Effects of atomic diffraction on the collective atomic recoil laser

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We formulate a wave-atom-optics theory of the collective atomic recoil laser (CARL) where the atomic center-of-mass motion is treated quantum mechanically. By comparing the predictions of this theory with those of the ray-atom-optics theory, which treats the center-of-mass atomic motion classically, we show that for the case of a far off-resonant pump laser the ray-optics model fails to predict the linear response of the CARL when the temperature is of the order of the recoil temperature or less. This is due to the fact that in this temperature regime one can no longer ignore the effects of matter-wave diffraction on the atomic center-of-mass motion. [S1050-2947(98)00510-1]

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I. INTRODUCTION

The collective atomic recoil laser, or CARL, is the atomic equivalent of the free-electron laser [1]. Developed theoretically by Bonifacio and co-workers [2–6], the CARL device has three main components: (1) the active medium, which consists of a gas of two-level atoms, (2) a strong pump laser which drives the two-level atomic transition, and (3) a ring cavity which supports an electromagnetic mode (the probe) counterpropagating with respect to the pump. Under suitable conditions, the operation of the CARL results in the generation of a coherent probe field due to the following mechanism. First, a weak probe field is initiated by noise, either optical in the form of spontaneously emitted light, or atomic in the form of density fluctuations in the atomic gas which backscatters the pump. Once initiated, the probe combines with the pump field to form a weak standing wave which acts as a periodic optical potential (light shift). The center-of-mass motion of the atoms on this potential results in a bunching (modulation) of their density, very much like the combined effects of the wiggler and the light field leads to electron bunching in the free-electron laser. This bunching process is then seen by the pump laser as the appearance of a polarization grating in the active medium, which results in stimulated backscattering into the probe field. The resulting increase in the probe strength further increases the magnitude of the standing wave field, resulting in more bunching followed by an increase in stimulated backscattering, etc. This positive feedback mechanism results in an exponential growth of both the probe intensity and the atomic bunching. This leads to the perhaps surprising result that the presence of the ring cavity turns the ordinarily stable system of an atomic gas driven by a strong pump laser into an unstable system.

Various experiments related to the CARL have been conducted recently. Using hot sodium atoms, Lippi *et al.* [7] observed amplification of an injected probe laser, which they interpreted in terms of scattering off an atomic density grating resulting from atomic recoil. Also using hot sodium atoms, Hemmer *et al.* [8] reported spontaneous probe oscillations in the absence of an injected signal. These were also interpreted as resulting from the CARL mechanism. There has, however, been some controversy regarding the interpretation of these experiments [9], mostly concerning the presence of a large Doppler broadening, and the possibility of gain mechanisms not necessarily related to atomic recoil. An unambiguous demonstration of the recoil related gain mechanism was performed by Courtois *et al.* [10], who observed small signal probe gain in a gas of cold cesium atoms. The absence of a ring cavity for probe feedback in this experiment, however, means that the observed gain was mainly a single-atom recoil effect, not the collective gain of the CARL system.

The CARL theory developed by Bonifacio *et al.* considers the atoms either as classical point particles moving in the optical potential generated by the light fields, or, in a ''hybrid'' version [11], as particles whose center of mass is labeled by their classical position, but with quantum fluctuations about that position included. From an atom-optics point of view, such theories can be described as ''ray-atom-optics'' treatments of the atomic field, in analogy with the ordinary ray-optics treatment of electromagnetic fields.

Like ordinary ray optics, the ray-atom-optics description of CARL is expected to be valid provided that the characteristic wavelength of the matter-wave field remains much smaller than the characteristic length scale of any atom-optical element in the system. The characteristic wavelength of the atomic field is its de Broglie wavelength, determined by the atomic mass and the temperature T of the atomic gas. The central atom-optical element of the CARL is the periodic optical potential, which acts as a diffraction grating for the atoms, and has the characteristic length scale of half the optical wavelength. Hence the classical ''ray-atom-optics'' description is intuitively expected to be valid provided that the temperature is high enough that the thermal de Broglie wavelength is much smaller than the optical wavelength. This gives the condition $T \gg T_R$, the recoil temperature of the atoms, as the domain of ray-atom optics. In particular, it is certainly expected to hold under the temperature conditions of the experiments performed so far.

However, the spectacular recent progress witnessed by atomic cooling techniques makes it likely that CARL experiments using ultracold atomic samples can and will be performed in the future. In particular, subrecoil temperatures can now be achieved almost routinely. The purpose of this paper is to extend the CARL theory to this ''wave-atom-optics'' regime [12]. In this regime it becomes necessary to treat the atomic center-of-mass motion fully quantum mechanically, in order to preserve the wave nature of the atomic motion. Thus the interaction between the atoms and the standing wave light field should no longer be thought of in terms of particles moving in a periodic potential, but instead as diffraction of matter waves by a grating. As the grating amplitude is initially asssumed to be zero (or at least infinitesimal), the system starts in the Bragg regime, where the free space evolution (kinetic energy term) plays an important dynamical role. Similar in origin to the free space diffraction which limits the spot size of a focused beam, this type of diffraction effect can counteract the bunching process by which the atomic density grating is established. It is our goal to determine the precise limitations which this aspect of matter-wave diffraction imposes on CARL operation.

The wave-optics theory of the CARL is similar to the analysis of atomic diffraction by standing waves [13], except that the electromagnetic field is now treated as a dynamical variable. It is also similar to the theory of recoil induced resonances [14], which describes the stimulated scattering of light off a standing wave induced polarization grating, but the absence of a feedback mechanism for the probe feedback in that case means that it models only single-atom gain effects, and not the collective gain of the CARL.

In this paper we focus on the case of a far off-resonant pump laser, thus permitting us to neglect the excited state population and therefore to ignore the effects of spontaneous emission (except as a hypothetical source of noise for probe initiation). We further concentrate on the linear regime, where both the probe field and the atomic bunching are considered as infinitesimal quantities, since it is this regime that determines whether or not the exponential instability occurs. Finally, we restrict our analysis to atomic densities low enough that collisions between atoms may be ignored, and neglect the transverse motion of the atoms, which in the absence of collisions is decoupled from the longitudinal degree of freedom along which bunching occurs.

We note at the outset that our theory is semiclassical in that it treats the electromagnetic field classically. While this approximation cannot fully describe the statistical properties of the CARL output, it is sufficient to describe the small-signal gain of the system, provided that one makes the implicit assumption that small fluctuations will trigger it, an approach familiar from conventional laser theory and nonlinear optics. We also emphasize that it is not inconsistent to treat the matter waves quantum mechanically while treating the light classically, since the limits under which a quantum description is required are independent. For light, this limit is usually associated with weak intensities, while for matter waves it is normally a low temperature limit.

The rest of this paper is organized as follows. Section II briefly reviews the ray-atom-optics model of the CARL, establishing the notation and setting the stage for a comparison of its predictions with those of the wave-atom-optics theory, which is introduced in Sec. III. Section IV discusses the collective instability leading to CARL operation, compares the ray-atom optics and the wave-atom optics predictions, and determines the domain of validity of the former theory. Section V is a discussion and Sec. VI offers a conclusion and outlook.

