Dynamical suppression of decoherence in two-state quantum systems

Lorenza Viola* and Seth Lloyd†

*d'Arbeloff Laboratory for Information Systems and Technology, Department of Mechanical Engineering,
Massachusetts Institute of Technology, Cambridge, Massachusetts 02139*

The dynamics of a decohering two-level system driven by a suitable control Hamiltonian is studied. The control procedure is implemented as a sequence of radio-frequency pulses that repetitively flip the state of the system, a technique that can be termed quantum ''bang-bang'' control after its classical analog. Decoherence introduced by the system's interaction with a quantum environment is shown to be washed out completely in the limit of continuous flipping and greatly suppressed provided the interval between the pulses is made comparable to the correlation time of the environment. The model suggests a strategy to fight against decoherence that complements existing quantum error-correction techniques. [S1050-2947(98)07109-1]

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I. INTRODUCTION

The design of strategies able to protect the evolution of a quantum system against irreversible corruption due to environmental noise represents a challenging conceptual issue. In particular, since maintaining quantum coherence is a crucial requirement for exploiting the novel possibilities opened up by quantum parallelism, practical implementations of quantum computation and communication proposals require methods to effectively resist the action of quantum decoherence and dissipation [1–3]. Roughly speaking, two classes of procedures are available to overcome the decoherence problem: either passive stabilization or active manipulation of the quantum state. The first kind of solutions, recently formalized as *error-avoiding codes* [4], relies on the existence of a subspace of states that, owing to special symmetry properties, are dynamically decoupled from the environment. The second approach, pioneered in [5] and closer in spirit to quantum control theory [6,7], embraces today a variety of sophisticated schemes known as *error-correcting codes* [8–18]. Basically, loss of information is corrected by monitoring the system and conditionally carrying on suitable feedback operations.

In this work we investigate a third strategy for reducing noise and decoherence. This strategy, which can be termed quantum ''bang-bang'' control after its classical analog [19], works by averaging out the unwanted effects of the environmental interaction through the application of suitable openloop control techniques on the system. The basic idea is that open-system properties, specifically decoherence, may be modified if a time-varying control field acts on the dynamics of the system over time scales that are comparable to the memory time of the environment. In particular, we work out an exact model for a two-state quantum system [quantum bit (qubit)] coupled to a thermal bath of harmonic oscillators, where decoherence is dynamically suppressed through repeated effective time-reversal operations on the combined system plus bath. Although the phenomenon is mathemati-

cally reminiscent of the quantum Zeno effect [20], the essential physical idea comes from refocusing techniques in nuclear magnetic resonance spectroscopy (NMR) [21]. Since the discovery of spin echoes in 1950 [22], clever pulse methods have been developed in NMR to eliminate much of the dephasing arising from variations in the local magnetic field acting on each spin. Because the latter effect can be thought of in terms of an interaction with some classical environment, it is not obvious *a priori* whether similar techniques work in the presence of a quantum mechanical environment and purely nonclassical effects like entanglement. Our result answers this question in the affirmative, and points out the role of the reservoir correlation time as a further parameter to be engineered in the struggle for preserving quantum coherence.

The plan of the paper is the following. In Sec. II a general model of a two-state system interacting with a thermal environment is reviewed and its decoherence properties in the absence of control recalled. In Sec. III the evolution under the action of a sequence of perturbative kicks is analyzed. Complete quenching of decoherence is established as a limiting situation. In Sec. IV the conditions for an effective decoherence reduction are clarified with reference to a variety of possible environmental configurations and the method is compared to different quantum error-correction techniques. We close by discussing possibilities for future work.

II. SINGLE-QUBIT DEPHASING MECHANISM

Our goal is to investigate how decoherence properties of an open quantum system may be modified through the application of an external controllable interaction. Decoherence is a process whereby quantum systems lose their ability to exhibit coherent behavior such as interference [23–26]. We start by introducing a model that allows investigation of the problem in its simplest nontrivial configuration. The physical system we are interested in is a single two-state quantum system, representing the elementary memory cell of quantum information. Although not strictly necessary, it will be convenient to think of the physical qubit as realized by a spin-1/2 system, which will provide us with direct reference to the well established language of nuclear magnetic resonance [21] and the rapidly growing field of NMR quantum compu-

