Quantum mechanics can be formulated as a non-Markovian stochastic process

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Both quantum mechanics and stochastic-process theory deal with dynamical systems the behaviors of which can be predicted only probabilistically. It is shown here that the measurable behavior of an elementary quantum system can be modeled as a non-Markovian stochastic process. The conclusion that quantum behavior can generally be modeled as a real stochastic process eliminates some mysterious characteristics of quantum mechanics when viewed in Copenhagen-interpretation terms. [S1050-2947(98)07710-5]

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I. INTRODUCTION

The efforts to reformulate quantum mechanics in terms of deterministic hydrodynamics [1–10] or the theory of real stochastic processes [11–67] have a history that goes back approximately 70 years. Although hydrodynamic models have a certain heuristic value, they contradict the generally accepted statistical (according to Born) character of quantum-mechanical motion [68–72].

The stochastic approach is another case: if it is accepted that behavior of microparticles is inalienably statistical, why is a specific quantum theory [73] used for its mathematical modeling, but not a theory of stochastic processes intended for analysis of *any* probabilistic process? The description of microworld phenomena by the use of the stochastic processes theory would contradict neither Born's interpretation of the wave function, nor von Neumann's theorem about hidden parameters [74], because nonlocal hidden parameters are not at variance with quantum mechanics as is evident from the well-known Bell's theorem [75]. Indeed, a quantum-mechanical random walk is just a completely discontinuous process, not only in the phase, but also in the configuration space [76]—unlike the Brownian motion, which is completely continuous with probability 1 not only in the configuration (cf. the Doob theorem [77]) but also in the phase space [24].

Nevertheless, it is generally accepted [78–88] that this goal has not been achieved in any of the works [11–66], because each time such nondeductive and nonclassical procedures as the introduction of imaginary time [25], a negative diffusion coefficient [33,49,59], complex probability [73,86], or the diffusion equation for a non-ambiguous-discontinuous stochastic process [22,38,42,54,57], etc. are necessary to obtain the Schrödinger equation from the classical Brownian motion.

By now it seems to be accepted [52,89] that it is impossible to obtain anything from the Brownian motion except the diffusion Fokker-Plank equation. For the Schrödinger equation to be deduced, it is necessary to proceed from a (non-Brownian) Markovian process [52,64,90]. However, it was shown [82,85,87], that quantum-mechanical processes cannot be described by the real Markovian equations, and in [85] it is even manifested that no non-Markovian equation is sufficient for solving this problem. The goal of the present work is to make clear which stochastic equations (Markovian, non-Markovian, or neither) are suitable for mathematical modeling of the quantum-mechanical processes.

II. A TWO-DIMENSIONAL QUANTUM SYSTEM

In [87] Gillespie studied a possibility of modeling as a Markov process the temporal evolution of the simplest quantum-mechanical two-level system previously considered in [60]. Let ''unperturbed'' Hamiltonian H_0 have two orthonormal eigenstates $|1\rangle$ and $|2\rangle$:

$$\left.\begin{array}{c} H_0|k\rangle = E_0|k\rangle \\ \langle k|l\rangle = \delta_{k,l} \end{array}\right\} k,l=1,2. \tag{1}$$

Oscillations between eigenstates $|1\rangle$ and $|2\rangle$ are induced by a ''perturbing'' Hamiltonian H', which is defined by its representation in the Hamiltonian H_0 eigenbasis according to

$$\langle k|H'|l\rangle = \begin{cases} E', & k=l \\ \Delta E, & k\neq l \end{cases} (k,l=1,2). \tag{2}$$

The time evolution of the whole system is determined by the Schrödinger equation:

$$i\hbar \frac{d}{dt}|\Psi(t)\rangle = (H_0+H')|\Psi(t)\rangle \tag{3}$$

with the initial condition, for example,

$$|\Psi(t_0)\rangle = |1\rangle. \tag{4}$$

If we expand $\Psi(t)$ in the eigenbasis of H_0,

$$|\Psi(t)\rangle = a_1(t)|1\rangle + a_2(t)|2\rangle, \tag{5}$$

then, from the solution set of Eq. (3) under initial condition (4), we can obtain the probabilities of finding the system in states $|1\rangle$ or $|2\rangle$ at any time $t \geqslant t_0$, respectively:

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$$P(1,t|1,t_0)=|\langle 1|\Psi(t)\rangle|^2=|a_1(t)|^2 =\cos^2[\Delta E(t-t_0)/\hbar], \tag{6a}$$

$$P(2,t|1,t_0)=|\langle 2|\Psi(t)\rangle|^2=|a_2(t)|^2 =\sin^2[\Delta E(t-t_0)/\hbar]. \tag{6b}$$

According to the theory of Markov processes [91] the time evolution of probabilities (6) should be modeled by the ''master equation'' set

$$\frac{d}{dt}P(1,t|1,t_0)=W_-(2,t)P(2,t|1,t_0) -W_+(1,t)P(1,t|1,t_0), \tag{7a}$$

$$\frac{d}{dt}P(2,t|1,t_0)=W_+(1,t)P(1,t|1,t_0) -W_-(2,t)P(2,t|1,t_0), \tag{7b}$$

where $W_\pm(k,t)$ are the ''stepping functions'' satisfying the Markovian consistency conditions [87]:

$$W_+(2,t)=W_-(1,t)\equiv 0, \tag{8}$$

$$W_+(1,t)dt=P(2,t+dt|1,t), \tag{9}$$

$$W_-(2,t)dt=P(1,t+dt|2,t). \tag{10}$$

By direct substitution, it is easy to show that functions (6) do not solve the equation set (7) under the conditions (8)–(10). Hence, Gillespie concluded: *the fundamental quantum-mechanical evolution-measurement process cannot generally be modeled as a jump Markov process [87]. It is necessary to agree completely with this conclusion.*

The question now arises as to whether there is a possibility of modeling the quantum-mechanical evolution-measurement process by means of the non-Markovian generalized master equation (GME), so that the probabilities (6) are satisfied. Gillespie [87] does not give any definite conclusions about this. Grabert, Hänggi, and Talkner [85], however, insist on the impossibility in general for quantum-mechanical probabilities to be modeled by the classical stochastic theory equations—neither Markovian, nor any non-Markovian equations. However, this notion is refuted by our success in finding a non-Markovian GME set such that probabilities (6) are satisfied. By the direct substitution alone it can be proved that probabilities (6) satisfy the following generalization of the Kolmogorov-Chapman equation:

$$\tfrac{1}{2}(t-t_0)\frac{\partial}{\partial t}P(n,t|k,t_0) =\sum_{m=1,2}\int_{t_0}^{t}P(n,t|m,\tau)d_\tau P(m,\tau|k,t_0) =-\sum_{m=1,2}\int_{t_0}^{t}P(m,\tau|k,t_0)d_\tau P(n,t|m,\tau), \tag{11}$$

where $k=1,2$; $n=1,2$. Differentiating Eq. (11) with respect to t, one obtains the desired GME set, which probabilities (6) satisfy:

$$\frac{d}{dt}P(1,t|1,t_0)+(t-t_0)\frac{d^2}{dt^2}P(1,t|1,t_0) =2\int_{t_0}^{t}P(1,\tau|1,t_0)d_\tau W_+(t-\tau) -2\int_{t_0}^{t}P(2,\tau|1,t_0)d_\tau W_-(t-\tau), \tag{12a}$$

$$\frac{d}{dt}P(2,t|1,t_0)+(t-t_0)\frac{d^2}{dt^2}P(2,t|1,t_0) =2\int_{t_0}^{t}P(2,\tau|1,t_0)d_\tau W_-(t-\tau) -2\int_{t_0}^{t}P(1,\tau|1,t_0)d_\tau W_+(t-\tau), \tag{12b}$$

where the stepping functions

$$W_+(t-\tau)=W_-(t-\tau)=\frac{\partial}{\partial t}P(2,t|1,\tau)=\frac{\partial}{\partial t}P(1,t|2,\tau) \tag{13}$$

are introduced.