II. RAY-ATOM-OPTICS MODEL

The ray-atom-optics (RAO) model of the CARL has been developed and extensively studied by Bonifacio and coworkers [2–6]. It begins with the classical N-particle Hamiltonian

$$H_N = \sum_{j=1}^{N} H_1(z_j, p_j), \tag{1}$$

where z_j and p_j are the classical position and momentum of the jth atom, obeying the canonical equations of motion $dz_j/dt = \partial H_N/\partial p_j$ and $dp_j/dt = -\partial H_N/\partial z_j$. The single-particle Hamiltonian H_1 is given explicitly by

$$H_1(z_j, p_j) = \frac{p_j^2}{2m} + \frac{\hbar\omega_0}{2}\sigma_{zj} + i\hbar[g_1 a_1^* e^{-ik_1 z_j}\sigma_{-j} + g_2 a_2^* e^{-ik_2 z_j}\sigma_{-j} - \text{c.c.}], \tag{2}$$

where m is the atomic mass, ω_0 is the natural frequency of the atomic transition being driven by the pump and probe lasers, and g_1 is the atom-probe electric dipole coupling constant. It is given by $g_1 = \mu_1[ck_1/(2\hbar\epsilon_0 V)]^{1/2}$, where μ_1 is the projection of the atomic dipole moment along the probe polarization, k_1 is the probe wave number, and V is the quantization volume. The atom-pump coupling constant g_2 is defined analogously to g_1, but depending on μ_2, the projection of the atomic dipole moment along the pump polarization, and k_2 the pump wave number. The normal variables a_1 and a_2 describe the probe and pump laser fields, respectively. They obey Maxwell's equation

$$\frac{d}{dt}a_i = -ic|k_i|a_i + g_i\sum_{j=1}^{N} e^{-ik_i z_j}\sigma_{-j}, \tag{3}$$

where $c|k_i|$ is the natural frequency of the probe $(i=1)$ or the pump $(i=2)$ field. Note that these equations are also valid for quantized electromagnetic fields, provided that a_i are interpreted as annihilation operators, but we describe the light fields classically in this paper.

The variables σ_{-j} and σ_{zj} are the expectation values of the quantum mechanical Pauli pseudospin operators which describe the internal state of the jth atom. They obey the familiar optical Bloch equations, appropriately modified to include the center-of-mass motion of the atoms and with spontaneous emission neglected,[1]

$$\frac{d}{dt}\sigma_{-j} = -i\omega_0\sigma_{-j} + [g_1^* a_1 e^{ik_1 z_j} + g_2^* a_2 e^{ik_2 z_j}]\sigma_{zj} \tag{4}$$

and

$$\frac{d}{dt}\sigma_{zj} = -2[g_1 a_1^* e^{-ik_1 z_j} + g_2 a_s^* e^{-ik_2 z_j}]\sigma_{-j} + \text{c.c.} \tag{5}$$

[1]Spontaneous emission is neglected in anticipation of the future approximation that the pump lasers are far off-resonant, and therefore the excited state population may be safely neglected.

It is convenient to introduce slowly varying variables via the transformations $a_1=a_1'e^{-i\omega_2 t}$, $a_2=a_2'e^{-i\omega_2 t}$, and $\sigma_{-j}=\sigma'_{-j}e^{-i(\omega_2 t-k_2 z_j)}$, where ω_2 is the pump frequency. The relation between ω_2 and k_2 will be derived shortly in a self-consistent manner, so as to include the dispersive effects of the polarized atoms on the pump propagation. These new variables obey the equations of motion

$$\frac{d}{dt}z_j=\frac{p_j}{m}, \tag{6}$$

$$\frac{d}{dt}p_j=-\hbar[g_1k_1a_1'^*e^{-i(k_1-k_2)z_j}+g_2k_2a_2'^*]\sigma'_{-j}+\text{c.c.}, \tag{7}$$

$$\frac{d}{dt}a_1'=i(\omega_2-c|k_1|)a_1'+g_1\sum_{j=1}^{N}e^{-i(k_1-k_2)z_j}\sigma'_{-j}, \tag{8}$$

$$\frac{d}{dt}a_2'=i(\omega_2-c|k_2|)a_2'+g_2\sum_{j=1}^{N}\sigma'_{-j}, \tag{9}$$

$$\frac{d}{dt}\sigma_{zj}=-2[g_1a_1'^*e^{-i(k_1-k_2)z_j}+g_2a_2'^*]\sigma'_{-j}+\text{c.c.}, \tag{10}$$

and

$$\frac{d}{dt}\sigma'_{-j}=i\left(\omega_2-\omega_0-\frac{k_2}{m}p_j\right)\sigma'_{-j}+[g_1^*a_1'e^{i(k_1-k_2)z_j}+g_2^*a_2']\sigma_{zj}. \tag{11}$$

In the case where the lasers are tuned far off-resonance, and the atoms are initially in the ground state, the excited state population remains small and can be neglected. This is equivalent to describing the atoms as classical Lorentz atoms, and is accomplished by setting $\sigma_{zj}=-1$ in Eq. (11). Assuming further that the detuning $\omega_2-\omega_0$ is much larger than any other frequency in Eq. (11) allows one to adiabatically eliminate σ'_{-j} with

$$\sigma'_{-j}\approx-\frac{i}{(\omega_2-\omega_0)}[g_1^*a_1'e^{i(k_1-k_2)z_j}+g_2^*a_2'], \tag{12}$$

where we have in addition neglected the Doppler shift k_2p_j/m compared to $\omega_2-\omega_0$. This leads to the reduced set of equations

$$\frac{d}{dt}z_j=\frac{p_j}{m}, \tag{13}$$

$$\frac{d}{dt}p_j=-i\frac{2\hbar k_0}{(\omega_2-\omega_0)}[g_1^*g_2a_2'^*a_1'e^{i2k_0z_j}-\text{c.c.}], \tag{14}$$

$$\frac{d}{dt}a_1'=i\left[\omega_2-\frac{N|g_1|^2}{(\omega_2-\omega_0)}-c|k_1|\right]a_1'-i\frac{g_2^*g_1}{(\omega_2-\omega_0)}a_2'\sum_{j=1}^{N}e^{-i2k_0z_j}, \tag{15}$$

and

$$\frac{d}{dt}a_2'=i\left[\omega_2-\frac{N|g_2|^2}{(\omega_2-\omega_0)}-c|k_2|\right]a_2'-i\frac{g_1^*g_2}{(\omega_2-\omega_0)}a_1'\sum_{j=1}^{N}e^{i2k_0z_j}, \tag{16}$$

where we have introduced $k_0=(k_1-k_2)/2$.

We now introduce the undepleted pump approximation, valid in the linear regime where a_1' remains small. This is achieved by dropping the term proportional to a_1' in Eq. (16). This yields

$$\frac{d}{dt}a_2'=i\left[\omega_2-\frac{N|g_2|^2}{(\omega_2-\omega_0)}-c|k_2|\right]a_2', \tag{17}$$

which has the steady state solution $a_2'(t)=a_2(0)$ provided that the dispersion relation

$$c|k_2|=\omega_2-\frac{N|g_2|^2}{(\omega_2-\omega_0)} \tag{18}$$

is satisfied. Equation (18) thus determines the magnitude of the pump wave number as a function of the pump frequency and other known experimental parameters.

