Levinson's theorem for the Schrödinger equation in two dimensions

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Levinson's theorem for the Schrödinger equation with a cylindrically symmetric potential in two dimensions is reestablished by the Sturm-Liouville theorem. The critical case, where the Schrödinger equation has a finite zero-energy solution, is analyzed in detail. It is shown that, in comparison to Levinson's theorem in the noncritical case, the half bound state for the P wave, in which the wave function for the zero-energy solution does not decay fast enough at infinity to be square integrable, will cause the phase shift of the P wave at zero energy to increase an additional π. [S1050-2947(98)08908-2]

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I. INTRODUCTION

In 1949, an important theorem in quantum mechanics was established by Levinson [1], who set up a relation between the total number n_ℓ of bound states with angular momentum ℓ and the phase shift $\delta_\ell(0)$ of the scattering state at zero momentum for the Schrödinger equation with a spherically symmetric potential $V(r)$ in three dimensions:

$$\delta_\ell(0)-\delta_\ell(\infty)=\begin{cases}(n_\ell+1/2)\pi & \text{when } \ell=0 \\ \quad \text{and a half bound state occurs} & \\ n_\ell\pi & \text{the remaining cases,}\end{cases} \tag{1}$$

here the potential $V(r)$ satisfies the asymptotic conditions

$$r^2|V(r)|dr\to 0 \quad \text{at} \quad r\to 0, \tag{2a}$$

$$r^3|V(r)|dr\to 0 \quad \text{at} \quad r\to\infty. \tag{2b}$$

The first condition is necessary for the nice behavior of the wave function at the origin and the second one is necessary for the analytic property of the Jost function, which was used in his proof. The first line in Eq. (1) was shown by Newton [2] for the case where a half bound state of S wave occurs. A zero-energy solution to the Schrödinger equation is called a half bound state if its wave function is finite, but does not decay fast enough at infinity to be square integrable. As is well known, there is degeneracy of states for the magnetic quantum number due to the spherical symmetry. Usually, this degeneracy is not expressed explicitly in the statement of Levinson's theorem. Due to the wide interest in lower-dimensional field theories recently, it may be worthwhile to study Levinson's theorem in two dimensions. The purpose of the present paper is to reestablish the Levinson theorem for the Schrödinger equation in two dimensions in terms of the Sturm-Liouville theorem.

Many papers [2–10] have been devoted to the different proofs and generalizations of Levinson's theorem, for example, to noncentral potentials [2], nonlocal interactions [4], relativistic equations [7,10], and electron-atom scattering [9]. Roughly speaking, there are three main methods for the proof of Levinson's theorem. One [1] is based on an elaborate analysis of the Jost function. This method requires good behavior of the potential. For example, as pointed out by Newton [11], when the asymptotic condition (2b) is not satisfied, Levinson's theorem is violated. The second one is the Green's-function method [5], where the total number of physical states, which is infinite, is proved to be independent of the potential and the number of bound states is the difference between the infinite numbers of the scattering states without and with the potential. Since the number of states in a continuous spectrum is uncountable, a simple model is usually used to discretize the continuous part of the spectrum by requiring the wave functions to be vanishing at a sufficiently large radius. We recommend the third method to prove Levinson's theorem by the Sturm-Liouville theorem [6–8]. For the Sturm-Liouville problem, the fundamental trick is the definition of a phase angle that is monotonic with respect to the energy [12]. This method is very simple, intuitive, and easy to generalize. In this proof it is demonstrated explicitly that as the potential changes, the phase shift at zero momentum jumps by π while a scattering state becomes a bound state, or vice versa. Newton's counterexamples [11], where the condition (2b) is violated, can be proved to satisfy the modified Levinson theorem [6].

Recently, Lin [13] established a two-dimensional analog of Levinson's theorem for the Schrödinger equation with a cylindrically symmetric potential by the Green's-function

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method and declared that, unlike the case in three dimensions, the half bound state did not modify Levinson's theorem in two dimensions:

$$\eta_m(0)-\eta_m(\infty)=n_m\pi, \quad m=0,1,2,\ldots, \tag{3a}$$

where $\eta_m(0)$ is the limit of the phase shifts at zero momentum for the mth partial wave and n_m is the total number of bound states with the angular momentum $m\hbar$. Both η_m and n_m are independent of the sign of the angular momentum $\pm m\hbar$ so that only non-negative m needs to be discussed. The experimental study [14] of Levinson's theorem in two dimensions has appeared in the literature.

