Optical measurements as projection synthesis

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We present an extension of the projection synthesis technique which allows us, at least in principle, to determine the probability distribution for any physical observable associated with a quantized optical field mode. This probability can be inferred from the photodetection statistics obtained by causing the field mode to be measured to interfere with a second mode prepared in a suitable reference state. We give an explicit expression for the required reference state. Weak-field homodyne detection using photon counting is, perhaps, the simplest system to which projection synthesis can be applied. We find the complete set of synthesized projectors for this system and show that they form a probability operator measure. We apply this set of projectors to study the discrimination between coherent states and to measure a quasiprobability distribution. [S1050-2947(98)01110-X]

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I. INTRODUCTION

Homodyne detection [1–3] is a measurement scheme that can be used to detect nonclassical phase dependent features of a quantum optical field. The measurement is performed by coupling the input field to a coherent state local oscillator using a beam splitter. This local oscillator acts as a phase reference which enables the phase-sensitive properties of the signal field to be obtained by realizing a measurement of the quadrature phases [4–7]. Homodyne detection has also found applications in, for example, the direct measurement of quasiprobability distribution functions [8–10] and in the deduction of the photon statistics [11,12] of a field mode. Homodyne detection has been applied to obtain a complete description of the state of a field mode by constructing either its Wigner function [13–17] or its density matrix [18–21]. In conventional homodyne detection the local oscillator field is prepared in a strong coherent state. It follows that the beam splitter output beams are large in amplitude and approximately equal in intensity, allowing for their detection as photocurrents. In balanced homodyne detection the difference between the two output photocurrents is taken to obtain the difference photoelectron count probability distribution. This difference photocurrent distribution is proportional to the quadrature probability distribution for the input light [1–4]. It was shown in a seminal paper by Yuen and Chan [22] that the local oscillator noise in the output is eliminated when a 50-50 beam splitter is used. Any resulting quantum fluctuations in the output of this balanced homodyne detector are necessarily related to the quantum noise of the input state. This highlights the usefulness of homodyne detection in that it allows the quantum properties of a system to be amplified enabling a measurement in the macroscopic environment to be performed.

Projection synthesis can be used to determine the probability distributions of quantum optical observables. The method was originally proposed as a means for measuring the optical phase probability distribution [23,24]. It involves the synthesis of a projection operator, the expectation value of which is proportional to the canonical phase probability distribution. A measurement of the Q function by projection synthesis has also been proposed [25]. We have also demonstrated physical truncation of an optical state by projection synthesis using a ''quantum scissors'' device which acts to prepare a chosen superposition of the vacuum and one-photon states [26]. Projection synthesis, like homodyne detection, exploits the interference between the field mode to be measured and a second field mode prepared in a reference state. With projection synthesis, however, the strong (quasiclassical) coherent state local oscillator is replaced by a mode prepared in a known quantum state with small mean photon number. This means that the signal comprises small numbers of photons rather than a macroscopic photocurrent and photon counting techniques can be employed.

In this paper we shall describe the application of projection synthesis to obtain the probability associated with any eigenvalue of any observable of a single field mode. Weak coherent states are the most readily available pure quantum states and are, therefore, most readily available for use as reference states in projection synthesis. We will analyze projection synthesis with weak coherent states using inefficient photodetectors and compare and contrast our results with homodyne detection using strong coherent states. We present the probability operator measure (POM) [27–29] which characterizes the entire measurement process. In Sec. IV we will apply this POM to the problem of discriminating between (nonorthogonal) coherent states [30] and to the measurement of a quasiprobability distribution.

II. MEASUREMENTS USING PROJECTION SYNTHESIS

Once the state of a quantum system is known, it is possible to deduce all of its physical properties. That is, we can reproduce the probability distributions associated with the measurement of any observable of that system. With the ad-

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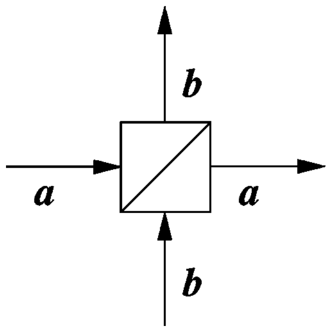

FIG. 1. Schematic representation of a beam splitter with input and output modes a and b, as indicated by the arrows.

vent of optical homodyne tomography [14–17], a determination of the Wigner function is performed to eventually yield the density matrix elements of the state. Recently, it has been shown that the density matrix can be constructed directly, without the need to evaluate the Wigner function beforehand, through the use of more efficient algorithms [18–21]. These processes, however, are extremely information intensive since a measurement of a large number of quadrature probability distributions is required to completely characterize the state [13]. For some applications, a more direct approach to measuring a specific property of the state would be desirable. Also, of course, it is of interest to know how particular physical properties can in principle be measured without obtaining sufficient information to determine the state.

Projection synthesis provides the means to perform a measurement of the probability distribution associated with any given observable. This is achieved by preparing a special reference state so that the probability of finding specific numbers of photons in the two output modes is proportional to the expectation value of the projector formed from a particular eigenstate of the chosen observable. Measurement of the expectation values associated with all of the eigenstates then gives the associated probability distribution. We have previously described the application of projection synthesis to the measurement of the canonical phase probability distribution [23,24]. Here we derive the required form of the reference state associated with measurement of any eigenstate of a given observable.