To proceed analytically past this point, it is convenient to introduce the dimensionless variables $\theta_j\equiv 2k_0z_j$, $P_j=p_j/\hbar k_0$, $A=g_1^*g_2a_2^*(0)a_1'/[2\omega_r(\omega_2-\omega_0)]$, and $\tau=4\omega_r t$, where the recoil frequency ω_r is given by

$$\omega_r=\hbar k_0^2/2m. \tag{19}$$

These variables obey the equations of motion

$$\frac{d}{d\tau}\theta_j=P_j, \tag{20}$$

$$\frac{d}{d\tau}P_j=-iAe^{i\theta_j}+\text{c.c.}, \tag{21}$$

and

$$\frac{d}{d\tau}A=i\Delta A-i\alpha\frac{1}{N}\sum_{j=1}^{N}e^{-i\theta_j}, \tag{22}$$

where we have introduced the dimensionless control parameters

$$\Delta=\left(\omega_2-\frac{N|g_1|^2}{\omega_2-\omega_0}-c|k_1|\right)\Big/4\omega_r \tag{23}$$

and

$$\alpha=N|g_1|^2|g_2|^2|a_2(0)|^2/8\omega_r^2(\omega_2-\omega_0)^2. \tag{24}$$

We note that both Δ and α are real numbers, and furthermore that $\alpha\geqslant 0$. The term $c|k_1|+N|g_1|^2/(\omega_2-\omega_0)$ in Eq. (23) is simply the natural frequency of the probe plus the shift to atomic dispersion, i.e., it is the frequency of the probe field ω_1. This means that $\Delta=(\omega_2-\omega_1)/4\omega_r$ is simply the pump-probe detuning in units of $4\omega_r$.

We seek solutions of these equations which are perturbations about the case $A=0$. Thus we make the substitutions

$$\theta_j=\theta_j(0)+P_j(0)\tau+\delta\theta_j \tag{25}$$

and

$$P_j=P_j(0)+\delta P_j, \tag{26}$$

where $\theta_j(0)$ is randomly taken from a uniform distribution and $P_j(0)$ is randomly taken from the initial momentum distribution. The new variables $\delta\theta_j$ and δP_j give the perturbations on the atomic center-of-mass motion due to a nonzero $A(0)$. We introduce finally the linearized velocity group bunching parameter and its ''conjugate'' momentum according to

$$B(k)=\frac{1}{N}\sum_{j=1}^{N}\delta_{P_j(0),2k}(1-i\delta\theta_j)e^{-i[\theta_j(0)+P_j(0)\tau]} \tag{27}$$

and

$$\Pi(k)=\frac{1}{N}\sum_{j=1}^{N}\delta_{P_j(0),2k}\delta P_j e^{-i[\theta_j(0)+P_j(0)\tau]}+2kB(k). \tag{28}$$

We note that

$$\sum_k B(k)=\langle e^{-i2k_0z}\rangle, \tag{29}$$

and the amplitude of Eq. (29) is a measure of the degree of bunching of the atomic gas. A magnitude of zero indicates no bunching, while a magnitude of one indicates maximum bunching. This leads to the equations

$$\frac{d}{d\tau}B(k)=-i\Pi(k), \tag{30}$$

$$\frac{d}{d\tau}\Pi(k)=i\left[4k^2B(k)-4k\Pi(k)-\frac{N(k)}{N}A\right], \tag{31}$$

and

$$\frac{d}{d\tau}A=i\left[\Delta A-\alpha\sum_k B(k)\right], \tag{32}$$

where $N(k)$ is the number of atoms in the velocity group with momentum $2\hbar k_0 k$ and we have assumed that

$$\sum_{j=1}^{N}\delta_{P_j(0),2k}e^{-i2\theta_j(0)}=0, \tag{33}$$

an assumption that requires that $N(k)\gg 1$. Note that this formulation implies a discretization of the initial momentum distribution, and furthermore assumes that the atomic positions in each velocity group are initially randomly distributed along the CARL cavity. Fluctuations in the initial distributions can of course readily be included into the initial conditions of the perturbation variables.

III. WAVE-ATOM-OPTICS MODEL

In order to quantize the center-of-mass motion of a gas of bosonic atoms, one may either utilize first quantization, and replace the variables z_j and p_j in the N-particle Hamiltonian (1) with operators satisfying the canonical commutation relations $[\hat{z}_j,\hat{p}_{j'}]=i\hbar\delta_{jj'}$, or equivalently we can second quantize the single-particle Hamiltonian (2), introducing creation and annihilation operators for excited and ground state atoms of a given center-of-mass momentum. It is this second method which we will adopt in deriving the wave-atom-optics (WAO) model. In the absence of collisions, the second-quantized Hamiltonian is simply

$$\hat{H}=\sum_k \hat{H}(k), \tag{34}$$

where $\hat{H}(k)$ is given by

$$\begin{aligned}\hat{H}(k)=&\frac{\hbar^2k^2}{2m}\hat{c}_g^\dagger(k)\hat{c}_g(k)+\left(\frac{\hbar^2k^2}{2m}+\hbar\omega_0\right)\hat{c}_e^\dagger(k)\hat{c}_e(k)\\&+i\hbar[g_1a_1^*\hat{c}_g^\dagger(k-k_1)\hat{c}_e(k)+g_2a_2^*\hat{c}_g^\dagger(k-k_2)\hat{c}_e(k)\\&-\text{H.c.}],\end{aligned} \tag{35}$$

The field operator $\hat{c}_g(k)$ annihilates a ground state atom of momentum $\hbar k$, and $\hat{c}_e(k)$ annihilates an excited atom of momentum $\hbar k$. We assume that the atoms in the sample are bosonic, so that these operators obey the commutation relations

$$[\hat{c}_g(k),\hat{c}_g^\dagger(k')]=[\hat{c}_e(k),\hat{c}_e^\dagger(k')]=\delta_{kk'}, \tag{36}$$

all other commutators being equal to zero.

With the atomic polarization now expressed in terms of field operators, Maxwell's equations (3) for the classical laser fields become

$$\frac{d}{dt}a_i=-ic|k_i|a_i+g_i\sum_k\langle\hat{c}_g^\dagger(k-k_i)\hat{c}_e(k)\rangle. \tag{37}$$

Hence, all that is required to determine the field evolution are the expectation values of bilinear combinations of atomic creation and annihilation operators. The evolution of these expectation values is easily obtained by introducing the ''single-particle'' atomic density operators[2]

$$\hat{\rho}_{gg}(k,k')=\hat{c}_g^\dagger(k')\hat{c}_g(k), \tag{38}$$

$$\hat{\rho}_{eg}(k,k')=[\hat{\rho}_{ge}(k',k)]^\dagger=\hat{c}_g^\dagger(k')\hat{c}_e(k), \tag{39}$$

and

$$\hat{\rho}_{ee}(k,k')=\hat{c}_e^\dagger(k')\hat{c}_e(k). \tag{40}$$

[2]These are single-particle operators in the sense of many-body theory, since they only involve the annihilation of an atom in a given state and its creation in some other state.