This form of Levinson's theorem for two dimensions [13] conflicts with an early result by Bollé, Gesztesy, Danneels, and Wilk (BGDW) [15] in 1986, who overcame the difficulty with the logarithmic singularity of the two-dimensional free Green's function at zero energy and proved ''surprisingly'' (see ''A surprise'' in the title of [15]) that the half bound state of P wave causes the phase shift $\eta_1(0)$ at zero momentum to increase an additional π, exactly like the zero-energy bound states:

$$\eta_m(0)-\eta_m(\infty)=(n_m+1)\pi$$

$$\text{when} \quad m=1 \quad \text{and a half bound state occurs.} \tag{3b}$$

The critical case where the Schrödinger equation has a finite zero-energy solution is very sensitive and worthy of some careful analysis, especially when two conflicting versions of Levinson's theorem in two dimensions were presented. The Sturm-Liouville theorem provides a powerful tool for this analysis. In the present paper we reestablish the Levinson theorem for the Schrödinger equation in two dimensions by the Sturm-Liouville theorem, which coincides with the version by BGDW [15]. It seems to us that the problem in the proof by Lin [13] may be whether or not the set of physical solutions to the Schrödinger equation in two dimensions is complete when a half bound state of P wave occurs because the corresponding wave function for the half bound state tends to zero at infinity, although it does not decay fast enough at infinity to be square integrable. It is different for the S wave because the wave function of the half bound state of the S wave is finite but does not tend to zero at infinity.

This paper is organized as follows. We first assume that the potential vanishes beyond a sufficiently large radius r_0 for simplicity and leave the discussion of the general potentials for Sec. V. In Sec. II we choose the logarithmic derivative of the radial wave function of the Schrödinger equation as the ''phase angle'' [12] and prove by the Sturm-Liouville theorem that it is monotonic with respect to the energy. In terms of this monotonic property, in Sec. III the number of bound states is proved to be related to the logarithmic derivative of zero energy at r_0 as the potential changes. In Sec. IV we further prove that the logarithmic derivative of zero energy at r_0 also determines the limit of the phase shifts at zero momentum, so that Levinson's theorem is proved. The critical case, where a zero-energy solution occurs, is analyzed carefully there. The problem that the potential has a tail at infinity will be discussed in Sec. V.

II. NOTATION AND THE STURM-LIOUVILLE THEOREM

Consider the Schrödinger equation with a potential $V(r)$ that depends only on the distance r from the origin

$$H\psi=-\frac{\hbar^2}{2\mu}\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}\right)\psi+V(r)\psi=E\psi,$$

where μ denotes the mass of the particle. For simplicity, we first discuss the case with a cutoff potential

$$V(r)=0 \quad \text{when} \quad r\geqslant r_0, \tag{4}$$

where r_0 is a sufficiently large radius. The general case where the potential $V(r)$ has a tail at infinity will be discussed in Sec. V.

We introduce a parameter λ for the potential $V(r)$:

$$V(r,\lambda)=\lambda V(r). \tag{5}$$

As λ increases from zero to one, the potential $V(r,\lambda)$ changes from zero to the given potential $V(r)$. Owing to the symmetry of the potential, we have

$$\psi(r,\varphi,\lambda)=r^{-1/2}R_m(r,\lambda)e^{\pm im\varphi}, \quad m=0,1,2,\ldots, \tag{6}$$

where the radial wave function $R_m(r,\lambda)$ satisfies the radial equation

$$\frac{\partial^2R_m(r,\lambda)}{\partial r^2}+\left\{\frac{2\mu}{\hbar^2}[E-V(r,\lambda)]-\frac{m^2-1/4}{r^2}\right\}R_m(r,\lambda)=0. \tag{7}$$

Now we are going to solve Eq. (7) in two regions and match two solutions at r_0. Since the Schrödinger equation is linear, the wave function ψ can be multiplied by a constant factor. Removing the effect of the factor, we only need one matching condition at r_0 for the logarithmic derivative of the radial function:

$$A_m(E,\lambda)\equiv\left\{\frac{1}{R_m(r,\lambda)}\frac{\partial R_m(r,\lambda)}{\partial r}\right\}_{r=r_0-}$$

$$=\left\{\frac{1}{R_m(r,\lambda)}\frac{\partial R_m(r,\lambda)}{\partial r}\right\}_{r=r_0+}. \tag{8}$$

Due to the condition (2a), only one solution is convergent at the origin. For example, for the free particle ($\lambda=0$), the solution to Eq. (7) at the region $0\leqslant r\leqslant r_0$ is proportional to the Bessel function $J_m(x)$:

$$R_m(r,0)=\begin{cases}\sqrt{\dfrac{\pi kr}{2}}J_m(kr) & \text{when} \quad E>0,\ k=(2\mu E)^{1/2}/\hbar \\ e^{-im\pi/2}\sqrt{\dfrac{\pi\kappa r}{2}}J_m(i\kappa r) & \text{when} \quad E\leqslant 0,\ \kappa=(-2\mu E)^{1/2}/\hbar.\end{cases} \tag{9}$$

The solution $R_m(r,0)$ given in Eq. (9) is a real function. A constant factor on the radial function $R_m(r,0)$ is not important.