Consider a lossless beam splitter with input and output modes a and b, as shown in Fig. 1. If the input fields are quantized, then the field in each arm can be characterized by an appropriate single-mode annihilation operator. The action of a beam splitter is to transform the input field states linearly, and for a 50-50 beam splitter the unitary transformation relating the output and input states is [31]

$$\hat{R}=\exp\left[\frac{i\pi}{4}(\hat{b}^\dagger\hat{a}+\hat{a}^\dagger\hat{b})\right]$$
$$=\exp(i\hat{b}^\dagger\hat{a})\exp\left[\frac{\ln 2}{2}(\hat{b}^\dagger\hat{b}-\hat{a}^\dagger\hat{a})\right]\exp(i\hat{a}^\dagger\hat{b}). \qquad (1)$$

The signal state $|f\rangle$ input in mode a is coupled to a general reference state $|B\rangle=\Sigma b_r|r\rangle$ entering mode b of the beam splitter. The probability of detecting n_1 and n_2 photons in the output modes a and b can be found from an ensemble of measurements using two photodetectors. This is the usual way of analyzing measurements with beam splitters in quantum optics and answers the following question: given two known input states into a beam splitter, what is the probability of the photon number at each detector in the output? Projection synthesis is perhaps most readily appreciated by working *backwards* from the measured photocounts. Given the output photon number counts, we can perform a transformation *back* in time through the beam splitter to deduce that state $|\psi_{\rm in}\rangle$ which would produce these counts with unit probability. We then ask the question: what is the probability that the combined input signal and reference state $|f\rangle_a\otimes|B\rangle_b$ will be found in $|\psi_{\rm in}\rangle$? This leads us to associate the detected numbers of photons with the action of a projector on the input state. The form of this projector depends on both the numbers of photons detected and the form of input reference state $|B\rangle_b$. Upon evolution of the output photon number states back through the beam splitter, we can deduce the form of the reference state needed to synthesize the required projector. The primary aim of this section is to find the general form of this reference state that will enable the measurement of the probability associated with any value of any observable to be performed.

The appearance of the measured photon numbers n_1 in output arm a and n_2 in output arm b can be associated with a combined output number state $|\psi_{\rm out}\rangle$ given by

$$|\psi_{\rm out}\rangle=|n_1\rangle_a\otimes|n_2\rangle_b. \qquad (2)$$

This combined output state can be evolved back through the beam splitter to form an expression for the inferred entangled state of input modes

$$|\psi_{\rm in}\rangle=\hat{R}^\dagger|n_1\rangle_a\otimes|n_2\rangle_b$$
$$=\hat{R}^\dagger\frac{1}{\sqrt{n_1!n_2!}}(\hat{a}^\dagger)^{n_1}(\hat{b}^\dagger)^{n_2}|0\rangle_a\otimes|0\rangle_b$$
$$=\hat{R}^\dagger\frac{1}{\sqrt{n_1!n_2!}}(\hat{a}^\dagger)^{n_1}(\hat{b}^\dagger)^{n_2}\hat{R}|0\rangle_a\otimes|0\rangle_b, \qquad (3)$$

where we have used the unitarity of $\hat{R}$ and the fact that $\hat{R}^\dagger$ acting on the two-mode vacuum leaves the state unchanged. It can be shown that [31]

$$\hat{R}^\dagger\hat{a}^\dagger\hat{R}=\frac{1}{\sqrt{2}}(\hat{a}^\dagger-i\hat{b}^\dagger), \qquad (4)$$

and

$$\hat{R}^\dagger\hat{b}^\dagger\hat{R}=\frac{1}{\sqrt{2}}(\hat{b}^\dagger-i\hat{a}^\dagger). \qquad (5)$$

These relations allow us to write the input state as

$$|\psi_{\rm in}\rangle=\frac{2^{-(n_1+n_2)/2}}{\sqrt{n_1!n_2!}}(\hat{a}^\dagger-i\hat{b}^\dagger)^{n_1}(\hat{b}^\dagger-i\hat{a}^\dagger)^{n_2}|0\rangle_a\otimes|0\rangle_b. \qquad (6)$$

This is the entangled input state required to produce the measured photoelectron counts at the outputs of a 50-50 beam splitter. It can be readily seen that the probability of finding n_1 and n_2 photocounts in the output modes a and b is simply the modulus squared of the projection of the combined input state $|f\rangle_a \otimes |B\rangle_b$ onto the entangled state $|\psi_{\mathrm{in}}\rangle$. This probability is given by $P(n_1,n_2) = {}_a\langle f|\hat{\Pi}(n_1,n_2)|f\rangle_a$, the expectation value of the projection operator $\hat{\Pi}(n_1,n_2) = \langle B|\psi_{\mathrm{in}}\rangle\langle\psi_{\mathrm{in}}|B\rangle$. This in turn is equal to $|A|^{-2}$ multiplied by the projection operator onto a pure state $A\langle B|\psi_{\mathrm{in}}\rangle$ where the factor A normalizes the state and, from Eq. (6)

$$\langle B|\psi_{\mathrm{in}}\rangle = \frac{2^{-(n_1+n_2)/2}}{\sqrt{n_1!n_2!}} \sum_{l=0}^{n_1} \sum_{m=0}^{n_2} \binom{n_1}{l}\binom{n_2}{m} \times b^*_{l+n_2-m}(-i)^{l+m} \times \sqrt{(n_1+m-l)!(l+n_2-m)!}\,|n_1+m-l\rangle_a, \tag{7}$$

where

$$\binom{n_1}{l}$$

is the binomial coefficient. To obtain the probability associated with the desired eigenvalue of the chosen observable we set

$$A\langle B|\psi_{\mathrm{in}}\rangle = |\phi\rangle = \sum_p c_p|p\rangle, \tag{8}$$

where $|\phi\rangle$ is the eigenstate of the Hermitian operator for the chosen observable that corresponds to this eigenvalue. The required probability is then equal to $|A|^2$ multiplied by the measured output photon number probability $P(n_1,n_2)$. Projecting both sides of Eq. (8) onto the number state $|r\rangle$ yields, using Eq. (7),

$$c_r = A\frac{2^{-(n_1+n_2)/2}}{\sqrt{n_1!n_2!}} b^*_{n_1+n_2-r}\sqrt{(n_1+n_2-r)!r!} \times \sum_{m=M_1}^{M_2} \binom{n_1}{n_1+m-r}\binom{n_2}{m}(-i)^{n_1+2m-r}, \tag{9}$$

where $M_1 = \max(0, r-n_1)$ that is, the larger of 0 and $r-n_1$, and $M_2 = \min(r,n_2)$. Thus the coefficients of the required input reference state are

$$b_r = \frac{2^{(n_1+n_2)/2}\sqrt{n_1!n_2!}\,c^*_{n_1+n_2-r}}{A^*\sqrt{(n_1+n_2-r)!r!}} \times \left[\sum_{m=M_3}^{M_4} \binom{n_1}{m+r-n_2}\binom{n_2}{m} i^{2m-n_2+r}\right]^{-1}, \tag{10}$$