Note that, e.g., the expectation value of the diagonal operator $\langle\hat{\rho}_{gg}(k,k)\rangle$ gives the mean number of ground state atoms with momentum $\hbar k$. The expectation values of these operators obey the equations of motion

$$\frac{d}{dt}\rho_{jj'}(k,k')=\frac{i}{\hbar}\langle[\hat{H},\hat{\rho}_{jj'}(k,k')]\rangle, \tag{41}$$

where $\rho_{jj'}(k,k')=\langle\hat{\rho}_{jj'}(k,k')\rangle$. The full form of these equations is given in the Appendix. The important point is that they depend only on $\rho_{jj'}(k,k')$, hence they form a closed set of equations which describe the response of the atomic field to the driving laser fields. We note that had we included collisions in our model, this would no longer be the case.

Introducing in analogy to the ray-optics description the rotating variables $a_1=a_1'e^{-i\omega_2 t}$, $a_2=a_2'e^{-i\omega_2 t}$, and $\rho_{eg}(k,k')=\rho_{eg}'(k-k_2,k')e^{-i\omega_2 t}$, neglecting the excited state population, and solving adiabatically for $\rho_{eg}'(k,k')$ yields

$$\rho_{eg}'(k,k')\approx-\frac{i}{(\omega_2-\omega_0)}[g_1^*a_1'\rho_{gg}(k-2k_0,k') +g_2^*a_2'\rho_{gg}(k,k')]. \tag{42}$$

Substituting Eq. (42) into Maxwell's equation (37) for the pump and making once more the undepleted pump approximation leads to the solution $a_2'(t)=a_2(0)$ provided that $|k_2|$ satisfies the dispersion relation (18). We then substitute Eq. (42) into the equation of motion for $\rho_{gg}(k,k')$, and introduce the dimensionless wave number $\kappa=k/2k_0$ and the mean density $\rho(\kappa,\kappa')=\rho_{gg}(k,k')/N$, in addition to the dimensionless variables already defined in the ray-atom-optics model. We arrive at the wave-optics equations of motion

$$\begin{aligned}\frac{d}{d\tau}\rho(\kappa,\kappa')=&-i(\kappa^2-\kappa'^2)\rho(\kappa,\kappa')\\&+\frac{i}{2}A^*[\rho(\kappa,\kappa'-1)-\rho(\kappa+1,\kappa')]\\&-\frac{i}{2}A[\rho(\kappa-1,\kappa')-\rho(\kappa,\kappa'+1)]\end{aligned} \tag{43}$$

and

$$\frac{d}{d\tau}A=i\Delta A-i\alpha\sum_\kappa \rho(\kappa,\kappa+1), \tag{44}$$

where the parameters Δ and α are given by Eqs. (23) and (24), respectively.

As in Sec. II, we seek a solution which is a perturbation about the case $A=0$. From Eq. (43), the unperturbed solution is readily found to be

$$\rho(\kappa,\kappa',\tau)=\rho(\kappa,\kappa',0)e^{-i(\kappa^2-\kappa'^2)\tau}. \tag{45}$$

We consider specifically an atomic sample initially in thermal equilibrium, so that Eq. (45) becomes

$$\rho(\kappa,\kappa',\tau)=\frac{N(\kappa)}{N}\delta_{\kappa,\kappa'}, \tag{46}$$

where $N(\kappa)$, the number of atoms with initial wave number $2k_0\kappa$, is given by a thermal distribution function. We introduce the perturbation variables $\delta\rho(\kappa,\kappa')$ according to

$$\rho(\kappa,\kappa')=\frac{N(\kappa)}{N}\delta_{\kappa,\kappa'}+\delta\rho(\kappa,\kappa') \tag{47}$$

and observe that Maxwell's equation (44), which becomes

$$\frac{d}{d\tau}A=i\Delta A-i\alpha\sum_\kappa \delta\rho(\kappa,\kappa-1), \tag{48}$$

together with the linearized equation

$$\begin{aligned}\frac{d}{d\tau}\delta\rho(\kappa,\kappa-1)=&-i(2\kappa-1)\delta\rho(\kappa,\kappa-1)\\&+i\frac{[N(\kappa)-N(\kappa-1)]}{2N}A,\end{aligned} \tag{49}$$

form a closed set of equations which underlies the dynamics of the CARL in the linear regime of wave-atom optics. We note that the sum of the spatial coherence terms has the physical interpretation

$$\sum_\kappa \delta\rho(\kappa,\kappa-1)=\langle e^{-i2k_0 z}\rangle, \tag{50}$$

which in analogy to Eq. (29) measures the degree of bunching of the atomic gas.

IV. COLLECTIVE INSTABILITY

The most important feature of the CARL is the appearance of a collective instability, which gives rise to exponential gain under appropriate parameter settings. This instability is characterized by an imaginary frequency component in the spectrum of the probe field $A(\tau)$. As has been demonstrated in Ref. [4], one needs not solve the complete set of equations derived in the previous sections in order to determine the necessary conditions for the collective instability. Instead, by taking the Laplace transform of these equations one can derive a ''characteristic equation'' which allows one to determine whether exponential gain occurs, and if so what the exponential growth rate is.

For the ray-atom-optics model, the Laplace transform of Eq. (32) yields

$$\tilde{A}_R(s)=\frac{A(0)}{R(s)}, \tag{51}$$

where $R(s)$ is given by

$$R(s)=\left[s-i\Delta-i\alpha\int\frac{f(k)dk}{(s+i2k)^2}\right]. \tag{52}$$

In obtaining this result we have taken the continuum limit and assumed that $B(k)$ and $\Pi(k)$ vanish at $\tau=0$, which corresponds to an initially homogeneous (unbunched) distribution of atoms. Here $f(k)$ is simply the normalized thermal distribution function for the dimensionless wave number k. We note that in original units an atom with dimensionless wave number k has momentum $2\hbar k_0 k$. The roots of $R(s)$ give the characteristic exponents of the CARL. Stability requires that all roots be purely imaginary. When the collective instability occurs, however, there will be one root with a positive real part. This real part is the RAO exponential growth rate Γ_R. This result is identical to that obtained by Bonifacio and De Salvo [4].

The wave-atom-optics model, which includes the effects of atomic diffraction, yields the Laplace transform

$$\tilde{A}_W(s)=\frac{A(0)}{W(s)}. \tag{53}$$

$W(s)$ is given by

$$W(s)=\left[s-i\Delta-i\alpha\int\frac{f(k)dk}{[s+i(2k-1)][s+i(2k+1)]}\right], \tag{54}$$

where we have again taken the continuum limit and assumed that $\delta\rho(k,k+1)$ vanishes at $\tau=0$, which corresponds to an initially unbunched atomic sample. If a root of $W(s)$ with a positive real part exists, that real part is the WAO exponential growth rate Γ_W.

We see by comparing Eqs. (51) and (53) that the effect of atomic diffraction is to lift the degeneracy of the singularity under the integral. This expression also leads us immediately to the conclusion that if the width of the momentum distribution $f(k)$ is large compared to $2k$, then the singularity will appear as essentially degenerate, and the effects of matter-wave diffraction will be negligible. Thus the RAO and WAO models should agree for large enough temperatures.