In the region $r_0\leqslant r<\infty$, we have $V(r)=0$. For $E>0$, there are two oscillatory solutions to Eq. (7). Their combination can always satisfy the matching condition (8), so that there is a continuous spectrum for $E>0$:

$$\begin{aligned}R_m(r,\lambda)&=\sqrt{\frac{\pi kr}{2}}\{\cos\eta_m(k,\lambda)J_m(kr)\\&\quad-\sin\eta_m(k,\lambda)N_m(kr)\}\\&\sim\cos\left(kr-\frac{m\pi}{2}-\frac{\pi}{4}+\eta_m(k,\lambda)\right)\quad\text{when}\quad r\to\infty,\end{aligned} \tag{10}$$

where $N_m(kr)$ is the Neumann function. From the matching condition (8) we have

$$\tan\eta_m(k,\lambda)=\frac{J_m(kr_0)}{N_m(kr_0)}\,\frac{A_m(E,\lambda)-kJ_m'(kr_0)/J_m(kr_0)-1/(2r_0)}{A_m(E,\lambda)-kN_m'(kr_0)/N_m(kr_0)-1/(2r_0)}. \tag{11}$$

$$\eta_m(k)\equiv\eta_m(k,1), \tag{12}$$

where the prime denotes the derivative of the Bessel function, the Neumann function, and later the Hankel function with respect to their argument.

The phase shift $\eta_m(k,\lambda)$ is determined from Eq. (11) up to a multiple of π due to the period of the tangent function. Levinson determined the phase shift $\eta_m(k)$ with respect to the phase shift $\eta_m(\infty)$ at infinite momentum. For any finite potential, the phase shift $\eta_m(\infty)$ will not change and is always equal to the phase shift of zero potential. Therefore, Levinson's definition for the phase shift is equivalent to the convention that the phase shift $\eta_m(k)$ is determined with respect to the phase shift $\eta_m(k,0)$ for the free particle, where $\eta_m(k,0)$ is defined to be zero:

$$\eta_m(k,0)=0 \quad \text{where} \quad V(r,0)=0. \tag{13}$$

We prefer to use this convention where the phase shift $\eta_m(k,\lambda)$ is determined completely as λ increases from zero to one. It is the reason why we introduce the parameter λ.

Since there is only one convergent solution at infinity for $E\leqslant 0$ the matching condition (8) is not always satisfied:

$$\begin{aligned}R_m(r,\lambda)&=e^{i(m+1)\pi/2}\sqrt{\frac{\pi\kappa r}{2}}H_m^{(1)}(i\kappa r)\\&\sim e^{-\kappa r}\quad\text{when}\quad r\to\infty,\end{aligned} \tag{14}$$

where $H_m^{(1)}(x)$ is the Hankel function of the first kind. When the condition (8) is satisfied, a bound state appears at this energy. It means that there is a discrete spectrum for $E\leqslant 0$.

Now we turn to the Sturm-Liouville theorem. Denote by $\overline{R}_m(r,\lambda)$ the solution to Eq. (7) for the energy $\overline{E}$,

$$\frac{\partial^2}{\partial r^2}\overline{R}_m(r,\lambda)+\left\{\frac{2\mu}{\hbar^2}[\overline{E}-V(r,\lambda)]-\frac{m^2-1/4}{r^2}\right\}\overline{R}_m(r,\lambda)=0. \tag{15}$$

Multiplying Eqs. (7) and (15) by $\overline{R}_m(r,\lambda)$ and $R_m(r,\lambda)$, respectively, and calculating their difference, we have

$$\begin{aligned}&\frac{\partial}{\partial r}\left\{R_m(r,\lambda)\frac{\partial\overline{R}_m(r,\lambda)}{\partial r}-\overline{R}_m(r,\lambda)\frac{\partial R_m(r,\lambda)}{\partial r}\right\}\\&\quad=-\frac{2\mu}{\hbar^2}(\overline{E}-E)\overline{R}_m(r,\lambda)R_m(r,\lambda).\end{aligned} \tag{16}$$