where $M_3 = \max(0, n_2-r)$ and $M_4 = \min(n_1+n_2-r, n_2)$, provided the term in the square brackets is not zero. If it is zero for a particular choice of n_1 and n_2 then a different choice of n_1 or n_2 should be made. It has been shown that for the particular case of the measurement of the canonical phase distribution with the choice $n_2=0$ a reciprocal binomial state is needed for the reference state [23,24]. The values of r range from zero to n_1+n_2 so only the first n_1+n_2+1 terms in the photon number expansion of the signal state will contribute to $P(n_1,n_2)$. Thus n_1+n_2 must be chosen large enough for this effective truncation of the signal state to have a negligible effect. As the sum of $|b_r|^2$ must be unity we find from Eq. (10) that

$$|A|^2 = 2^{(n_1+n_2)} n_1!n_2! \sum_{r=0}^{n_1+n_2} \frac{|c_{n_1+n_2-r}|^2}{(n_1+n_2-r)!r!} \times \left|\left[\sum_{m=M_3}^{M_4}\binom{n_1}{m+r-n_2}\binom{n_2}{m} i^{2m-n_2+r}\right]^{-1}\right|^2. \tag{11}$$

This allows us to find b_r and also provides us with the factor we need to multiply the measured probability $P(n_1,n_2)$ to obtain the probability associated with the desired eigenvalue of the chosen observable. This factor is expressed in terms of the coefficients of the corresponding eigenstate $|\phi\rangle$. Conversely, if we have a known reference state combined with the signal state, expression (9) can be exploited to deduce the nature of the observable whose distribution is measured by the beam splitter and photodetectors.

In summary, we have found that the technique of projection synthesis proposed to measure canonical phase can be extended to determine probabilities associated with any physical observable of a quantized field mode. Projection synthesis has a distinct advantage over other schemes in that only one device is needed to measure any physical property of the system; it provides a means to generalize the measurement process. However, this benefit is offset by the necessity of having to prepare general reference states. This problem has been addressed in our previous paper [24] where it was shown that, in principle, any reference state can be fabricated provided one can generate an arbitrary superposition of the $|0\rangle$ and $|1\rangle$ states. We have also proposed a scheme to physically truncate quantum states [26]. These techniques, based upon the ideas of conditional output measurements, have the potential for fabricating exotic quantum states [11,32].

III. HOMODYNE MEASUREMENT AS PROJECTION SYNTHESIS

The signal in balanced homodyne detection is the difference between two photocurrents produced by the interference between the field mode being measured and an intense, coherent local oscillator. In projection synthesis, both the field under investigation and the reference state are in the quantum regime of low photon number and the number of photocounts registered at each detector forms the signal. In this section we investigate the weak local oscillator limit of homodyne detection as an example of projection synthesis.

As with conventional balanced homodyne detection, we have a local oscillator in a coherent state $|\alpha\rangle$, acting as a reference state, input into arm b of a beam splitter. This local oscillator is combined with the signal state input into arm a of the beam splitter. We do not subtract the output photodetector counts but instead we record individual photons in each output arm. The technique of projection synthesis will be used to obtain an expression for the probability operator measure [27–29] characterizing the homodyne detection process using photoelectron counting techniques. The probability operator measure approach is an extension of the standard von Neumann picture of measurement in quantum optics in which the initial system is coupled to a probe system. A standard von Neumann measurement is performed on this combined system and provides a more general view of measurement (see Appendix A). Once the POM for a system has been found we can find the probabilities associated with all possible experimental outcomes. It should be noted that this set of possible outcomes, and therefore the form of the associated POM, is different from that found for balanced homodyne detection, which depends only on the difference between the two photocurrents. It follows that, unlike balanced homodyne detection [22], projection synthesis is sensitive to noise associated with the reference state.

In projection synthesis with a weak coherent reference state, the detected signal is in the form of discrete photoelectron counts and this can be measured using perfect photodetectors. By applying the analysis of the preceding section we find a representation of the POM element formed using perfect detectors $\hat{\Pi}_P(n_1,n_2)$ in the form

$$\hat{\Pi}_P(n_1,n_2) = {}_b\langle\alpha|\psi_{\rm in}\rangle\langle\psi_{\rm in}|\alpha\rangle_b = {}_b\langle\alpha|\hat{R}^\dagger|n_1\rangle_a|n_2\rangle_{bb}\langle n_2|_a\langle n_1|\hat{R}|\alpha\rangle_b$$
$$= \frac{1}{n_1!n_2!}{}_b\langle\alpha|\hat{R}^\dagger(\hat{a}^\dagger)^{n_1}(\hat{b}^\dagger)^{n_2}:\exp(-\hat{a}^\dagger\hat{a})\exp(-\hat{b}^\dagger\hat{b}):\hat{a}^{n_1}\hat{b}^{n_2}\hat{R}|\alpha\rangle_b, \tag{12}$$

where the colons denote normal ordering. We can write the operator product appearing in Eq. (12) as

$$(\hat{a}^\dagger)^{n_1}(\hat{b}^\dagger)^{n_2}:\exp(-\hat{a}^\dagger\hat{a})\exp(-\hat{b}^\dagger\hat{b}):\hat{a}^{n_1}\hat{b}^{n_2}$$
$$= \left(\frac{-\partial}{\partial\lambda}\right)^{n_1}\left(\frac{-\partial}{\partial\mu}\right)^{n_2}:\exp(-\lambda\hat{a}^\dagger\hat{a})\exp(-\mu\hat{b}^\dagger\hat{b}):|_{\lambda=\mu=1}, \tag{13}$$

so that Eq. (12) can be expressed in terms of a moment generating operator $\hat{L}$,

$$\hat{\Pi}_P(n_1,n_2) = \frac{1}{n_1!n_2!}\left(\frac{-\partial}{\partial\lambda}\right)^{n_1}\left(\frac{-\partial}{\partial\mu}\right)^{n_2}\hat{L}(\alpha,\lambda,\mu)|_{\lambda=\mu=1}, \tag{14}$$

where

$$\hat{L}(\alpha,\lambda,\mu) = \exp\left[-\frac{1}{2}(\lambda-\mu)i\alpha\hat{a}^\dagger\right]$$
$$\times:\exp\left[-\frac{1}{2}(\lambda+\mu)\hat{a}^\dagger\hat{a}\right):\exp\left[\frac{1}{2}(\lambda-\mu)i\alpha^*\hat{a}\right]$$
$$\times\exp\left[-\frac{1}{2}(\lambda+\mu)|\alpha|^2\right]. \tag{15}$$