A. Finite temperatures

In the absence of quantum degeneracies, the thermal momentum distribution is given by the Maxwell-Boltzmann distribution

$$f(k)=\frac{2\beta}{\sqrt{\pi}}e^{-4k^2\beta^2}, \tag{55}$$

where $\beta^2=T_R/T$ and $T_R=\hbar\omega_r/k_B$ is the recoil temperature, K_B being the Boltzmann constant. By substituting Eq. (55) into Eq. (52) and using the Fourier convolution theorem we find that the RAO exponential growth rate Γ_R is determined by the equation

$$s-i\Delta-i\alpha\int_0^\infty pe^{-p^2/4\beta^2-ps}dp=0, \tag{56}$$

which can be integrated to give the transcendental equation

$$s-i\Delta-i2\alpha\beta^2+i2\sqrt{\pi}\alpha\beta^3e^{\beta^2s^2}\mathrm{erfc}(\beta s)=0. \tag{57}$$

In contrast, substituting Eq. (55) into Eq. (54) and again using the convolution theorem we find that the WAO exponential growth rate Γ_W is determined by the equation

$$s-i\Delta-i\alpha\int_0^\infty e^{-p^2/4\beta^2-ps}\sin(p)dp=0. \tag{58}$$

By examining Eq. (58) we see that in the case $\beta\ll1$ we are justified in expanding sin(p) to lowest order in p. This exactly reproduces Eq. (56), thus showing that the WAO and RAO descriptions make indistinguishable predictions about the exponential growth rate in the limit $T\gg T_R$. However, for temperatures comparable to or less than the recoil temperature, we will see that the RAO theory fails to correctly predict the behavior of the CARL in the linear regime. Physically, this is due to the fact that it does not account for the effects of atomic diffraction, which tends to counteract the bunching process. Finally, we note that upon integration, Eq. (58) becomes the transcendental equation

$$s-i\Delta+\frac{\sqrt{\pi}}{2}\alpha\beta e^{\beta^2(s^2-1)}\{e^{i2\beta^2s}\mathrm{erfc}[\beta(s+i)] \\ -e^{-i2\beta^2s}\mathrm{erfc}[\beta(s-i)]\}=0. \tag{59}$$

In the next subsection we will examine in more detail the precise manner in which diffraction interferes with the bunching process for the special case of a zero temperature atomic gas. But before turning to this extreme situation, we present numerical results comparing RAO and WAO models at nonzero temperature, as determined by solving Eqs. (57) and (59).

Figures 1(b)–1(d) compare Γ_R with Γ_W at $\alpha=10$ for the three different temperature regimes $T=T_R$, $T=10T_R$, and $T=100T_R$, respectively. Figures 2(b)–2(d) show the same comparison for $\alpha=10^{-1}$. While we see that the behavior of Γ_R and Γ_W depends strongly on α (recall that α is proportional to both the pump intensity and the atomic density), the discrepancies between the two models as a function of temperature are very similar. At $T=T_R$ there are significant differences between the predictions of the RAO and WAO models, but these differences become minimal at $T=10T_R$, and insignificant at $T=100T_R$. We also observe that the differences are more pronounced for lower values of α, meaning that at lower densities and/or pump intensities, the quantum mechanical behavior becomes more apparent. The reason for this is that at high intensities the bunching process, driven by the probe field, dominates, while at low intensities the antibunching effects of atomic diffraction play a larger role.

B. The $T=0$ limit

For a typical atom, the recoil temperature is of the order of microkelvins, e.g., for sodium we have $T_R=2.4$ μK. However, recent advances in cooling techniques have led to measured temperatures as low as the picokelvin regime. At these extreme temperatures the condition $T\ll T_R$ is satisfied, i.e., we are effectively in the $T\rightarrow0$ limit. In this section we study the $T=0$ case in detail in order to gain further insight into the exact role of matter-wave diffraction in the CARL system.

For the RAO model, we have a single velocity group at $k=0$. Thus by differentiating Eq. (30) with respect to τ and using Eq. (31), we see that the bunching parameter B

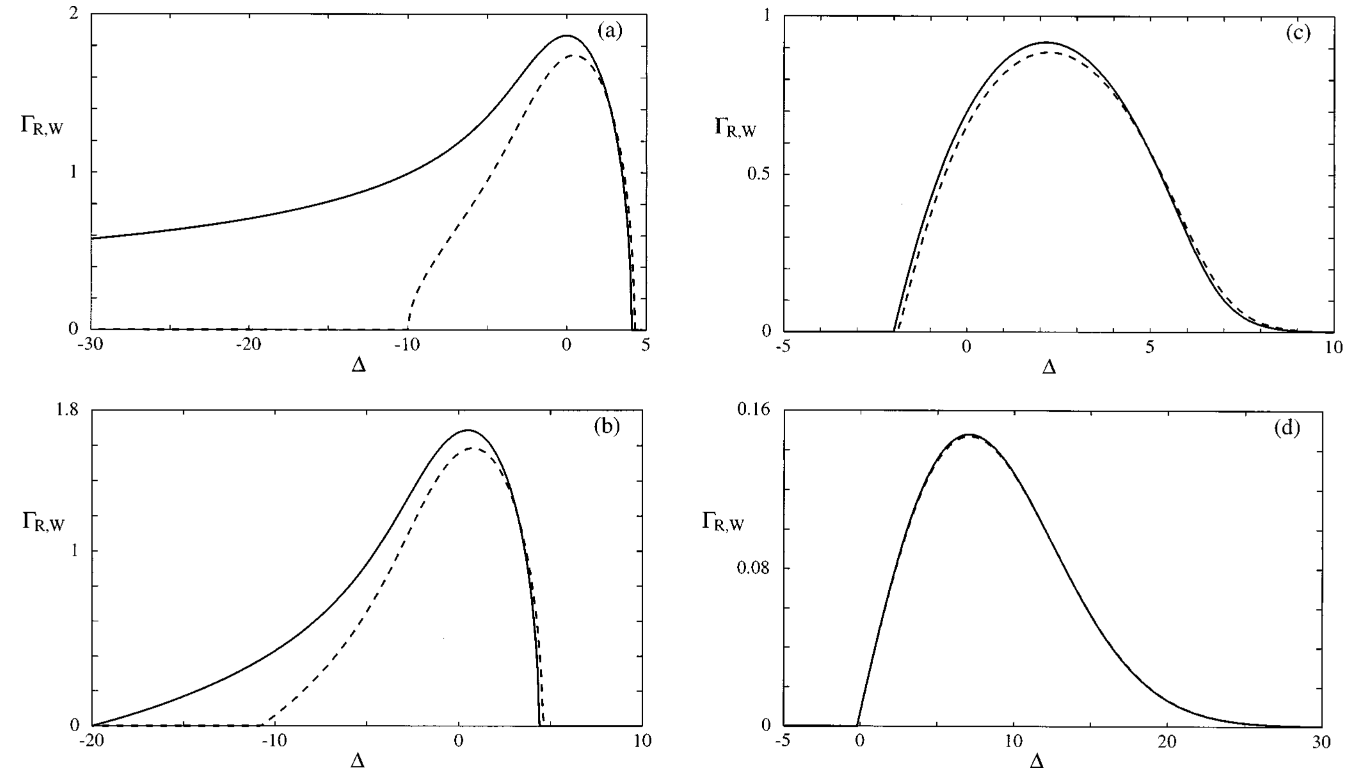

FIG. 1. Comparison of the exponential growth rate as a function of pump-probe detuning Δ between the RAO (solid line) and the WAO (dashed line) models, for the case $\alpha=10$. (a) shows the results for $T=0$ (see Sec. III B), (b) shows the case $T=T_R$, (c) shows $T=10T_R$, and (d) shows $T=100T_R$. We see that the ray-atom-optics model gives the correct result only in the limit $T\gg T_R$.