According to the boundary condition, both solutions $R_m(r,\lambda)$ and $\overline{R}_m(r,\lambda)$ should be vanishing at the origin. Integrating Eq. (16) in the region from 0 to r_0, we have

$$\begin{aligned}&\frac{1}{\overline{E}-E}\left\{R_m(r,\lambda)\frac{\partial\overline{R}_m(r,\lambda)}{\partial r}-\overline{R}_m(r)\frac{\partial R_m(r,\lambda)}{\partial r}\right\}_{r=r_0-}\\&\quad=-\frac{2\mu}{\hbar^2}\int_0^{r_0}\overline{R}_m(r,\lambda)R_m(r,\lambda)dr.\end{aligned}$$

Taking the limit, we obtain

$$\begin{aligned}\frac{\partial A_m(E,\lambda)}{\partial E}&=\frac{\partial}{\partial E}\left(\frac{1}{R_m(r,\lambda)}\frac{\partial R_m(r,\lambda)}{\partial r}\right)_{r=r_0-}\\&=-\frac{2\mu}{\hbar^2}R_m(r_0,\lambda)^{-2}\int_0^{r_0}R_m(r,\lambda)^2dr<0.\end{aligned} \tag{17}$$

Similarly, from the boundary condition that when $E\leqslant 0$ the radial function $R_m(r,\lambda)$ tends to zero at infinity, we have

$$\frac{\partial}{\partial E}\left(\frac{1}{R_m(r,\lambda)}\frac{\partial R_m(r,\lambda)}{\partial r}\right)_{r=r_0+}$$

$$=\frac{2\mu}{\hbar^2}R_m(r_0,\lambda)^{-2}\int_{r_0}^{\infty}R_m(r,\lambda)^2dr>0. \qquad (18)$$

Therefore, when $E\leqslant 0$, both sides of Eq. (8) are monotonic with respect to the energy E: As energy increases, the logarithmic derivative of the radial function at r_0- decreases monotonically, but that at r_0+ increases monotonically. This is the essence of the Sturm-Liouville theorem.

III. THE NUMBER OF BOUND STATES

In this section we will relate the number of bound states with the logarithmic derivative $A_m(0,\lambda)$ of the radial function at r_0- for zero energy when the potential changes, in terms of the monotonic property of the logarithmic derivative of the radial function with respect to the energy E. From Eq. (14) we have

$$\left(\frac{1}{R_m(r,\lambda)}\frac{\partial R_m(r,\lambda)}{\partial r}\right)_{r=r_0+}$$

$$=\frac{i\kappa H_m^{(1)}(i\kappa r_0)'}{H_m^{(1)}(i\kappa r_0)}-\frac{1}{2r_0}$$

$$=\begin{cases}(-m+1/2)/r_0 & \text{when} \quad E\sim 0\\ -\kappa\sim-\infty & \text{when} \quad E\to-\infty.\end{cases} \qquad (19)$$

The logarithmic derivative given in Eq. (19) does not depend on λ. On the other hand, when $\lambda=0$ we obtain from Eq. (10)

$$A_m(E,0)=\left(\frac{1}{R_m(r,0)}\frac{\partial R_m(r,0)}{\partial r}\right)_{r=r_0-}=\frac{i\kappa J_m'(i\kappa r_0)}{J_m(i\kappa r_0)}-\frac{1}{2r_0}$$

$$=\begin{cases}(m+1/2)/r_0 & \text{when} \quad E\sim 0\\ \kappa\sim\infty & \text{when} \quad E\to-\infty.\end{cases} \qquad (20)$$

It is evident from Eqs. (19) and (20) that as the energy increases from $-\infty$ to 0, there is no overlap between two variant ranges of two logarithmic derivatives such that there is no bound state when $\lambda=0$ except for the S wave where there is a half bound state at $E=0$. The half bound state for the S wave will be discussed in Sec. IV.

If $A_m(0,\lambda)$ decreases across the value $(-m+1/2)/r_0$ as λ changes, an overlap between the variant ranges of two logarithmic derivatives of two sides of r_0 appears. Since the logarithmic derivative of the radial function at r_0- decreases monotonically as the energy increases and that at r_0+ increases monotonically, the overlap means that there must be one and only one energy where the matching condition (8) is satisfied, namely, a bound state appears. From the viewpoint of node theory, when $A_m(0,\lambda)$ decreases across the value $(-m+1/2)/r_0$, a node for the zero-energy solution to the Schrödinger equation moves inward from infinity, namely, a scattering state changes to a bound state.

As λ changes, $A_m(0,\lambda)$ decreases to $-\infty$, jumps to ∞, and then decreases again across the value $(-m+1/2)/r_0$, so that another overlap occurs and another bound state appears. Note that when the zero point in the zero-energy solution $R_m(r,\lambda)$ goes to r_0, $A_m(0,\lambda)$ goes to infinity. It is not a singularity.