To prove that the projection operators $\hat{\Pi}(n_1,n_2)$ form a POM that describes the photoelectron count measurement in the presence of the reference state, we must ensure that $\hat{\Pi}(n_1,n_2)$ forms a resolution of the identity

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\hat{\Pi}(n_1,n_2) = \hat{1}. \tag{16}$$

Upon expansion of the left hand side of Eq. (16) and application of the resolution of the identity on the number state basis, we obtain [11]

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\hat{\Pi}(n_1,n_2)$$
$$= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\langle\alpha|\psi_{\rm in}\rangle\langle\psi_{\rm in}|\alpha\rangle$$
$$= {}_b\langle\alpha|\hat{R}^\dagger\left(\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}|n_1\rangle_a|n_2\rangle_{bb}\langle n_2|_a\langle n_1|\right)\hat{R}|\alpha\rangle_b$$
$$= \hat{1}. \tag{17}$$

The formation of the resolution of the identity proves that $\hat{\Pi}(n_1,n_2)$ is indeed a POM element.

For practical purposes, it is necessary to account for the effects of finite detector efficiencies. It is possible to recover the ideal detector statistics from the detected statistics using sufficiently good detectors [33] although here we adopt a different approach (see Appendix B). The detection of n photons is equivalent to projecting the state entering the nonideal photodetector onto the mixed-state projector [34]

$$\Pi_n = \frac{1}{n!}(\eta\hat{a}^\dagger)^n:\exp(-\eta\hat{a}^\dagger\hat{a}):\hat{a}^n. \tag{18}$$

For an ideal photodetector with $\eta=1$ the state that enters the detector is projected down onto the $|n\rangle\langle n|$ projector as expected. By applying the result in Eq. (18) to the analysis of homodyne measurement using projection synthesis, Eq. (14) becomes

$$\hat{\Pi}_I(n_1,n_2) = {}_b\langle\alpha|\Pi_{n_1}\Pi_{n_2}|\alpha\rangle_b$$
$$= \frac{1}{n_1!n_2!}\left(\frac{-\partial}{\partial\lambda}\right)^{n_1}\left(\frac{-\partial}{\partial\mu}\right)^{n_2}\hat{L}(\alpha,\eta\lambda,\eta\mu)|_{\lambda=\mu=1}. \tag{19}$$

The POM element formed, $\hat{\Pi}_I(n_1,n_2)$, is the projector that corresponds to the finite-efficiency detection of n_1 counts in detector c and n_2 counts in detector d for a weak homodyne measurement. Equivalently, this result can be found by considering the effect of finite-efficiency detectors on the form of the moment-generating operator in Eq. (15).

We have seen that the individual photon number counts can be used to form a POM in a variant of conventional homodyne detection. The projection synthesis description differs from that associated with balanced homodyne detection in that only a weak local oscillator is employed and that the measurement comprises the numbers of photons obtained in both output modes. We can obtain the projection synthesis description of balanced homodyne detection by adding together all of the POM elements corresponding to the same photon number difference (n_1-n_2) and then taking the limit of a large amplitude coherent reference state. In effect, the projectors in balanced homodyne detection form a subset of the elements formed in the above example of projection synthesis. We obtain more information in our method than just the difference in the output photon counts and therefore it is possible to perform more general measurements on the input state. In the next section we will apply this POM to characterize a measurement that cannot be performed by measuring a quadrature component of the input state.

IV. DISTINGUISHING BETWEEN COHERENT STATES

In this section we apply the results derived above to form a POM that fully characterizes the outcomes of measurements on a nontrivial system used to discriminate between (nonorthogonal) coherent states. The difficulties inherent in the measurement of nonorthogonal states ensure the security of certain quantum cryptographic protocols [35]. If a system is known to be in either of two nonorthogonal states, then it is impossible to determine with certainty which one of these states the system is in. Consider a single system prepared in one of two nonorthogonal states $|a\rangle$ and $|b\rangle$, each with an *a priori* probability of occurrence of one-half. These are not the nondegenerate eigenstates of a Hermitian operator and therefore cannot be distinguished with complete certainty. There are two approaches that can be used to discriminate between these nonorthogonal states; one is by allowing for a minimum probability of error and the other is by performing an optimum error-free measurement [36]. With the first scheme, we can determine in which state the system was prepared, leaving us with a minimum probability of error given by the Helstrom bound [27]. The second scheme was devised by Ivanovic [37] and subsequently refined by Dieks [38] and Peres [39]. This measurement scheme leads to no errors but has the possibility that the result of the measurement will be inconclusive. The minimum allowed probability for the inconclusive outcome is equal to the modulus of the overlap between $|a\rangle$ and $|b\rangle$. This bound provides a way to determine whether a measurement scheme is an optimal one.

Huttner *et al.* [30] have shown that an optimal error-free discrimination between two coherent states $|\alpha\rangle$ and $|-\alpha\rangle$ can be performed using a 50-50 beam splitter to superpose the field mode known to be in one of these states with a mode prepared in the coherent state $|i\alpha\rangle$ (see Fig. 2). We can view this state discrimination scheme as a special case of the analysis presented in Sec. III with the input state being one of the two coherent states $|\alpha\rangle$ and $|-\alpha\rangle$. Instead of using the photon number distribution to synthesize a probability distribution, we simply distinguish between the two possible pre-

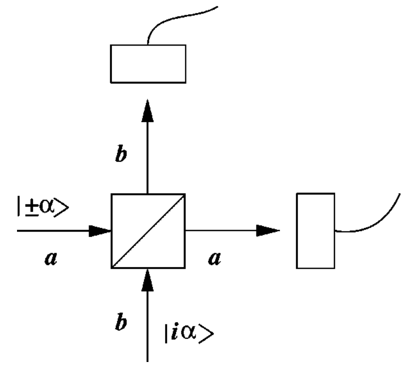

FIG. 2. Schematic representation of the Huttner-Imoto-Gisin-Tor beam splitter arrangement for discriminating between coherent states.

measurement input states. We analyze this state discrimination technique by using projection synthesis to form the POM that fully describes the possible results of the measurement and show that these POM elements can be recast into an expression for a quasiprobability function of the input state.

