the external perturbation δU can be solved [2] by introducing a Fourier representation of the velocity fluctuations [compare with Eq. (5)] and the external perturbation

$$\mathbf{v}(\mathbf{r},t)=\int_{-\infty}^\infty \frac{dk}{2\pi}e^{ikz}(xf(k,r_\perp,t),yf(k,r_\perp,t),h(k,r_\perp,t)),$$
$$\delta U(z)=\int_{-\infty}^\infty \frac{dk}{2\pi}e^{ikz}\delta U(k). \tag{41}$$

Taking the Fourier transform of Eq. (41) and using the radial variable s defined in Eq. (9), we obtain a coupled set of equations [compare with Eqs. (12a) and (12b)]

$$\frac{\partial^2\bar{f}}{\partial t^2}+\omega_0^2\left[\hat{L}[\bar{f}]-\bar{k}\left(\frac{5}{3}B(z_0)\frac{\partial\bar{h}}{\partial s}-\frac{2}{3}\bar{h}\right)\right]=0, \tag{42a}$$

$$\frac{\partial^2\bar{h}}{\partial t^2}+\omega_0^2\left\{\frac{5}{6}B(z_0)\bar{k}^2\bar{h}-\bar{k}\left[\frac{5}{3}B(z_0)s\frac{\partial\bar{f}}{\partial s}+\left(\frac{5}{3}B(z_0)-s\right)\bar{f}\right]\right\}$$
$$=-i\frac{k}{m}\delta U(k)\delta(t). \tag{42b}$$

Here we have used notation analogous to that in Eq. (11), but now $\bar{f}$ and $\bar{h}$ also depend on t and are for a particular k component.

In order to solve these coupled equations, we expand $\bar{f}$ and $\bar{h}$ as follows:

$$\bar{f}(k,s,t)=\sum_m b_m(k,t)\bar{f}_m(k,s),$$
$$\bar{h}(k,s,t)=\sum_m b_m(k,t)\bar{h}_m(k,s), \tag{43}$$

where the basis functions $\bar{f}_m$ and $\bar{h}_m$ are the normal mode solutions of Eq. (12). These have eigenvalues $\bar{\omega}_m^2(k)$ and form an orthonormal set [see Eq. (8b)],

$$\int_0^\infty ds\ \ g_{3/2}(z_0)[s\bar{f}_m^*\bar{f}_n+\bar{h}_m^*\bar{h}_n]=\delta_{mn}. \tag{44}$$

Substituting the expression (43) into (42) and using (44), we obtain

$$\frac{\partial^2 b_n}{\partial t^2}+\omega_n^2(k)b_n=F_n(k)\delta(t), \tag{45}$$

where $\omega_n(k)\equiv\bar{\omega}_n(k)\omega_0$ and

$$F_n(k)\equiv -ik\frac{\delta U(k)}{m}\int_0^\infty ds\ \ g_{3/2}(z_0)\bar{h}_n^*(k,s). \tag{46}$$

The solution of Eq. (45) with the boundary condition $b_n(t<0)=0$ is given by

$$b_n(k,t)=\theta(t)\frac{F_n(k)}{\omega_n(k)}\sin[\omega_n(k)t], \tag{47}$$

where the quantum number n refers to the radial degree of freedom.

In order to analyze the time evolution of the associated density fluctuations, it is convenient to work with the radially averaged density

$$\overline{\delta n}(z,t) \equiv \int d\mathbf{r}_\perp \, \delta n(\mathbf{r},t). \tag{48}$$

Using Eq. (14), one finds

$$\begin{aligned}\frac{\partial}{\partial t}\overline{\delta n} &= -\int d\mathbf{r}_\perp \nabla\cdot(n_0\mathbf{v}) = -\int d\mathbf{r}_\perp n_0 \frac{\partial v_z}{\partial z}\\ &= -i\frac{2\pi}{\Lambda^3}\int_0^\infty r_\perp dr_\perp g_{3/2}(z_0)\int_{-\infty}^{\infty}\frac{dk}{2\pi}ke^{ikz}h(k,r_\perp,t)\\ &= -\frac{\pi R^2}{\Lambda^3}\sum_n \int_{-\infty}^{\infty}\frac{dk}{2\pi}e^{ikz}\frac{k^2\delta U(k)}{m\omega_n(k)}\theta(t)\\ &\quad\times \sin\omega_n(k)t\left|\int_0^\infty ds\ \ g_{3/2}(z_0)\overline{h}_n(k,s)\right|^2.\end{aligned} \tag{49}$$

Integrating Eq. (49) over t, one finds

$$\begin{aligned}\overline{\delta n}(z,t) &= -\frac{\pi R^2}{\Lambda^3}\sum_n \int_{-\infty}^{\infty}\frac{dk}{2\pi}e^{ikz}\frac{k^2\delta U(k)}{m\omega_n^2(k)}\theta(t)\\ &\quad\times[1-\cos\omega_n(k)t]\left|\int_0^\infty ds\ \ g_{3/2}(z_0)\overline{h}_n(k,s)\right|^2.\end{aligned} \tag{50}$$

One can see that a low frequency phonon mode makes a large contribution to Eq. (50). To illustrate the contribution to Eq. (50), which is associated with the phonon density fluctuations $\omega_n = ck$ in the classical limit, we use $\overline{h}_n(k,s) = A_h \exp(2s/5)$ [see Eq. (23)] where the normalization condition (44) gives $A_h^2 = 1/5\tilde{z}_0$. The contribution to Eq. (50) of this classical sound wave is given by [we use $N/L = \pi R^2/\Lambda^3(\tilde{z}_0)$ appropriate to the classical limit]

$$\begin{aligned}\overline{\delta n}(z,t) &= -\frac{5}{9}\frac{N}{L}\frac{\theta(t)}{mc^2}\{\delta U(z) - \tfrac{1}{2}[\delta U(z-ct)\\ &\quad+\delta U(z+ct)]\},\end{aligned} \tag{51}$$

which has the form of a propagating pulse moving with a speed $\pm c$.