Each time $A_m(0,\lambda)$ decreases across the value $(-m+1/2)/r_0$, a new overlap between the variant ranges of two logarithmic derivatives appears such that a scattering state changes to a bound state. At the same time, a new node moves inward from infinity in the zero-energy solution to the Schrödinger equation. Conversely, each time $A_m(0,\lambda)$ increases across the value $(-m+1/2)/r_0$, an overlap between those two variant ranges disappears such that a bound state changes back to a scattering state and, simultaneously, a node goes outward and disappears in the zero-energy solution. The number of bound states n_m is equal to the times that $A_m(0,\lambda)$ decreases across the value $(-m+1/2)/r_0$ as λ increases from zero to one minus the times that $A_m(0,\lambda)$ increases across the value $(-m+1/2)/r_0$. It is also equal to the number of nodes in the zero-energy solution. In the next section we will show that this number is nothing but the phase shift $\eta_m(0)$ at zero momentum divided by π.

IV. LEVINSON'S THEOREM

In order to determine the phase shift $\eta_m(k)$ completely, we have introduced the convention for the phase shift $\eta_m(k,\lambda)$, where $k>0$, which changes continuously as λ increases from zero to one and $\eta_m(k,0)$ is defined to be vanishing. The phase shift $\eta_m(k,\lambda)$ is calculated by Eq. (11). It is easy to see from Eq. (11) that the phase shift $\eta_m(k,\lambda)$ increases monotonically as the logarithmic derivative $A_m(E,\lambda)$ decreases:

$$\left.\frac{\partial\eta_m(k,\lambda)}{\partial A_m(E,\lambda)}\right|_k=\frac{-8r_0\cos^2\eta_m(k,\lambda)}{\pi\{2r_0A_m(E,\lambda)N_m(kr_0)-2kr_0N_m'(kr_0)-N_m(kr_0)\}^2}\leqslant 0, \qquad (21)$$

where $k=(2\mu E)^{1/2}/\hbar$.

The phase shift $\eta_m(0,\lambda)$ is the limit of the phase shift $\eta_m(k,\lambda)$ as k tends to zero. Therefore, we are interested in the phase shift $\eta_m(k,\lambda)$ at a sufficiently small momentum k, $k\ll 1/r_0$. For the small momentum we obtain from Eq. (11)

$$\tan\eta_m(k,\lambda)=\begin{cases}\dfrac{-\pi(kr_0)^{2m}}{2^{2m}m!(m-1)!}\dfrac{A_m(0,\lambda)-(m+1/2)/r_0}{A_m(0,\lambda)-c^2k^2-\dfrac{-m+1/2}{r_0}\left(1-\dfrac{(kr_0)^2}{(m-1)(2m-1)}\right)} & \text{when} \quad m\geqslant 2\\ \dfrac{-\pi(kr_0)^2}{4}\dfrac{A_m(0,\lambda)-3/(2r_0)}{A_m(0,\lambda)-c^2k^2+\dfrac{1}{2r_0}[1+2(kr_0)^2\ln(kr_0)]} & \text{when} \quad m=1\\ \dfrac{\pi}{2\ln(kr_0)}\dfrac{A_m(0,\lambda)-c^2k^2-\dfrac{1}{2r_0}[1-(kr_0)^2]}{A_m(0,\lambda)-c^2k^2-\dfrac{1}{2r_0}\left(1+\dfrac{2}{\ln(kr_0)}\right)} & \text{when} \quad m=0.\end{cases} \tag{22}$$

where the expansion for $A_m(E,\lambda)$, calculated from Eq. (17), is used:

$$A_m(E,\lambda)=A_m(0,\lambda)-c^2k^2+\cdots, \quad c^2>0, \quad E=\frac{\hbar^2k^2}{2\mu}. \tag{23}$$

In addition to the leading terms, we include in Eq. (22) some next leading terms, which are useful only for the critical case where the leading terms cancel each other.

First of all, it can be seen from Eq. (22) that $\tan\eta_m(k,\lambda)$ tends to zero as k goes to zero, namely, $\eta_m(0,\lambda)$ is always equal to a multiple of π. In other words, if the phase shift $\eta_m(k,\lambda)$ for a sufficiently small k is expressed as a positive or negative acute angle plus $n\pi$, its limit $\eta_m(0,\lambda)$ is equal to $n\pi$, where n is an integer. This means that $\eta_m(0,\lambda)$ changes discontinuously. In addition, in three dimensions, the tangent of the phase shift may go to infinity for the critical case of S wave.