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*Electronic address: vlorenza@mit.edu
†Electronic address: slloyd@mit.edu

tation [27,28]. Decoherence arises due to the coupling to a quantized environment, here schematized as a continuum of harmonic modes. We assume that the dynamics of the overall qubit plus bath is ruled by the following Hamiltonian:

$$H_0 = H_S + H_B + H_{SB} = \hbar \omega_0 \frac{\sigma_z}{2} + \sum_k \hbar \omega_k b_k^\dagger b_k$$

$$+ \sum_k \hbar \sigma_z (g_k b_k^\dagger + g_k^* b_k), \qquad (1)$$

where the first and second contribution H_S and H_B describe, respectively, the free evolution of the qubit and the environment, and the third term H_{SB} describes a bilinear interaction between the two. σ_z is the standard diagonal Pauli matrix, with qubit basis states denoted as $|i\rangle$, $i=0,1$, while $b_k^\dagger, b_k$ are bosonic operators for the kth field mode, characterized by a generally complex coupling parameter g_k. In the Schrödinger picture, the state of the combined system $(S+B)$ is represented by a density operator $\rho_{\text{tot}}(t)$ and the reduced qubit dynamics is thereupon recovered from a partial trace over the environment degrees of freedom:

$$\rho_S(t) = \sum_{i,j=0,1} \rho_{ij}(t) |i\rangle\langle j| = \text{Tr}_B\{\rho_{\text{tot}}(t)\}. \qquad (2)$$

Hamiltonian (1), which corresponds to a special case of the so-called spin-boson problem [29], has been used by many authors to model decoherence in quantum computers [2–4]. In particular, we adhere closely to the notations of [3]. The basic fact about the dynamics induced by Eq. (1) is that, since $[\sigma_z, H_0] = 0$, the interaction with the environment has the two memory states $|0\rangle, |1\rangle$ as eigenstates. In other words, the model describes a purely decohering mechanism, where no energy exchange between qubit and bath is present. In NMR terminology, this implies that no T_1 type of decay takes place [21]. Equivalently, in terms of errors, only phase errors are introduced. However, neglecting the effects associated to quantum dissipation is justified, in a sense, by two related reasons: energy exchange processes not only produce amplitude errors which need to be corrected even in the classical computation, but they typically involve time scales much longer than decoherence mechanisms. In addition, being exactly soluble, the model (1) has the advantage of allowing a clear picture of the decoherence properties in the absence of control. To this end, since spin populations are not affected by the environment, the relevant quantity is the qubit coherence $\rho_{01}(t)$ [of course, $\rho_{10}(t) = \rho_{01}^*(t)$].

It will be convenient to move to the interaction picture associated to the free dynamics $(H_S + H_B)$, corresponding to the transformed state vector

$$\tilde{\rho}_{\text{tot}}(t) = e^{i(H_S+H_B)t/\hbar} \rho_{\text{tot}}(t) e^{-i(H_S+H_B)t/\hbar} \qquad (3)$$

and to the effective Hamiltonian

$$\tilde{H}(t) = \tilde{H}_{SB}(t) = \hbar \sigma_z \sum_k (g_k b_k^\dagger e^{i\omega_k t} + g_k^* b_k e^{-i\omega_k t}). \qquad (4)$$

Time evolution is determined by the time-ordered unitary operator

$$\tilde{U}_{\text{tot}}(t_0, t) = T \exp\left\{ -\frac{i}{\hbar} \int_{t_0}^t ds \tilde{H}(s) \right\}, \qquad (5)$$

which can be evaluated exactly and can be written, up to a global c-number phase factor, in the following form:

$$\tilde{U}_{\text{tot}}(t_0, t) = \exp\left\{ \frac{\sigma_z}{2} \sum_k [b_k^\dagger e^{i\omega_k t_0} \xi_k(t-t_0) \right.$$

$$\left. - b_k e^{-i\omega_k t_0} \xi_k^*(t-t_0)] \right\}, \qquad (6)$$

where the following complex function has been introduced:

$$\xi_k(\Delta t) = \frac{2g_k}{\omega_k}(1 - e^{i\omega_k \Delta t}). \qquad (7)$$

Note that the separate dependence of Eq. (6) on both initial time t_0 and evolution interval $(t-t_0)$ is consistent with the time-reversal property $\tilde{U}_{\text{tot}}(t, t_0)\tilde{U}_{\text{tot}}(t_0, t) = 1$. A nice discussion on the entanglement generated by $\tilde{U}_{\text{tot}}(t_0, t)$ between qubit and field states is given in [3]. We are interested in calculating

$$\tilde{\rho}_{01}(t) = \langle 0 | \text{Tr}_B\{\tilde{U}_{\text{tot}}(t_0, t)\tilde{\rho}_{\text{tot}}(t_0)\tilde{U}_{\text{tot}}^\dagger(t_0, t)\} | 1 \rangle. \qquad (8)$$