The 50-50 beam splitter depicted in Fig. 2 transforms the input state $|\pm\alpha\rangle_a|i\alpha\rangle_b$ into the output state

$$\hat{R}|\pm\alpha\rangle_a|i\alpha\rangle_b=\left|\frac{1}{\sqrt{2}}(\pm\alpha-\alpha)\right\rangle_a\left|\frac{i}{\sqrt{2}}(\alpha\pm\alpha)\right\rangle_b. \quad (20)$$

One of the resulting output modes is in its vacuum state, while the other is in a coherent state with mean photon number $2|\alpha|^2$. It follows that ideal photon counting will unambiguously reveal the unknown input state, $|\alpha\rangle$ or $|-\alpha\rangle$, unless there are no photons detected at either detector. The probability for this inconclusive result to occur is simply

$$|\langle 0|-\sqrt{2}\alpha\rangle|^2=|\langle 0|i\sqrt{2}\alpha\rangle|^2=\exp(-2|\alpha|^2). \quad (21)$$

For perfect photodetectors, this is equal to the modulus of the overlap of the two coherent states $|\alpha\rangle$ and $|-\alpha\rangle$ and therefore achieves the Ivanovic-Dieks-Peres bound for error-free state discrimination [30]. There are, in principle, four possible outcomes of the measurement, although only three of these are actually possible. These outcomes can be characterized by a POM.

We can form the nonorthogonal projector corresponding to n_1 counts recorded in the detector in output arm a and n_2 detected in arm b, as described in the preceding section. This is given by

$$\hat{\Pi}_I(n_1,n_2)={}_b\langle i\alpha|\hat{R}^\dagger\hat{\Pi}_{n_1}\hat{\Pi}_{n_2}\hat{R}|i\alpha\rangle_b$$
$$=\frac{1}{n_1!n_2!}\left(\frac{-\partial}{\partial\lambda}\right)^{n_1}\left(\frac{-\partial}{\partial\mu}\right)^{n_2}\hat{L}(i\alpha,\eta\lambda,\eta\mu)|_{\lambda=\mu=1}, \quad (22)$$

where α is replaced with $i\alpha$ in Eq. (19). The POM projectors so formed can be grouped together and associated with four different conclusions. When we observe counts in output arm b and none in output arm a, then we can be certain that our original input state was $|\alpha\rangle$. For brevity, this outcome can be denoted by the POM elements $\hat{\Pi}_I(\alpha)$. Similarly, for the situations where we measure zero counts in arm b with $n_1 \neq 0$ counts in arm a, the premeasurement state is $|-\alpha\rangle$ with certainty and the corresponding POM elements are labeled $\hat{\Pi}_I(-\alpha)$. As we have seen, the inconclusive outcome occurs when we observe zero counts in both output arms and this POM element is termed $\hat{\Pi}_I(?)$. Finally, for completeness, we include the physically unrealistic case where the POM element $\hat{\Pi}_I(\emptyset)$ describes the measurement when we detect photoelectrons in *both* output arms. The first two POM elements are clearly sums of individual projectors but nevertheless, we can construct each of these elements explicitly and then verify the probabilities obtained for each different outcome. The inconclusive POM element is found simply by setting n_1 and n_2 equal to zero in Eq. (22),

$$\hat{\Pi}_I(?) = \hat{\Pi}_I(0,0) = \hat{L}(i\alpha, \eta\lambda, \eta\mu)|_{\lambda=\mu=1}$$
$$= :\exp(-\eta\hat{a}^\dagger\hat{a}):\exp(-\eta|\alpha|^2). \qquad (23)$$

The POM element representing measurement of the initial state $|\alpha\rangle$ can be found from

$$\hat{\Pi}_I(\alpha) = \hat{\Pi}_I(n_1=0, n_2\neq 0) = \hat{\Pi}_I(0,\text{any}) - \hat{\Pi}_I(0,0). \quad (24)$$

The POM element $\hat{\Pi}_I(0,\text{any})$, associated with zero counts recorded in output mode a and any number of counts in mode b, can be obtained from Eq. (22) as

$$\hat{\Pi}(0,\text{any}) = \sum_{n_2=0}^{\infty} \frac{1}{n_2!}\left(\frac{-\partial}{\partial\mu}\right)^{n_2}\hat{L}(i\alpha, \eta\lambda, \eta\mu)|_{\lambda=\mu=1}$$
$$= \exp\left(\frac{-\partial}{\partial\mu}\right)\hat{L}(i\alpha, \eta\lambda, \eta\mu)|_{\lambda=\mu=1}$$
$$= \hat{L}(i\alpha, \eta\lambda, \eta(\mu-1))|_{\lambda=\mu=1}$$
$$= :\exp\left[-\frac{\eta}{2}(\hat{a}^\dagger - \alpha^*)(\hat{a}-\alpha)\right]:. \qquad (25)$$

Hence the POM element associated with the conclusive result that the input state was $|\alpha\rangle$ is

$$\hat{\Pi}_I(\alpha) = :\exp\left[-\frac{\eta}{2}(\hat{a}^\dagger - \alpha^*)(\hat{a}-\alpha)\right]:$$
$$- :\exp[-\eta(\hat{a}^\dagger\hat{a} + |\alpha|^2)]:. \qquad (26)$$

Similarly, the POM element which is realized in a measurement of the $|-\alpha\rangle$ input state is

$$\hat{\Pi}_I(-\alpha) = :\exp\left[-\frac{\eta}{2}(\hat{a}^\dagger + \alpha^*)(\hat{a}+\alpha)\right]:$$
$$- :\exp[-\eta(\hat{a}^\dagger\hat{a} + |\alpha|^2)]:. \qquad (27)$$