The following reasons led us to deduce the equations (11)–(13). We have previously established [92] that, in the case of a purely discontinuous stochastic process, a non-Markovian property may be introduced in different ways: by introduction of a nonlinear combination of probabilities, or by integration over a ''memory function,'' etc. However, the operation of integration over a memory function alone is compatible with the quantum-mechanical systems. The matter is that even the simplest differential equation with an argument lag

$$Y'(x)=-\int_0^\infty Y(x-s)dr(x,s) \tag{14}$$

possesses the remarkable property: under sufficiently large argument lag, every solution of Eq. (14) has the number of roots unlimited to its right [93]. Meanwhile, the quantum-mechanical probabilities are characterized by the oscillations unbounded in time. Proceeding from this property of Eq. (14), we introduced a distributed time lag both in the income (positive), and in the outcome (negative) item in the right side of GME (12). Moreover, the form of delay function is defined strictly by the ''consistency relationships'' proposed by Gillespie [87], compelling us to choose the stepping function in the unique possible form (13).

The non-Markovian extension (11) of the Markovian Kolmogorov-Chapman equation is defined uniquely by the GME [Eq. (12)], which is consistent with the time evolution of quantum-mechanical probabilities (6).

Let us show that Eqs. (11)–(12) are the minimal non-Markovian extension of Markovian equations (7). Indeed, for

sufficiently small difference $t-t_0$, it is possible to limit oneself by the first three terms in the power decomposition:

$$P(n,t|k,t_0)=P(n,t_0|k,t_0)+\left.\frac{\partial P(n,t|k,t_0)}{\partial t}\right|_{t=t_0}(t-t_0)+\frac{1}{2}\left.\frac{\partial^2 P(n,t|k,t_0)}{\partial t^2}\right|_{t=t_0}(t-t_0)^2+\cdots. \quad (15)$$

For probabilities (6) one has $\partial P/\partial t|_{t=t_0}=0$ and $P(n,t|k,t_0)=\delta_{n,k}$. As a result, we get from Eq. (15)

$$P(n,t|k,t_0)\cong\delta_{n,k}+\frac{1}{2}\left.\frac{\partial^2 P}{\partial t^2}\right|_{t=t_0}(t-t_0)^2. \quad (16)$$

By differentiating Eq. (16), one obtains, using once again Eq. (16),

$$\frac{\partial}{\partial t}P(n,t|k,t_0)\cong\left.\frac{\partial^2 P}{\partial t^2}\right|_{t=t_0}(t-t_0)=\frac{2}{t-t_0}[P(n,t|k,t_0)-\delta_{n,k}]. \quad (17)$$

In another way, Eq. (17) can be rewritten as

$$\tfrac{1}{2}(t-t_0)\frac{\partial}{\partial t}P(n,t|k,t_0)=P(n,t|k,t_0)-\delta_{n,k}. \quad (18)$$

Substituting Eq. (18) into Eq. (11), one gets two identical Kolmogorov-Chapman equations for the both cases ($n=k=1$ and $n=2$, $k=1$):

$$P(n,t|k,t_0)=\sum_{m=1,2}\int_{t_0}^{t}P(m,\tau|k,t_0)d_\tau P(n,t|m,\tau). \quad (19)$$

It is well known [91] that the Markov character of the process is summarized in the corresponding Kolmogorov-Chapman equation.

In a similar manner, at the small difference $t-t_0$, Eq. (12) become the ordinary Markovian master equation [94]:

$$\frac{\partial}{\partial t}P(1,t|1,t_0)=\int_{t_0}^{t}P(1,\tau|1,t_0)d_\tau W_+(t-\tau)-\int_{t_0}^{t}P(2,\tau|1,t_0)d_\tau W_-(t-\tau), \quad (20a)$$

$$\frac{\partial}{\partial t}P(2,t|1,t_0)=\int_{t_0}^{t}P(2,\tau|1,t_0)d_\tau W_-(t-\tau)-\int_{t_0}^{t}P(1,\tau|1,t_0)d_\tau W_+(t-\tau), \quad (20b)$$

where $dW_\pm/d\tau=\text{const}$, as it follows from Eq. (13) due to the small difference $(t-t_0)$.

Thus, Eq. (11) is the minimal non-Markovian generalization of Kolmogorov-Chapman equation (19) necessary to model the quantum-mechanical probabilities time evolution. Similarly, GME set (12) is the minimal non-Markovian generalization for master equation set (20) or (7) necessary for the same goal.