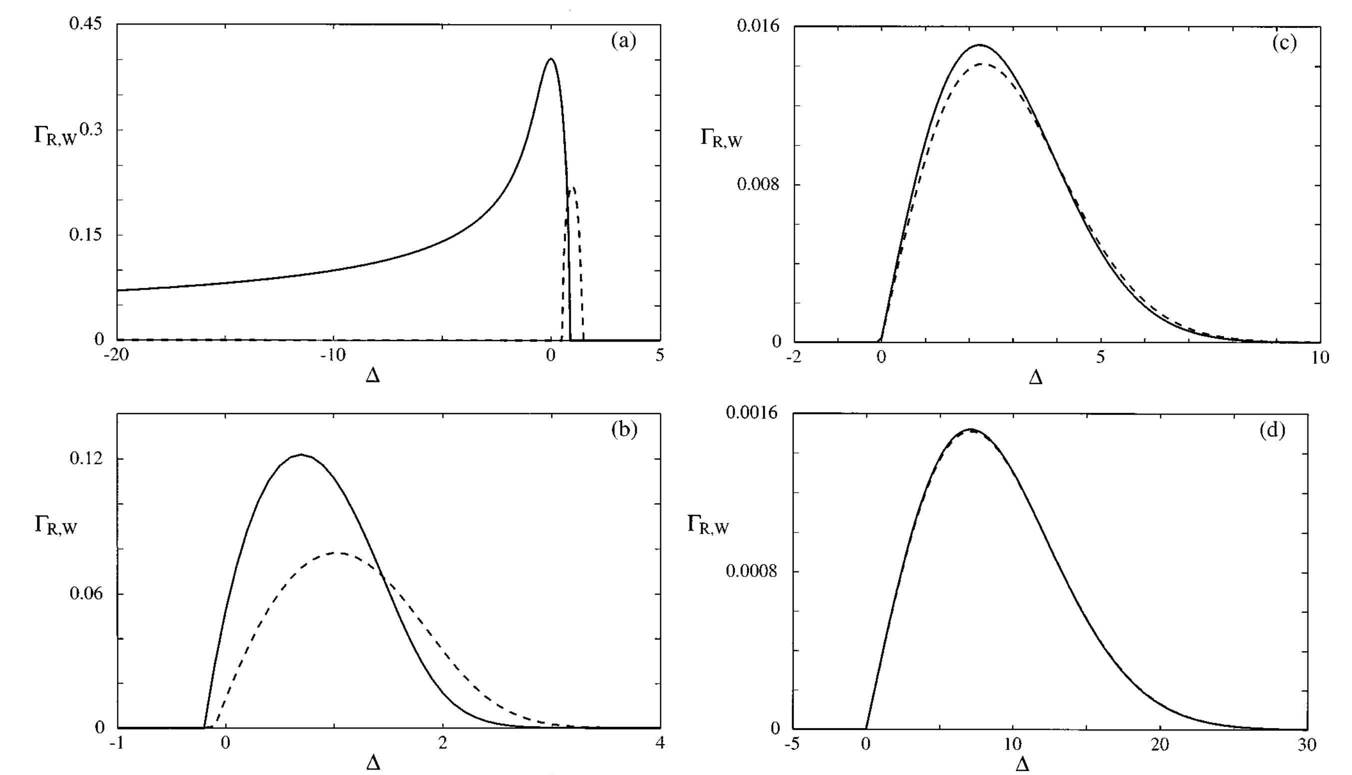

FIG. 2. Identical to Fig. 1, except we have now taken $\alpha=10^{-1}$. Since α gives the strength of the bunching process, when it is small the effects of atomic diffracton play a larger role, leading to stronger discrepancies between the predictions of wave-atom optics and ray-atom optics. However, we see that the RAO limit, given by $T\gg T_R$, is independent of α.

$\equiv B(0)=\langle\exp(-i2k_0z)\rangle$ obeys the equation of motion

$$\frac{d^2}{d\tau^2}B=-A, \tag{60}$$

where we have taken $P(0)=0$ and $N(0)=N$ to indicate that all atoms are initially at rest.

In the WAO description, setting $N(\kappa)/N=\delta_{\kappa,0}$ in Eq. (49) shows that two variables are coupled to the probe field, $\delta\rho(-1,0)$, and $\delta\rho(0,1)$. They describe the recoil of atoms initially at rest as a result of their interaction with the light fields. We proceed then by introducing the new variable $B\equiv\delta\rho(-1,0)+\delta\rho(0,1)$, which has the same physical meaning as in the RAO model, namely, $B=\langle\exp(-i2k_0z)\rangle$. But in contrast to that case, the time evolution of B is now governed by the equation of motion

$$\frac{d^2}{d\tau^2}B=-B-A. \tag{61}$$

This result shows that in contrast to the predictions of classical mechanics, where the bunching parameter B has dynamics similar to a *free particle* driven by the probe field A, quantum mechanically B behaves as a *simple harmonic oscillator* of frequency $4\omega_r$ (in original time units), and subject to that same driving force. In the linear regime, B is assumed to be a small perturbation about its initial value of zero, and the forces resulting from a nonzero probe field A tend to cause B to increase. But this mechanism is opposed by the ''restoring force'' due to atomic diffraction.

In addition to opposing any increase in the magnitude of B, the diffraction term also modifies its phase, which may upset any phase relation between A and B which might be required for the collective instability to occur.

The RAO model only makes accurate predictions at $T=0$ in the limit $\omega_r\to 0$. Therefore, if we were to increase the mass of the atoms, thus decreasing ω_r, the behavior at $T=0$ would become more and more classical. This is because heavier atoms suffer less diffraction than lighter atoms. We also note that the correspondence principle states that quantum mechanics should agree with classical mechanics in the limit $\hbar\to 0$, which would also cause ω_r to tend to zero. These considerations can also be derived from the statement that the RAO model is valid when $T\gg T_R$, if we note that as $\omega_r\to 0$ the recoil temperature also goes to zero.

In both the RAO and WAO models, the probe field A obeys the equation

$$\frac{d}{d\tau}A=i(\Delta A-\alpha B). \tag{62}$$

For the RAO model we combine Eq. (62) with Eq. (60), and find that the solutions are exponentials with exponents given by the roots of the cubic equation

$$s^3-i\Delta s^2-i\alpha=0. \tag{63}$$

This is exactly the ''cold-beam'' cubic equation of Bonifacio and De Salvo [4]. However, with the inclusion of atomic diffraction effects, we now see that the correct ''cold-beam'' cubic equation, derived from Eqs. (62) and (61), is given by the WAO model to be

$$s^3-i\Delta s^2+s-i(\alpha+\Delta)=0. \tag{64}$$

These equations can also be derived from the Laplace transform method of Sec. III, with the substitution $f(k)=\delta(k)$, indicating a zero temperature momentum distribution.