We note that $\hat{\Pi}_I(-\alpha)$ can be obtained from $\hat{\Pi}_I(\alpha)$ by the simple replacement of α with $-\alpha$. Since POM elements form the resolution of the identity, the last remaining element is

$$\hat{\Pi}_I(\emptyset) = \hat{1} - \hat{\Pi}_I(?) - \hat{\Pi}_I(\alpha) - \hat{\Pi}_I(-\alpha), \qquad (28)$$

where $\emptyset$ denotes a zero probability of occurring, and ? denotes an inconclusive result. We can now deduce the probabilities of the various outcomes for the two possible input states. The probabilities resulting from an input state of $|\alpha\rangle$ are simply given by the expectation value of the relevant POM elements and these are

$$P(?) = \exp(-2\eta|\alpha|^2), \qquad (29)$$

$$P(\alpha) = 1 - \exp(-2\eta|\alpha|^2), \qquad (30)$$

$$P(-\alpha) = 0, \qquad (31)$$

$$P(\emptyset) = 0. \qquad (32)$$

These probabilities, for unit detector efficiency, are exactly the same as those given in [30] and verify that the POM elements constructed above are indeed correct. Similarly, for an input state of $|-\alpha\rangle$, the expected probabilities are obtained. The POM elements associated with projection synthesis using weak coherent states can therefore be combined in order to coincide with the outcomes of the state discrimination scheme devised by Huttner *et al.* [30]. The inclusion of the factor η in the moment-generating operator $\hat{L}$ has a detrimental effect on the possibility of discriminating between coherent states. We note, however, that finite detector efficiency does not introduce any errors in that $P(-\alpha)=0$. We find that the inconclusive probability $P(?)$ no longer reaches the Ivanovic-Dieks-Peres bound. For $|\alpha|^2=1$, the probability of the inconclusive result for the ideal detector is 0.135, whereas the corresponding probability with a detector efficiency of one-half is 0.368. The latter probability is more than twice the probability required to satisfy the Ivanovic-Dieks-Peres bound.

It has recently been proposed that a measurement of the s-parametrized quasiprobability distributions can be performed using simple photon counting techniques [8–10]. It is interesting to note that the POM element $\hat{\Pi}_I(0,\text{any})$ obtained from the sum of synthesized projectors $\Sigma_n\hat{\Pi}_I(0,n)$ is also related to a quasiprobability distribution. A general expression for the s-parametrized quasiprobability distributions $W(\alpha,s)$, for $s<1$, is given by [31],

$$W(\alpha,s) = \langle\hat{D}(\alpha)\hat{T}(s)\hat{D}^\dagger(\alpha)\rangle, \qquad (33)$$

where

$$\hat{T}(s) = \frac{2}{\pi(1-s)}:\exp\left(-\frac{2}{1-s}\hat{a}^\dagger\hat{a}\right):. \qquad (34)$$

$W(\alpha,s)$ can be reexpressed in terms of the expectation value of a normally ordered operator [8],

$$W(\alpha,s) = \frac{2}{\pi(1-s)}\left\langle:\exp\left[-\frac{2}{1-s}(\hat{a}^\dagger-\alpha^*)(\hat{a}-\alpha)\right]:\right\rangle. \qquad (35)$$

The Wigner function and the Q function can be constructed when the parameter s has the values of 0 and -1, respectively. The formation of the probability operator measure in the coherent state discrimination scheme enables us to construct a quasiprobability distribution function characterizing the input state. The expectation value of the projector given in Eq. (25) is

$$P(0,\text{any}) = \langle \hat{\Pi}_I(0,\text{any}) \rangle = \left\langle :\exp\left[-\frac{\eta}{2}(\hat{a}^\dagger - \alpha^*)(\hat{a}-\alpha) \right]: \right\rangle$$
$$= \frac{2\pi}{\eta} W\left(\alpha, \frac{\eta-4}{\eta} \right). \tag{36}$$

Upon rearrangement, we can deduce the $s = 1 - 4/\eta$ quasiprobability distribution function

$$W\left(\alpha, \frac{\eta-4}{\eta} \right) = \frac{\eta}{2\pi} P(0,\text{any}). \tag{37}$$

The probability operator measure representation of the above scheme has provided us with enough information to reconstruct the input state via its quasiprobability distribution. The quasiprobability distribution $W(\alpha,-3)$ can be constructed when using ideal photodetectors. This is a very broad quasiprobability distribution, even more so than the Q function, but it is nevertheless a complete description of the state. For a 50% efficient photodetector, the quasiprobability distribution function measured is $W(\alpha,-7)$. This distribution is smoother and broader than the $s=-3$ parametrized quasiprobability distribution measured using ideal detectors.

V. CONCLUSION

The projection synthesis approach involves deducing the probability distribution for any observable of an unknown field by measuring the properties of a two-field system coupled by a beam splitter, where the state of the probe field is known. In this context it can be seen that there is some parallel with a recent suggestion [40] for deducing an atomic density matrix by measuring properties of a coupled atom-field system, where the initial state of the probe field is known.

In this paper we have shown how projection synthesis can be applied to obtain the probability associated with any eigenstate of any observable associated with a single field mode. A series of such measurements will yield the probability distribution for any observable. The method relies on coherently mixing the field to be measured with a suitably prepared reference state using a 50-50 beam splitter followed by high efficiency photon counting. We have given an explicit expression for the number state amplitudes of the required reference state. Such states can, in principle, be synthesized using a succession of beam splitters and a superposition of the vacuum state $|0\rangle$ and the one-photon number state $|1\rangle$ [24]. The necessary state preparation can also be performed using a ''quantum scissors'' device [26]. We have examined the connection between homodyne detection and projection synthesis using weak coherent states. The projection synthesis differs from balanced homodyne detection in that the reference coherent state for projection synthesis must be in the quantum regime of low photon number and in that the photocounts registered at both detectors are required and not just the difference between them. We have given the POM elements associated with projection synthesis using coherent states and have shown how these can be applied to analyze the discrimination between two possible coherent states [30]. We have also shown how they can be applied to make a connection between the measured photon statistics and a field mode quasiprobability distribution (see also [8–10,25]). An analysis of the effects of using finite-efficiency detectors in projection synthesis has been performed.