The Kolmogorov-Chapman equation (19) contains the transition probability $P(n,t|k,t_0)$ on its left side and all combinations of transition probabilities over the intermediate state m in the intermediate time τ on its right side. As to our non-Markovian generalization (11) of Markovian equation (19), it contains on the left side the derivation from the transition probability $P(n,t|k,t_0)$ that does not allow us to see the essence of introduced non-Markovity. To eliminate this ''defect,'' let us rewrite Eq. (11) in the equivalent integral form:

$$P(n,t|k,t_0)-\delta_{nk}=2\int_{t_0}^{t}\frac{1}{s-t_0}\left[\sum_m\int_s^t P(n,s|m,\tau)d_\tau P(m,\tau|k,t_0)\right]ds. \quad (21)$$

Equation (21) is the minimal non-Markovian extension of Markovian equation (19), and its right side demonstrates explicitly the memory and time lag introduced in the standard Kolmogorov-Chapman equation (19) to achieve adequate mathematical modeling of quantum-mechanical probabilities.

Gillespie [87] had entitled his paper ''Why quantum mechanics cannot be formulated as a Markov process'' and Grabert, Hänggi, and Talkner [85] had entitled their paper ''Is quantum mechanics equivalent to a classical stochastic process?'' The title of our article is explained by the results of the next section.

III. A GENERALIZED TWO-LEVEL SYSTEM

It is possible to suppose that non-Markovian stochastic equations for quantum probabilities were found in the previous section only for the simplest example of quantum problem (1)–(3), which may not even be of real interest from a physical point of view. That is why in this section we shall analyze the quantum two-level system, which can be evidently realized physically. Let us consider a process of photon emission (absorption) by a two-level atom interacting with monochromatic electromagnetic field. The solution of this quantum-mechanical problem in semiclassical approximation was discussed in the article [95], according to which the Hamiltonian of such a system is

$$H(t)=\begin{Vmatrix} E_1-i\hbar\dfrac{\gamma}{2}, & Ve^{-i\omega t} \\ Ve^{i\omega t}, & E_2+i\hbar\dfrac{\gamma}{2}\end{Vmatrix}, \quad (22)$$

where E_1 is the upper level energy, E_2 is the lower level energy, γ^{-1} is a radiation lifetime of the upper level, V is the amplitude of monochromatic electromagnetic wave, and ω, $\hbar$, and t denote the wave frequency, Plank's constant, and time, respectively.

The full wave function $\Psi(t)$ is determined from the time-dependent Schrödinger equation,

$$i\hbar\,\frac{\partial}{\partial t}\,|\Psi(t)\rangle = H(t)|\Psi(t)\rangle. \tag{23}$$

Two-component wave function $\Psi(t)$ has two eigenvalues (λ_1,λ_2) and two eigenvectors

$$|\varphi_1\rangle = \begin{Vmatrix} a_{11} \\ a_{12} \end{Vmatrix}, \quad |\varphi_2\rangle = \begin{Vmatrix} a_{21} \\ a_{22} \end{Vmatrix},$$

satisfying the Schrödinger equations

$$H|\varphi_1\rangle = \lambda_1|\varphi_1\rangle, \quad H|\varphi_2\rangle = \lambda_2|\varphi_2\rangle \tag{24}$$

with time-independent Hamiltonian

$$H = T(t)H(t)T^{+}(t) - i\hbar T(t)\,\frac{dT^{+}(t)}{dt},$$

where $T(t)$ is the unitary transformation.

The resulting expressions for the transition amplitudes are [95]

$$A_{21} = \frac{\overline{V}}{V}\exp(i\omega t)A_{12} = -2ie^{(i/2)\omega t}\,\frac{\overline{V}}{\hbar\mu}\sin\left(\frac{\mu t}{2}\right), \tag{25a}$$

$$A_{11} = \overline{A}_{22} = e^{-(i/2)\omega t}\left[\cos\left(\frac{\mu}{2}\,t\right) + i\,\frac{\Delta\omega}{\mu}\sin\left(\frac{\mu}{2}\,t\right)\right], \tag{25b}$$

where

$$\mu = \frac{2}{\hbar}\,\sqrt{|V|^2 + \tfrac{1}{4}\hbar^2(\Delta\omega)^2}, \tag{26}$$

and the detuning of the system from exact resonance is designated $\hbar\Delta\omega$ with

$$\hbar\Delta\omega = (E_1 - E_2) - \hbar\omega = \hbar\omega_0 - \hbar\omega. \tag{27}$$