From these cubic equations it is possible to determine the point of transition between the stable and the unstable regimes of the CARL. For the RAO model the collective instability occurs provided that the threshold condition

$$\alpha>\frac{4\Delta^2}{27} \tag{65}$$

is satisfied, and above threshold the exponential growth rate is given by

$$\Gamma_R=\frac{\sqrt{3}}{2}\left(\frac{\alpha}{4}\right)^{1/3}\left|(1+\sqrt{C})^{2/3}-(1-\sqrt{C})^{2/3}\right|, \tag{66}$$

where $C=1-4\Delta^3/27\alpha$. For the WAO theory the threshold condition is

$$\alpha>\frac{2}{27}[(3+\Delta^2)^{3/2}-9\Delta+\Delta^3], \tag{67}$$

and above threshold the exponential growth rate is given by

$$\Gamma_W=\frac{\sqrt{3}}{2}\left(\frac{\alpha}{4}\right)^{1/3}\left|\left[(1+\sqrt{D})^2+\frac{4}{27\alpha^2}(1-\Delta^2)^2\right]^{1/3}-\left[(1-\sqrt{D})^2+\frac{4}{27\alpha^2}(1-\Delta^2)^2\right]^{1/3}\right|, \tag{68}$$

where

$$D=1-\frac{4\Delta}{3\alpha}\left(1-\frac{\Delta^2}{9}\right)-\frac{4}{27\alpha^2}(1-\Delta^2)^2. \tag{69}$$

In Fig. 3(a) we examine the CARL operating regime, defined as the region in parameter space where the exponential instability occurs, at $T=0$ as it would be if ray-atom optics were valid. We contrast this with Fig. 3(b) which shows the actual CARL operating regime at $T=0$, as calculated using wave-atom optics. From this figure we see that the operating regime of the CARL is drastically reduced at low pump intensities and/or atomic densities when the effects of atomic diffraction are included.

Figure 1(a) compares Γ_R with Γ_W for the case $\alpha=10$ at $T=0$, and Fig. 2(a) shows the same comparison for $\alpha=10^{-1}$. We see that atomic diffraction leads to the appearance of a second threshold below which the collective instability does not occur. From Fig. 2(a) we see that this second threshold may even be above $\Delta=0$ for low intensities and/or densities. In fact, the threshold crosses $\Delta=0$ at precisely $\alpha=2/3\sqrt{3}$.

Figure 2(a) shows that in the limit of weak pump intensities and/or atomic densities the peak gain for the WAO

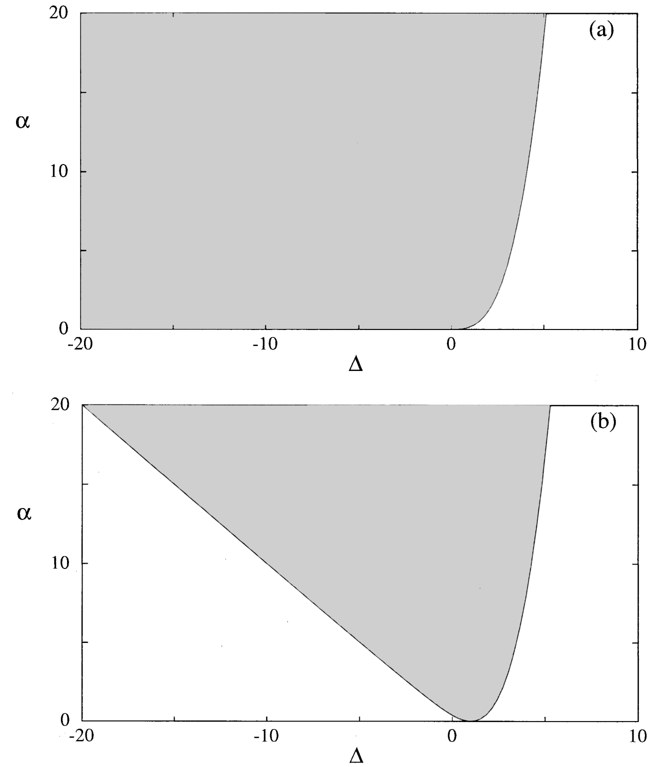

FIG. 3. The CARL operating regime (shaded region) as predicted by the RAO model (a), and the actual operating regime (b), as given by the WAO model.

model tends to $\Delta=1$, while that of the RAO model is at $\Delta=0$. This result can actually be understood quite simply: The atomic center-of-mass dispersion curve tells us that the absorption of a pump photon and the emission of a probe photon by an atom initially at rest creates an energy defect of $4\omega_r$ due to atomic recoil. This defect can be compensated by a detuning between the pump and probe, which in dimensionless units occurs at $\Delta=1$. Therefore the fact that Γ_W is a sharply peaked function around $\Delta=1$ is simply an expression of energy-momentum conservation. If we are to take the ray-atom-optics model seriously at $T=0$, then we must concede that we are in the limit where $\omega_r\rightarrow 0$, therefore energy-momentum conservation would predict the maximum of Γ_R to occur at $\Delta=0$. In other words, in that limit the center-of-mass dispersion curve is flat over the range of a few photon momenta.

V. DISCUSSION

We have argued that the classical description of atomic center-of-mass motion actually corresponds to a ray-optics description of the atomic Schrödinger field. At high temperatures it adequately describes the CARL dynamics, however, at subrecoil temperatures the wave nature of the atomic field becomes apparent, and the ray-optics approximation no longer suffices. Even at high temperatures, where the ray and wave pictures make indistinguishable predictions, their physical interpretations are different: the first considers the atoms as localized distinguishable particles which follow trajectories in phase space, and the other considers the collection of atoms as a quantum Schrödinger field in which the various normal modes of oscillation are coupled via the atom-photon interactions.

Quantum thermodynamics tells us that in thermal equilibrium, the uncertainty in position of each atom completely fills the volume of its container, independent of the temperature. Therefore, from the quantum field point of view, even at high temperatures it makes no sense *in principle* to consider the atoms as localized or even as distinguishable particles, even though *in practice* such a picture works quite well. The differences between the two points of view lead to different physical interpretations of CARL behavior, even though quantitatively they agree completely in the proper limit.

For example, in the classical model the dynamics of a single atom differs vastly from the dynamics of a large number of atoms, hence leading to a distinction between effects which rely on the presence of many atoms and effects which would occur for even a single atom. We note that in our derivation of the RAO model we have made averaging approximations which assume that the atom number is very large, thus it cannot be used as a single-atom theory just by setting $N=1$. In contrast, in the quantum picture the atom number appears only as the amplitude of the Schrödinger field, and due to the fact that the atoms are delocalized, no averaging is necessary and the WAO model *can* be used as a single-atom theory by simply setting $N=1$. Thus we see that, excluding high intensity effects and collisions not included in our model, a single-atom CARL will exhibit all possible CARL behavior, provided that the pump intensity is increased to compensate for the small atom number.

Using the quantum picture, it is relatively easy to understand the effects of Doppler broadening on the CARL. These effects have been studied in detail within the framework of the RAO model by Bonifacio and co-workers and Brown *et al.* [4,6,9], and we present arguments here only to illustrate the utility of the quantum picture as well as to discuss what happens to the Doppler broadening effect as one enters the subrecoil regime.