We have described projection synthesis in terms of the evolution of a state corresponding to the detected photon numbers back through the beam splitter and then projecting this onto the product state formed from the state to be measured and the reference state. This view of measurement is in some ways reminiscent of retrodiction [41] and we shall explore this idea more fully elsewhere.

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APPENDIX A: PROBABILITY OPERATOR MEASURES

In quantum mechanics, an observable is any physical quantity that can be measured. Such a quantity can be represented by a Hermitian operator $\hat{A}$ that acts on its associated eigenvectors $|a_i\rangle$ to form the eigenvalue equation $\hat{A}|a_i\rangle = a_i|a_i\rangle$ in the Hilbert space $\mathcal{H}$. Since $\hat{A}$ is Hermitian, the eigenvalues a_i are real numbers and the eigenvectors $|a_i\rangle$ form a complete set allowing a pure state to be represented as a linear expansion of eigenvectors. The eigenvectors are orthonormal and complete so that they form the resolution of the identity in an N-dimensional Hilbert space $\mathcal{H}_N$ given by

$$\sum_{i=0}^{N} |a_i\rangle\langle a_i| = \hat{1}, \tag{A1}$$

where $|a_i\rangle\langle a_i|$ are the projection operators that characterize the measurement.

The standard quantum mechanical picture of a measurement is based upon the ideas of von Neumann [42]. An ideal von Neumann measurement transforms the premeasurement state to the eigenstate of the measured observable which corresponds to the observed eigenvalue. The probability of obtaining the eigenvalue a_i is

$$P(a_i) = \mathrm{Tr}_N(\hat{\rho}|a_i\rangle\langle a_i|), \tag{A2}$$

where $\hat{\rho}$ is the density operator of the state immediately before measurement and the trace is performed over the N-dimensional Hilbert space $\mathcal{H}_N$ of the system. With this conventional von Neumann view of the measurement process in quantum mechanics, we can obtain all of the statistical data needed to determine the probability distributions of observables of the quantum state under investigation.

The idea of a probability operator measure can be used to provide a more general description of measurements

[27–29]. Essentially, a POM is an extension of the von Neumann approach in which the measurement is described by the coupling of the quantum system under investigation to a probe system P. The combined density matrix of the coupled system is the tensor product of the initial quantum state and the probe state $\hat{\rho}\otimes\hat{\rho}_P$. We have, in effect, an extension of the Hilbert space from $\mathcal{H}_N$ to $\mathcal{H}_N\otimes\mathcal{H}_P$ upon which a von Neumann measurement of an observable associated with both systems is performed. The probability of obtaining the result b_i is now given by

$$P(b_i)=\mathrm{Tr}_N(\hat{\rho}\hat{\Pi}_i), \tag{A3}$$

where $\hat{\Pi}_i$ are the family of operators that form the POM which describes the measurement and are defined by

$$\hat{\Pi}_i=\mathrm{Tr}_P(\hat{\rho}_P|b_i\rangle\langle b_i|), \tag{A4}$$

and $|b_i\rangle$ is a state in the space $\mathcal{H}_N\otimes\mathcal{H}_P$.

In order to ensure that $P(b_i)$ represents a true probability, $\hat{\Pi}_i$ must be nonnegative definite and self-adjoint. The normalization of the probability distribution for all possible measured states implies the following resolution of the identity:

$$\sum_{i=0}^{M}\hat{\Pi}_i=\hat{1}, \tag{A5}$$

where M can be larger than the dimension of the Hilbert space $\mathcal{H}_N$ of the initial system.

In general, we have more POM elements $\hat{\Pi}_i$ than dimensions of the initial Hilbert space of the system under investigation. As such, the POM elements are not always orthogonal even though they correspond to distinct experimental outcomes. This highlights the crucial difference between the two classes of measurements. POM elements are in general nonorthogonal with $\hat{\Pi}_i\hat{\Pi}_j\neq\delta_{ij}\hat{\Pi}_i$ and in contrast with this, standard von Neumann measurements are strictly orthogonal projections with $\hat{\Pi}_i\hat{\Pi}_j=\delta_{ij}\hat{\Pi}_i$. At first glance, it seems as though the association of the nonorthogonal POM elements with different measurement outcomes is in disagreement with fundamental measurement theory in quantum mechanics; to measure an observable, we need to project onto an orthogonal basis set. This apparent paradox is resolved by invoking the Naimark theorem [43], which shows that a nonorthogonal POM in the Hilbert space of the initial state can always be reexpressed in terms of an *orthogonal* projection in the higher dimensional space $\mathcal{H}_N\otimes\mathcal{H}_P$ as described above. As a result, every POM corresponds to an orthogonal projection, albeit in a larger Hilbert space, in line with standard quantum measurement theory.

APPENDIX B: FINITE DETECTION EFFICIENCY

It is necessary to consider the problem of finite-efficiency detectors in projection synthesis. Consider a detector with an efficiency η used to measure a mode containing precisely m photons. The consequence of this inefficiency is that not all of the incident photons are converted into photoelectrons. This can be modeled by considering, in place of the nonideal detector, a combination of a lossless beam splitter and an ideal photodetector [31]. The photons are incident into one arm of the beam splitter and are coherently combined with a vacuum state input from the other port. The probability that a photon is detected is related to the transmission probability η of the beam splitter. Only photons that pass through the ideal beam splitter are measured by the photodetector and the loss is modeled by the existence of unmeasured reflected photons. The m photons incident on the nonideal detector will lead to n photoelectron pulse with probability

$$P_{\mathrm{det}}(n)=\binom{m}{n}\eta^n(1-\eta)^{m-n}. \tag{B1}$$

If, however, we have an input state with a probability of containing m photons given by the expectation value of the projector $|m\rangle\langle m|$, then the corresponding output photocount probability distribution is modified to [2,3,31,44]

$$P'_{\mathrm{det}}(n)=\sum_{m=0}^{\infty}\binom{m}{n}\eta^n(1-\eta)^{m-n}P(m). \tag{B2}$$

This is simply the sum of the probability of detecting n photons when m are present multiplied by the probability $P(m)$ that the input field had m photons. The detection of n photons is equivalent to projecting the state entering the nonideal photodetector onto the mixed-state projector [34]

$$\Pi_n=\frac{1}{n!}(\eta\hat{a}^\dagger)^n{:}\exp(-\eta\hat{a}^\dagger\hat{a}){:}\hat{a}^n. \tag{B3}$$

These are indeed POM elements since they are positive semidefinite and form a resolution of the identity. This result can be applied to determine the effect of nonideal photodetectors on projection synthesis schemes.