From Eq. (25) the corresponding transition probabilities can be obtained:

$$P(1,t|1,t_0) = |A_{11}|^2 = \cos^2\left(\frac{\mu}{2}\,(t-t_0)\right) + \frac{\Delta\omega^2}{\mu^2}\sin^2\left(\frac{\mu}{2}\,(t-t_0)\right), \tag{28a}$$

$$P(2,t|1,t_0) = |A_{21}|^2 = \frac{4|V|^2}{\hbar^2\mu^2}\sin^2\left(\frac{\mu}{2}\,(t-t_0)\right). \tag{28b}$$

Let us verify the equality to unit of the sum of probabilities (28):

$$P(1,t|1,0) + P(2,t|1,0) = \cos^2\left(\frac{\mu}{2}\,t\right) + \frac{(\hbar\Delta\omega)^2 + 4|V|^2}{\hbar^2\mu^2}\sin^2\left(\frac{\mu}{2}\,t\right) = \cos^2\left(\frac{\mu}{2}\,t\right) + \sin^2\left(\frac{\mu}{2}\,t\right) = 1, \tag{29}$$

where in the second equality the relationship (26) is used. From Eq. (29) it is evident that probabilities (28) are the classical probabilities varying in the range 0–1. They are the generalization of the probabilities (6) and can be reduced to them under the full resonance condition $\Delta\omega = 0$, that is when in Eq. (28b) the factor $(4|V|^2/\hbar^2\mu^2)$ equals 1.

We now turn to search for the non-Markovian GME, which the probabilities (28) satisfy. For this purpose, we rewrite the formulas (28) in the very compact form:

$$P(1,t|1,\tau) = |A_{11}|^2 = \cos^2\alpha(t-\tau) + q\,\sin^2\alpha(t-\tau), \tag{30a}$$

$$P(2,t|1,\tau) = |A_{21}|^2 = (1-q)\sin^2\alpha(t-\tau), \tag{30b}$$

with

$$\alpha \equiv \frac{\mu}{2}, \quad q \equiv \frac{\Delta\omega^2}{\mu^2}. \tag{31}$$

By the direct substitution, it is easy to check that probabilities (30) satisfy the following integrodifferential equation:

$$q[\,\delta_{n,k} - P(n,t|k,t_0)] = \tfrac{1}{2}(1-q)(t-t_0)\,\frac{\partial P(n,t|k,t_0)}{\partial t} + \sum_{m=1,2}\int_{t_0}^{t} P(m,\tau|k,t_0)\,d_\tau P(n,t|m,\tau), \tag{32}$$

where $n = 1,2$ and $k = 1,2$. If $q = 0$, when the problem (22)–(28) is reduced to the problem (1)–(6), the equation (32) has become equivalent to Eq. (11). Thus, Eq. (32) is just the non-Markovian generalization of the Kolmogorov-Chapman equation (19), modeling the time evolution of the quantum-mechanical probabilities (28) [or (30)].

With Eqs. (15)–(18), it is not difficult to show that non-Markovian equation (32) gets simplified and becomes the Markovian Kolmogorov-Chapman equation (19) under sufficiently small values of the difference $(t-t_0)$. Thus, the non-Markovian equation (32) is the minimal extension of the Markovian equation (19) allowing one to include the quantum-mechanical probabilities (28) [or (30)] into the theory of real stochastic processes.