VII. CONCLUDING REMARKS

In this paper, we have given a detailed analysis of the hydrodynamic normal modes of a Bose gas in a cigar-shaped trap above T_{BEC}. We have discussed the nonpropagating and propagating modes, both in the classical limit as well as in the degenerate Bose limit just above T_{BEC}. Our results complement the analogous studies [2,4–6] of such modes in the quantum hydrodynamic limit at $T=0$. In contrast with the $T=0$ analysis, which works with a single equation for the density fluctuations, we have to work with coupled equations for the velocity fluctuations. For simplicity, we have considered the limit of a uniform gas along the axial direction. We note that Stringari [4] has discussed the condensate modes at $T=0$ in the limit of a very weak trap in the axial direction ($\omega_z \ll \omega_0$).

As in Ref. [7], we have ignored the Hartree-Fock mean field contribution to the hydrodynamic equations. Such terms are given in Eq. (6) of Ref. [11]. In place of Eq. (1), we obtain the linearized velocity equation

$$\begin{aligned}m\frac{\partial^2\mathbf{v}}{\partial t^2} &= \frac{5P_0(\mathbf{r})}{3n_0(\mathbf{r})}\nabla(\nabla\cdot\mathbf{v}) - \nabla[\mathbf{v}\cdot\nabla U(\mathbf{r})] - \frac{2}{3}(\nabla\cdot\mathbf{v})\nabla U(\mathbf{r})\\ &\quad+2g\nabla(\nabla\cdot n_0\mathbf{v}) - \frac{\partial}{\partial t}\nabla\delta U(\mathbf{r},t).\end{aligned} \tag{52}$$

This now involves the effective trap potential

$$U(\mathbf{r}) = U_0(\mathbf{r}) + 2gn_0(\mathbf{r}), \tag{53}$$

which also appears in the equilibrium fugacity $z_0 = e^{\beta(\mu_0 - U)}$ in the expressions for $n_0(\mathbf{r})$ and $P_0(\mathbf{r})$. The usual s-wave scattering interaction is $g = 4\pi a\hbar^2/m$. The analysis given in this paper can be generalized [12] to include the effects of this HF mean field but it is much more complicated. We simply quote some final results for the classical limit. The $n=1$ nonpropagating mode has a frequency given by

$$\omega_1^2 = \frac{10}{3}\omega_0^2\left(1 - \frac{gn_0(\mathbf{r}=\mathbf{0})}{2k_{\mathrm{B}}T}\right), \tag{54}$$

where $n_0(\mathbf{r}=\mathbf{0})$ is the density at the center of the cylindrical trap. The sound velocity corresponding to Eq. (23) is given by

$$c^2 = \frac{5k_{\mathrm{B}}T}{3m} + \frac{gn_0(\mathbf{r}=\mathbf{0})}{3m}. \tag{55}$$

The two-fluid hydrodynamic equations for a trapped Bose-condensed gas ($T < T_{\mathrm{BEC}}$) have been recently discussed by Zaremba *et al.* [11]. These equations have been used to study first and second sound modes in a dilute uniform Bose gas [13] at finite temperatures. It is found that first sound corresponds mainly to an oscillation of the noncondensate, with a velocity given by

$$u_1^2 = \frac{5}{3}\frac{k_{\mathrm{B}}T}{m}\frac{g_{5/2}(z_0)}{g_{3/2}(z_0)} + \frac{2g\tilde{n}_0}{m}. \tag{56}$$

In contrast, the second sound mode mainly corresponds to an oscillation of the condensate, with a velocity given by

$$u_2^2 = \frac{g n_{c0}}{m}. \tag{57}$$

Here $n_{c0}(\tilde{n}_0)$ is the equilibrium condensate (noncondensate) density. As discussed in Ref. [13], to a good approximation, one can use

$$\tilde{n}_0 = \frac{1}{\Lambda^3} g_{3/2}(z_0), \tag{58}$$

where the equilibrium fugacity is $z_0 = e^{-\beta g n_{c0}}$.

In principle, we could use the equations in Ref. [11] to extend the analysis of the present paper and discuss the propagating first and second sound modes in a cigar-shaped trap. Here we limit ourselves to some qualitative remarks. One expects to find an expression similar to Eq. (51) for the propagation of a pulse, and there should be distinct first and second sound pulses moving with velocities quite close to u_1 and u_2 as given above. However, as the expression in Eq. (51) shows, the relative amplitude of these two modes is proportional to $1/u_i^2$. We conclude that if pulse experiments such as in Ref. [1] were done in the hydrodynamic region, most of the weight would be in the second sound pulse if $u_2^2 \ll u_1^2$. This mode, given by Eq. (57), is the natural hydrodynamic analogue of the Bogoliubov mode exhibited in the quantum hydrodynamic region at $T=0$ [1,2]. At temperatures close to T_{BEC}, the first sound pulse has a much faster speed and thus its intensity will be very weak. The observation of distinct first and second sound pulses in cigar-shaped traps would be very dramatic evidence for superfluid behavior in dilute Bose gases. The experiment would best be done at intermediate or lower temperatures, where u_1 and u_2 are more comparable in magnitude. Observation of the first sound pulse would be a way of measuring the noncondensate density ‘‘underneath’’ the condensate.

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