Second, if $A_m(E,\lambda)$ decreases as λ increases, $\eta_m(k,\lambda)$ increases monotonically. As $A_m(E,\lambda)$ decreases, each time $\tan\eta_m(k,\lambda)$ for a sufficiently small k changes sign from positive to negative (through a jump from positive infinity to negative infinity), $\eta_m(0,\lambda)$ jumps by π. However, each time $\tan\eta_m(k,\lambda)$ changes sign from negative to positive, $\eta_m(0,\lambda)$ remains invariant. Conversely, if $A_m(E,\lambda)$ increases as λ increases, $\eta_m(k,\lambda)$ decreases monotonically. As $A_m(E,\lambda)$ increases, each time $\tan\eta_m(k,\lambda)$ changes sign from negative to positive, $\eta_m(0,\lambda)$ jumps by $-\pi$, and each time $\tan\eta_m(k,\lambda)$ changes sign from positive to negative, $\eta_m(0,\lambda)$ remains invariant.

When $V(r,\lambda)$ changes from zero to the given potential $V(r)$ continuously, each time the $A_m(0,\lambda)$ decreases from near and larger than the value $(-m+1/2)/r_0$ to smaller than that value, the denominator in Eq. (22) changes sign from positive to negative and the remaining factor remains positive such that the phase shift at zero momentum $\eta_m(0,\lambda)$ jumps by π. Conversely, each time $A_m(0,\lambda)$ increases across that value, the phase shift at zero momentum $\eta_m(0,\lambda)$ jumps by $-\pi$. Note that when $A_m(0,\lambda)$ decreases from near and larger than the value $(m+1/2)/r_0$ to smaller than that value, the numerator in Eq. (22) changes sign from positive to negative and the remaining factor remains negative such that the phase shift at zero momentum $\eta_m(0,\lambda)$ does not jump. Conversely, when $A_m(0,\lambda)$ increases across the value $(m+1/2)/r_0$, the phase shift at zero momentum $\eta_m(0,\lambda)$ also remains invariant. This is the reason why we did not include the next leading terms in the numerator of Eq. (22) except for $m=0$.

Therefore, the phase shift $\eta_m(0)/\pi$ is just equal to the times $A_m(0,\lambda)$ decreases across the value $(-m+1/2)/r_0$ as λ increases from zero to one minus the times $A_m(0,\lambda)$ increases across that value. In Sec. III we have proved that the difference of the two times is nothing but the number of bound states n_m, namely, we proved the Levinson theorem for the Schrödinger equation in two dimensions for noncritical cases:

$$\eta_m(0)=n_m\pi. \tag{24}$$

We should pay some attention to the case of $m=0$. When $A_m(0)$ decreases across the value $1/2r_0$, both the numerator and denominator in Eq. (22) change signs, but not simultaneously because the next leading terms in the numerator and denominator of Eq. (22) are different. It is easy to see that the numerator changes sign first and then the denominator changes sign, namely, $\tan\eta_m(k)$ at small k changes first from negative to positive and then to negative again so that $\eta_m(0)$ jumps by π. Similarly, when $A_m(0)$ increases across the value $1/2r_0$, $\eta_m(0)$ jumps by $-\pi$.

For $\lambda=0[V(r,0)=0]$ and $m=0$, the numerator in Eq. (22) is equal to zero, the denominator is positive, and the phase shift $\eta_0(0)$ is defined to be zero. If $A_0(E)$ decreases as λ increases from zero, the numerator becomes negative first and then the denominator changes sign from positive to negative such that the phase shift $\eta_0(0,\lambda)$ jumps by π and simultaneously a bound state appears. If $A_0(E)$ increases as λ increases from zero, the numerator becomes positive and the remaining factor remains negative such that the phase shift $\eta_0(0,\lambda)$ is zero and no bound state appears.

Now we turn to discuss the critical case where the logarithmic derivative $A_m(0,1)(\lambda=1)$ is equal to the value $(-m+1/2)/r_0$. In the critical case, the following solution with zero energy in the region $r_0\leqslant r<\infty$ will match this $A_m(0,1)$ at r_0:

$$R_m(r)=r^{-m+1/2}. \tag{25}$$

It is a bound state when $m \geqslant 2$, but a half bound state when $m=1$ and 0. A half bound state is not a bound state because its wave function is finite but not square integrable. We are going to discuss the critical case where $A_m(0,\lambda)$ decreases (or increases) and reaches, but does not cross, the value $(-m+1/2)/r_0$ as $V(r,\lambda)$ changes from zero to the given potential $V(r)$. For definiteness, we discuss the case where $A_m(0,\lambda)$ decreases and reaches the value $(-m+1/2)/r_0$. In this case a new bound state with zero energy appears for $m \geqslant 2$, but not for $m=1$ and 0. We should check whether or not the phase shift $\eta_m(0)$ increases an additional π.