Differentiate Eq. (32) with respect to t, one obtains the following two integrodifferential equations:

$$(1+q)\frac{d}{dt}P(1,t|1,t_0)+(1-q)(t-t_0)\frac{d^2}{dt^2}P(1,t|1,t_0)$$
$$=2\int_{t_0}^{t}P(1,\tau|1,t_0)d_\tau W_+(t-\tau)$$
$$-2\int_{t_0}^{t}P(2,\tau|1,t_0)d_\tau W_-(t-\tau), \tag{33a}$$

$$(1+q)\frac{d}{dt}P(2,t|1,t_0)+(1-q)(t-t_0)\frac{d^2}{dt^2}P(2,t|1,t_0)$$
$$=-2\int_{t_0}^{t}P(1,\tau|1,t_0)d_\tau W_+(t-\tau)$$
$$+2\int_{t_0}^{t}P(2,\tau|1,t_0)d_\tau W_-(t-\tau), \tag{33b}$$

where notations

$$W_+(t-\tau)\equiv\frac{\partial}{\partial t}P(2,t|1,\tau), \tag{34a}$$

$$W_-(t-\tau)\equiv\frac{\partial}{\partial t}P(1,t|2,\tau) \tag{34b}$$

are used. At $q=0$ (the exact resonance), the stepping functions (34) are equal to Eq. (13), and equations (33) can be reduced to the equations (12). Thus, Eqs. (33) are the desired non-Markovian GME set satisfied by the quantum probabilities (28) [or (30)].

As in the Sec. II it is possible to check that under sufficiently small difference $(t-t_0)$ the non-Markovian GME set [Eq. (33)] transforms into the ordinary (Markovian) master equations set (20) or (7). This means that GME set (33) is the minimal non-Markovian extension of Markovian ME [Eq. (7)] allowing one to include the quantum-mechanical probabilities (28) [or (30)] into the theory of real stochastic processes. The transition functions $W_+(t-\tau)$ and $W_-(t-\tau)$ in the GME set (33) thus obey consistency relationships (34) established in [87]. In the limit of $q\to 1$ (that is, $\Delta\omega\to\infty$, the full absence of resonance), the GME set (33) becomes equal to the set of equations (20), that are the ordinary (Markovian) master equations of the macroscopic balance with positive ''birth'' and negative ''death'' terms. Thus once again non-Markovian GME set (33) is the desired minimal generalization of Markovian ME set (20), modeling the time evolution of quantum probabilities (28) [or (30)].

From Eq. (30b) for sufficiently small transition time $dt=t-\tau$ one obtains

$$P(1,t+dt|2,t)=P(2,t+dt|1,t)=(1-q)\alpha^2(dt)^2. \tag{35}$$

It is a particular manifestation of the general regularity in the quantum-mechanical systems behavior [87,96]:

$$1-|\langle\Psi|e^{-iHdt/\hbar}|\Psi\rangle|^2=[\Delta_\Psi H/\hbar]^2(dt)^2, \tag{36}$$

where H is the time-independent Hamiltonian.

IV. UNITY AND CONTRAST

In Ref. [49] one of the authors had shown that Brownian (classical) and quantum-mechanical random walks can be described by the almost identical equations—see Eqs. (8) and (11) in [49]:

$$\frac{\partial w}{\partial t}=-\mathrm{div}\left(w\nabla\frac{S}{m}\right), \tag{37a}$$

$$\frac{\partial S}{\partial t}+\frac{1}{2m}(\nabla S)^2-\kappa mb^2\left[\frac{\nabla^2 w}{w}-\frac{1}{2}\left(\frac{\nabla w}{w}\right)^2\right]+\frac{\partial U}{\partial w}=0, \tag{37b}$$

with

$$b=\frac{\hbar}{2m},\quad \kappa=+1 \text{ for quantum-mechanical particle,} \tag{38a}$$

$$b=D,\quad \kappa=-1 \text{ for Brownian particle,} \tag{38b}$$

where w is a probability density, S is full velocity potential, m denotes the particle's mass, and U is the external potential. It is evident from Eqs. (37) and (38) that quantum-mechanical random walks of microparticles (motion Q) and diffusion walks of macroscopic Brownian particles (motion B) are connected very tightly and, simultaneously, are antipodes, much like the two poles of a magnet or like bosons and fermions.

The similarity and even identity of these two types of random walks lies in the fact that *Eqs. (37) only for the two first moments*—probability density $w=|\Psi|^2$ and current $J=(\rho/m)\nabla\vartheta$ [2] (or full velocity potential S [12], or two-time correlation function $\langle x(t)x(t-\tau)\rangle$ [85], etc.)—*are necessary and sufficiently for its full and single-valued description.* Hence, both the *Q motion* and the *B motion* ranked among the normal (Gaussian) stochastic processes.