[1] H. P. Yuen and J. H. Shapiro, IEEE Trans. Inf. Theory **24**, 657 (1978).
[2] J. H. Shapiro, H. P. Yuen, and J. A. Machado Mata, IEEE Trans. Inf. Theory **25**, 179 (1979).
[3] H. P. Yuen and J. H. Shapiro, IEEE Trans. Inf. Theory **26**, 78 (1980).
[4] R. Loudon and P. Knight, J. Mod. Opt. **34**, 709 (1987), and references therein.
[5] S. L. Braunstein, Phys. Rev. A **42**, 474 (1990).
[6] W. Vogel and J. Grabow, Phys. Rev. A **47**, 4227 (1993).
[7] K. Banaszek and K. Wódkiewicz, Phys. Rev. A **55**, 3117 (1997).
[8] K. Banaszek and K. Wódkiewicz, Phys. Rev. Lett. **76**, 4344 (1996).
[9] S. Wallentowitz and W. Vogel, Phys. Rev. A **53**, 4528 (1996).
[10] S. Mancini, P. Tombesi, and V. I. Man'ko, Europhys. Lett. **37**, 79 (1997).
[11] M. Ban, J. Mod. Opt. **43**, 1281 (1996).

[12] W. Grice and I. A. Walmsley, J. Mod. Opt. **43**, 795 (1996).
[13] K. Vogel and H. Risken, Phys. Rev. A **40**, R2847 (1989).
[14] D. T. Smithey, M. Beck, M. G. Raymer, and A. Faridani, Phys. Rev. Lett. **70**, 1244 (1993).
[15] D. T. Smithey, M. Beck, J. Cooper, and M. G. Raymer, Phys. Rev. A **48**, 3159 (1993).
[16] M. Beck, D. T. Smithey, J. Cooper, and M. G. Raymer, Opt. Lett. **18**, 1259 (1993).
[17] D. T. Smithey, M. Beck, J. Cooper, M. G. Raymer, and A. Faridani, Phys. Scr. **T48**, 35 (1993).
[18] G. M. D'Ariano, C. Macchiavello, and M. G. A. Paris, Phys. Rev. A **50**, 4298 (1994).
[19] M. Munroe, D. Boggavarapu, M. E. Anderson, and M. G. Raymer, Phys. Rev. A **52**, R924 (1995).
[20] S. Schiller, G. Breitenbach, S. F. Pereira, T. Müller, and J. Mlynek, Phys. Rev. Lett. **77**, 2933 (1996).
[21] U. Leonhardt, M. Munroe, T. Kiss, Th. Richter, and M. G. Raymer, Opt. Commun. **127**, 144 (1996).
[22] H. P. Yuen and V. W. S. Chan, Opt. Lett. **8**, 177 (1983).
[23] S. M. Barnett and D. T. Pegg, Phys. Rev. Lett. **76**, 4148 (1996).
[24] D. T. Pegg, S. M. Barnett, and L. S. Phillips, J. Mod. Opt. **44**, 2135 (1997).
[25] B. Baseia, M. H. Y. Moussa, and V. S. Bagnato, Phys. Lett. A **231**, 331 (1997).
[26] D. T. Pegg, L. S. Phillips, and S. M. Barnett, Phys. Rev. Lett. (to be published).
[27] C. W. Helstrom, *Quantum Detection and Estimation Theory* (Academic Press, New York, 1976).
[28] G. M. D'Ariano, in *Quantum Optics and the Spectroscopy of Solids* (Kluwer Academic Publishers, Dordrecht, 1997), pp. 139–174.
[29] B. Huttner *et al.*, Phys. Rev. A **54**, 3783 (1996).
[30] B. Huttner, N. Imoto, N. Gisin, and T. Mor, Phys. Rev. A **51**, 1863 (1995).
[31] S. M. Barnett and P. M. Radmore, *Methods in Theoretical Quantum Optics* (Oxford University Press, Oxford, 1997).
[32] M. Dakna, T. Anhut, T. Opatrný, L. Knöll, and D. G. Welsch, Phys. Rev. A **55**, 3184 (1997).
[33] See, for example, C. T. Lee, Phys. Rev. A **48**, 2285 (1993).
[34] S. M. Barnett, L. S. Phillips, and D. T. Pegg (unpublished).
[35] For an extensive introduction to and review of quantum cryptography see S. J. D. Phoenix and P. D. Townsend, Contemp. Phys. **36**, 165 (1995). A short historical summary together with an extensive list of references may be found in G. Brassard and C. Crépeau, SIGACT News **27**, No. 3, 13 (1996).
[36] These two approaches to state discrimination are described in S. M. Barnett, Philos. Trans. R. Soc. London, Ser. A **355**, 2279 (1997).
[37] I. D. Ivanovic, Phys. Lett. A **123**, 257 (1987).
[38] D. Dieks, Phys. Lett. A **126**, 303 (1988).
[39] A. Peres, Phys. Lett. A **128**, 19 (1988).
[40] A. Luis and L. L. Sanchez-Soto, Phys. Rev. A **57**, 3105 (1998).
[41] Y. Aharonov, P. G. Bergmann, and J. L. Lebowitz, Phys. Rev. **134**, B1410 (1964).
[42] J. von Neumann, *Mathematical Foundations of Quantum Mechanics* (Princeton University Press, Princeton, NJ, 1955).
[43] M. A. Naimark, Izv. Akad. Nauk SSR Ser. Fiz. Mat. Nauk **4**, 227 (1940).
[44] R. Loudon, *The Quantum Theory of Light* (Oxford University Press, Oxford, 1983).