It is easy to see from the next leading terms in the denominator of Eq. (22) that the denominator for $m \geqslant 2$ has changed sign from positive to negative as $A_m(0,\lambda)$ decreases and reaches the value $(-m+1/2)/r_0$, namely, the phase shift $\eta_m(0)$ jumps by π and simultaneously a new bound state of zero-energy appears.

For $m=0$ the next leading term with $\ln(kr_0)$ in the denominator of Eq. (22) is positive and larger than the term $-c^2k^2$, such that the denominator does not change sign, namely, the phase shift $\eta_m(0)$ does not jump. This agrees with the fact that no new bound state appears.

For $m=1$ the next leading term in the denominator of Eq. (22) is negative such that the denominator does change sign as $A_m(0,\lambda)$ decreases and reaches the value $-1/2r_0$, namely, the phase shift $\eta_m(0)$ jumps by π. However, in this case no new bound state appears simultaneously.

The discussion for the cases where $A_m(0,\lambda)$ increases and reaches the value $(-m+1/2)/r_0$ is similar. Therefore, Levinson's theorem (24′) holds for the critical cases except for $m=1$. In the latter case, Levinson's theorem for the Schrödinger equation in two dimensions becomes

$$\eta_m(0)=(n_m+1)\pi$$

$$\text{when} \quad m=1 \quad \text{and a half bound state occurs.} \qquad (24')$$

Equations (24) and (24′) are the same as Eq. (3) because in our convention $\eta_m(\infty)=0$.

V. DISCUSSION

Now we discuss the general case where the potential $V(r)$ has a tail at $r \geqslant r_0$. Let r_0 be so large that only the leading term in $V(r)$ is in the region $r \geqslant r_0$:

$$V(r) \sim \frac{\hbar^2}{2\mu} b r^{-n} \quad \text{when} \quad r \to \infty, \qquad (26)$$

where b is a nonvanishing constant and n is a positive constant, not necessarily an integer. From the condition (2b), n should be larger than 3. Substituting Eq. (26) into Eq. (7) and changing the variable r to ξ,

$$\xi = \begin{cases} kr = r\sqrt{2\mu E}/\hbar & \text{when} \quad E>0 \\ \kappa r = r\sqrt{-2\mu E}/\hbar & \text{when} \quad E \leqslant 0, \end{cases} \qquad (27)$$

we get the radial equation at the region $r_0 \leqslant r < \infty$,

$$\frac{d^2R_m(\xi,\lambda)}{d\xi^2} + \left\{ 1 - \frac{b}{\xi^n} k^{n-2} - \frac{m^2-1/4}{\xi^2} \right\} \times R_m(\xi,\lambda) = 0 \quad \text{when} \quad E>0,$$

$$\frac{d^2R_m(\xi,\lambda)}{d\xi^2} + \left\{ -1 - \frac{b}{\xi^n} \kappa^{n-2} - \frac{m^2-1/4}{\xi^2} \right\} \times R_m(\xi,\lambda) = 0 \quad \text{when} \quad E \leqslant 0, \qquad (28)$$

where $R_m(\xi,\lambda)$ depends on λ through the matching condition (8).

As far as Levinson's theorem is concerned, we are interested only in the solutions with the sufficiently small k and κ. If $n \geqslant 3$, in comparison to the term of the centrifugal potential, the term with a factor k^{n-2} (or κ^{n-2}) is too small to affect the phase shift at a sufficiently small k and the variant range of the logarithmic derivative $[dR_m(r)/dr]/R_m(r)$ at r_0+. Therefore, the proof given in the previous sections is effective for those potentials with a tail so that Levinson's theorem (24) and (24′) and holds.

When $n=2$, we define

$$\nu^2 = m^2 + b. \qquad (29)$$

The radial equation (7) becomes

$$\frac{\partial^2 R_m(r,\lambda)}{\partial r^2} + \left\{ \frac{2\mu E}{\hbar^2} - \frac{\nu^2-1/4}{r^2} \right\} R_m(r,\lambda) = 0, \quad r \geqslant r_0. \qquad (30)$$