Nonsimilarity and full contrast of Q and B motions mean that Q motion is a purely discontinuous (jump) stochastic process, and B motion is purely continuous (diffusive) random walk. It was shown by Doob [77] and by Nelson [24], that both the Brownian trajectory and its derivative are everywhere continuous with probability 1. On the other hand, the quantum-mechanical walks have everywhere discontinuous derivatives and everywhere discontinuous trajectories [76]. In terms of the Ito stochastic differential equation [77],

$$d\xi=a(t,\xi)dt+b(t,\xi)dw(t)+\int c(t,\xi,u)\tilde{\nu}(dt,du), \tag{39}$$

this means the following: in the case of quantum mechanics, diffusional item $b(t,\xi)$ is equal to zero identically, but in the case of Brownian motion, the item of jumps $c(t,\xi,u)$ is equal to zero identically:

$$\text{motion } Q,\quad b(t,\xi)\equiv 0,\quad c(t,\xi,u)\neq 0; \tag{40a}$$

$$\text{motion } B,\quad b(t,\xi)\neq 0,\quad c(t,\xi,u)\equiv 0. \tag{40b}$$

In Eq. (39), the $a(t,\xi)$ is a drift item, $w(t)$ is the Wiener process, and $\tilde{\nu}(dt,du)$ is the Poisson process [77].

In the absence of drift ($a \equiv o$), an arbitrary function

$$u(t,x) = Mf(\xi_{xt}(s)), \quad t < s \tag{41}$$

is described by the diffusion equation

$$\frac{\partial u}{\partial t} = -\tfrac{1}{2} sp[b(t,x)b^*(t,x)\nabla^2 u] \tag{42a}$$

in the completely continuous case B, and it is described by GME,

$$\frac{\partial u}{\partial t} = \int [u(t,x) + [c(t,x,u)|\nabla u(t,x)] - u(t,x + c(t,x,u))]\Pi(du), \tag{42b}$$

where $\delta\Pi_{tx}(A) = M\nu_{tx}(\delta, A)$, in the completely discontinuous case Q [77].

Nelson was the first to establish a fact that velocity undergoes a jump at *any point* of quantum-mechanical trajectory [$\overline{v}_+(\xi,t) \neq \overline{v}_-(\xi,t)$]. It is evident from this, that it is necessary try to derive the Schrödinger equation from GME [Eq. (42b)] but not from the diffusion equation (42a). However, Nelson [22] started from the diffusion equation and all followers of his approach [27,38,44,45,54,57] continue to do so to our knowledge, the only work starting from GME is Ref. [76]. Our GME set (33) has the form of Eq. (42b) for completely discontinuous process.

In closing of this section, let us note that although hydrodynamic models [1–10] do not give an adequate model of quantum-mechanical motion, they are rather useful, because they allow us to find the two first moments of the corresponding stochastic process [2]. In particular, the critics of hydrodynamic models [68,69,71,72] considered the impossibility of specifying phenomena of a microparticle's spin as its main defect. However, this defect is overcome in the recent versions of hydrodynamics model [5–8].

V CONCLUSIONS

Jump Markov process theory is the quintessential realistic stochastic theory for mathematically modeling systems that randomly jump about in real time over discrete states. For reasons of self-consistency, the probability for any Markov process to jump in the next dt must be *linear* in dt [97]. In contrast, at the *unapproximated fundamental* level, quantum-mechanical systems have *quadratic-dt* transition probabilities (35),(36) and *so cannot* be modeled as jump Markov processes. Hence, quantum-mechanical systems *must* be modeled as a *jump non-Markov* process. It is axiomatic that, there still would be consistency conditions (9),(10), that would have to be satisfied [87]. Our conditions (34a),(34b) are just these consistency conditions (9),(10), and our Eqs. (33) are just the non-Markov GME set modeling the quantum-mechanical problem (22),(23) as a jump non-Markov process.

The problem of quantum-mechanical motion description as a real Markov process was recently discussed in [98,99]. However, the authors of both articles manifested again the positions of their previous works [60,87], which were discussed above. In our view, the understanding of the nature of quantum-mechanical motion was not clarified, hence the discussions in [98,99] are not relevant to the results presented here.

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