The fundamental interaction which gives rise to probe amplification in the CARL involves the absorption of a pump photon and the emission of a probe photon, together with the transfer of an atom from an initial mode with dimensionless wave number k to a final mode $k-1$. As can be seen from Eq. (49) this transition rate is proportional to the population difference between the initial and final atomic field modes. This population difference is maximized when the initial atomic field mode coincides with the maximum population gradient, given by $k=-\sigma_k+1/2$, where $\sigma_k=\sqrt{T/4T_R}$ is the half-width of the Maxwell-Boltzmann distribution function. However, this interaction carries an energy defect given in units of $4\hbar\omega_r$ by $\Delta E=2\sigma_k+\Delta$, where Δ is the pump-probe detuning defined by Eq. (23). Therefore the maximum gain will occur when the pump-probe detuning is chosen to conserve energy for transitions between atomic field modes centered around the maximum population gradient. This leads to the condition $\Delta=2\sigma_k=\sqrt{T/T_R}$ as maximizing the exponential growth rate for the CARL.

The previous argument applies only to atomic fields with a velocity spread much larger than the recoil velocity. For fields where the spread in velocity is small compared to the recoil velocity a slightly different mechanism occurs. Here, a transition involving atomic field modes separated by two recoil momenta can no longer be centered around the maximum population gradient, as this would result in both levels involved having virtually zero population. Instead, the maximum population difference occurs between the mode with the largest population ($k=0$) and a virtually empty mode ($k=-1$). This transition carries an energy defect given by $\Delta E=1-\Delta$, so the maximum exponential growth rate occurs for pump-probe detunings $\Delta=1$, as discussed in Sec. IV B.

Thus we see that the physics of the CARL at high temperatures is different from that at low temperatures. For $T \gg T_R$ maximum gain comes from transitions centered on the maximum population gradient, characterized by the condition $\Delta=\sqrt{T/T_R}$, while for $T \ll T_R$ maximum gain comes from transitions starting from the mode with the largest population, characterized by the condition $\Delta=1$.

Because the wave-optics picture involves transitions between center-of-mass modes with different atomic velocities, it raises the question of whether or not a $T\approx 0$ sample of atoms is actually destroyed due to the interaction with the pump and probe lasers. Consider that before the interaction, the atoms are practically at rest, but after absorbing a pump photon and emitting a probe photon, they are moving at twice the recoil velocity due to atomic recoil, and therefore will eventually leave the initial sample. This results in atom losses, causing the atomic bunching to decay and reducing the CARL gain. We argue, however, that these losses will remain negligible for a realistic experiment involving a Bose-Einstein condensate (BEC) with a size on the order of 100 μm. Here the decay rate should be approximately equal to the atomic velocity divided by the condensate size. For sodium, which has a recoil velocity of 3 cm/s, the loss rate works out to 10^2–10^3 s^{-1}. The CARL growth rate, on the other hand, can be of the order $4\omega_r$ or larger, which for sodium is of the order 10^5 s^{-1} or larger. Therefore, on the time scale of the exponential growth in the CARL, the loss rate due to atoms leaving the sample is negligible. Additionally, we note that in the linear regime we have shown that the population of the various atomic field modes remains unchanged, the effect of the light-matter interaction being solely to generate a spatial coherence, i.e., bunching, in the form of nonzero off-diagonal density matrix elements $\rho(k,k-1)$. This means that while the increase in the atomic bunching is a linear effect, the shift in atomic population is a second-order effect. The fractional population shift is in fact given by the square of the bunching parameter. So in the linear regime, we can consider the sample to experience bunching (spatial coherence) effectively without population transfer. Of course in the non-linear regime this would no longer be the case. But here we also need to consider probe reabsorption, which plays a crucial role in the saturation behavior of CARL, and can reverse the spread in the atomic sample.

VI. CONCLUSION AND OUTLOOK

The CARL system represents an experimentally realizable nontrivial example of dynamically coupled Maxwell and Schrödinger fields. Thus the theory of the CARL is actually a hybrid which combines ordinary nonlinear optics with nonlinear atom optics. In ordinary nonlinear optics, one benefits greatly when using laser light (characterized by coherent, ‘‘single-mode’’ optical fields) as opposed to incoherent light. By analogy, one should greatly benefit in atom optics when using coherent atomic fields. This has been the primary motivation behind the recent interest in developing the‘‘atom laser’’ as a source of coherent atomic fields. While the search for a cw atom laser continues, the current state of the art in coherent atomic field generation involves the creation of Bose-Einstein condensates. A trapped BEC can be thought of as a stationary ‘‘atom laser’’ pulse, and as such is ideal for studying systems of coupled atomic and electromagnetic fields, such as the CARL. However, the temperature of a Bose-Einstein condensate typically falls well below the atomic recoil temperature, and thus outside of the regime of the current CARL theory.

The main result of this paper has been to develop the wave-atom-optics model of the CARL, valid in the subrecoil regime, and to compare this theory to the previous ray-atom-optics model. We have shown that, as expected, in the subrecoil regime the behavior of the CARL is strongly influenced by matter-wave diffraction, which tends to counteract the atomic bunching process and reduce the operating regime of the system. However, for temperatures large compared to the recoil temperature we have shown that the two models make indistinguishable predictions.

The present theory quantizes the matter wave, but not the electromagnetic field. It will be of considerable interest to extend it to regimes where both fields need to be quantized. An analysis of the density regime where quantum degeneracy becomes important will also be a fascinating extension, in particular when two-body collisions are included. This study will allow one to investigate to what extent a Bose-Einstein condensate can be manipulated and modified in a far off-resonant CARL configuration. An intriguing possibility would be to generate in this fashion a coupled laser–‘‘atom-laser’’ system. The study of the coherence properties of this system will be the object of future investigations. Finally, a comparison between bosonic and fermionic CARL systems in the quantum degenerate regime should also be considered.

ACKNOWLEDGMENTS

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APPENDIX: HEISENBERG EQUATIONS OF MOTION FOR DENSITY OPERATORS

The full equations of motion for the expectation values of the density operators are

$$\frac{d}{dt}\rho_{gg}(k,k')=-\frac{i\hbar}{2m}(k^2-k'^2)\rho_{gg}(k,k') +g_1a_1^*\rho_{eg}(k-k_1,k')+g_2a_2^*\rho_{eg}(k-k_2,k') +g_1^*a_1\rho_{ge}(k,k'-k_1)+g_2^*a_2\rho_{eg}(k,k'-k_2), \tag{A1}$$

$$\frac{d}{dt}\rho_{eg}(k,k') = -i\left[\frac{\hbar}{2m}(k^2-k'^2)+\omega_0\right]\rho_{eg}(k,k') + g_1^* a_1[\rho_{ee}(k,k'-k_1)-\rho_{gg}(k+k_1,k')] + g_2^* a_2[\rho_{ee}(k,k'-k_2)-\rho_{gg}(k+k_2,k')], \tag{A2}$$

and

$$\frac{d}{dt}\rho_{ee}(k,k') = -\frac{i\hbar}{2m}(k^2-k'^2)\rho_{ee}(k,k') - g_1 a_1^* \rho_{eg}(k,k'+k_1) - g_2 a_2^* \rho_{eg}(k,k'+k_2) - g_1^* a_1 \rho_{ge}(k+k_1,k') - g_2^* a_2 \rho_{ge}(k+k_2,k'). \tag{A3}$$

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