If $\nu^2<0$, there are an infinite number of bound states. We will not discuss this case nor the case with $\nu=0$ here. When $\nu^2>0$, we take $\nu>0$. Some formulas given in the previous sections will be changed by replacing the angular quantum number m with ν. Equation (19) becomes

$$\left(\frac{1}{R_m(r,\lambda)} \frac{\partial R_m(r,\lambda)}{\partial r} \right)_{r=r_0+} = \frac{i\kappa H_\nu^{(1)}(i\kappa r_0)'}{H_\nu^{(1)}(i\kappa r_0)} - \frac{1}{2r_0} = \begin{cases} (-\nu+1/2)/r_0 & \text{when} \quad E \sim 0 \\ -\kappa \sim -\infty & \text{when} \quad E \to -\infty. \end{cases} \qquad (31)$$

The scattering solution (10) in the region $r_0 \leqslant r < \infty$ becomes

$$R_m(r,\lambda) = \sqrt{\frac{\pi k r}{2}} \{ \cos\delta_\nu(k,\lambda) J_\nu(kr) - \sin\delta_\nu(k,\lambda) N_\nu(kr) \} \sim \cos\left(kr - \frac{\nu\pi}{2} - \frac{\pi}{4} + \delta_\nu(k,\lambda) \right), \quad \text{when} \quad r \to \infty. \qquad (32)$$

Thus the phase shift $\eta_m(k)$ can be calculated from $\delta_\nu(k,1)$,

$$\eta_m(k) = \delta_\nu(k,1) + (m-\nu)\pi/2. \qquad (33)$$

$\delta_\nu(k,\lambda)$ satisfies

$$\tan\delta_\nu(k,\lambda)=\frac{J_\nu(kr_0)}{N_\nu(kr_0)}\,\frac{A_m(E,\lambda)-kJ'_\nu(kr_0)/J_\nu(kr_0)-1/(2r_0)}{A_m(E,\lambda)-kN'_\nu(kr_0)/N_\nu(kr_0)-1/(2r_0)} \tag{34}$$

and it increases monotonically as the logarithmic derivative $A_m(E,\lambda)$ decreases:

$$\left.\frac{\partial\delta_\nu(k,\lambda)}{\partial A_m(E,\lambda)}\right|_k=\frac{-8r_0\cos^2\delta_\nu(k,\lambda)}{\pi\{2r_0A_m(E,\lambda)N_\nu(kr_0)-2kr_0N'_\nu(kr_0)-N_\nu(kr_0)\}^2}\leqslant 0. \tag{35}$$

For a sufficiently small k we have

$$\tan\delta_\nu(k,\lambda)=\begin{cases}\dfrac{-\pi(kr_0)^{2\nu}}{2^{2\nu}\nu!(\nu-1)!}\,\dfrac{A_m(0,\lambda)-(\nu+1/2)/r_0}{A_m(0,\lambda)-c^2k^2-\dfrac{-\nu+1/2}{r_0}\left(1-\dfrac{(kr_0)^2}{(\nu-1)(2\nu-1)}\right)} & \text{when}\quad \nu>1\\[2ex] \dfrac{-\pi}{\nu\Gamma(\nu)^2}\left(\dfrac{kr_0}{2}\right)^{2\nu}\dfrac{A_m(0,\lambda)-(\nu+1/2)/r_0}{A_m(0,\lambda)-c^2k^2-\dfrac{-\nu+1/2}{r_0}+\dfrac{2\pi\cot(\nu\pi)}{r_0\Gamma(\nu)^2}\left(\dfrac{kr_0}{2}\right)^{2\nu}} & \text{when}\quad 0<\nu<1.\end{cases} \tag{36}$$

The asymptotic forms for the case $\nu=1$ have already been given in Eq. (22).

Now, repeating the proof for Levinson's theorem (24) and (24′), we obtain the modified Levinson theorem for the noncritical cases:

$$\eta_m(0)-(m-\nu)\pi/2=\delta_\nu(0,1)=n_m\pi. \tag{37}$$

For the critical case where $A_m(0,1)=(-\nu+1/2)/r_0$, the modified Levinson theorem (37) holds for $\nu>1$, where a new bound state appears and simultaneously $\eta_m(k)$ jumps by π, but the modified Levinson theorem (37) is violated for $0<\nu\leqslant 1$, where a half bound state appears and simultaneously $\eta_m(k)$ jumps by $\nu\pi$. In other words, the theorem needs to be modified further in these cases.

From the above discussion, we come to the conclusion that for the potential with a tail (26) at the infinity, when $n\leqslant 2$ Levinson's theorem (24) is violated and when $n>2$, even if it contains a logarithmic factor, Levinson's theorem (24) holds because in the latter case, for any arbitrarily given small ϵ, one can always find a sufficiently large r_0 such that $|V(r)|<\epsilon/r^2$ in the region $r_0<r<\infty$. Since $\nu^2=m^2+\epsilon\sim m^2$, Levinson's theorem (24) and (24′) holds for this case.

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