This can be done without approximations under two standard assumptions: (i) qubit and environment are initially uncorrelated, i.e.,

$$\rho_{\text{tot}}(t_0) = \rho_S(t_0) \otimes \rho_B(t_0), \qquad (9)$$

(ii) the environment is initially in thermal equilibrium at temperature T, i.e.,

$$\rho_B(t_0) = \prod_k \rho_{B,k}(T) = \prod_k (1 - e^{\beta\hbar\omega_k}) e^{-\beta\hbar\omega_k b_k^\dagger b_k}. \qquad (10)$$

In Eq. (10), $\beta = 1/k_B T$, with k_B the Boltzmann constant. For simplicity, we choose henceforth units such that $\hbar = k_B = 1$. Conditions (9) and (10) are easily translated to interaction picture and, inserting Eq. (6) into Eq. (8), the problem is reduced to a single-mode trace:

$$\tilde{\rho}_{01}(t) = \tilde{\rho}_{01}(t_0) \prod_k \text{Tr}_k\{\rho_{B,k}(T)\mathcal{D}[e^{i\omega_k t_0}\xi_k(t-t_0)]\}$$

$$= \tilde{\rho}_{01}(t_0) e^{-\Gamma_0(t_0, t)}, \qquad (11)$$

where, in the first equality, the harmonic displacement operator $\mathcal{D}(\xi_k)$ is given by

$$\mathcal{D}(\xi_k) = e^{b_k^\dagger \xi_k - b_k \xi_k^*}, \qquad (12)$$

and the second equality defines the time-dependent function $\Gamma_0(t_0, t)$. The final step is to recognize that, for each mode, the quantity in curly brackets in Eq. (11) is nothing but the symmetric order generating function for a thermal harmonic oscillator [30]. Thus the explicit expression for $\Gamma_0(t_0, t)$ is

$$\Gamma_0(t_0, t) = \Gamma_0(t-t_0) = \sum_k \frac{|\xi_k(t-t_0)|^2}{2} \coth\left(\frac{\omega_k}{2T}\right). \qquad (13)$$

Since Γ_0 is a real function, it corresponds to pure damping in Eq. (11). Accordingly, this function characterizes completely

the dynamics of the decoherence process destroying the qubit phase information. Of course, the complete evolution in the original Schrödinger picture includes oscillation at the natural frequency ω_0,

$$\rho_{01}(t) = e^{i\omega_0(t-t_0)-\Gamma_0(t-t_0)}\rho_{01}(t_0), \tag{14}$$

while, using Eq. (6), one can check that $\rho_{ii}(t) = \rho_{ii}(t_0)$, $i = 0,1$.

Deeper insight into the time dependence of the decoherence process is gained if the continuum limit is made explicit in Eq. (13). By substituting $|\xi_k(t-t_0)|^2$ through Eq. (7), we get

$$\Gamma_0(t-t_0) = \int_0^\infty d\omega \left[\sum_k \delta(\omega-\omega_k)|g_k|^2 \right]$$
$$\times 4 \coth\left(\frac{\omega}{2T}\right)\frac{1-\cos\omega(t-t_0)}{\omega^2}. \tag{15}$$

The quantity in square brackets is known as the spectral density $I(\omega)$ of the bath. It turns out that, once the initial state is specified, complete information about the effect of the environment is encapsulated in this single function. As a general feature, the spectral density is characterized by a certain ultraviolet cutoff frequency ω_c such that $I(\omega)\to0$ for $\omega > \omega_c$. Although the specific value of ω_c depends on a natural cutoff frequency varying from system to system, the existence of a finite ω_c is always demanded on physical grounds. Indeed, assuming that the environment does not have a high-frequency cutoff means, generally, that energy can be dissipated instantaneously [31]. If, for instance, decoherence arises from the coupling to a phonon field, the natural cutoff frequency can be identified with the Debye frequency. In general, $\tau_c \sim \omega_c^{-1}$ sets the fastest time scale (or the memory time) of the environment. When a specific choice will be useful, we will assume a spectral density with the following functional form:

$$I(\omega) = \frac{\alpha}{4}\omega^n e^{-\omega/\omega_c}, \tag{16}$$

the parameter $\alpha>0$ measuring (in suitable units) the strength of the system-bath coupling and the index $n>0$ classifying different environment behaviors [3,29,31].

In addition to τ_c, another time scale $\tau_\beta \sim T^{-1}$, associated with the temperature of the bath, is expected to play a major role in the evolution of the qubit coherence. This is manifest if Eq. (15) is written in the equivalent form

$$\Gamma_0(t-t_0) = 4\int_0^\infty d\omega I(\omega)[2\bar{n}(\omega,T)+1]\frac{1-\cos\omega(t-t_0)}{\omega^2}, \tag{17}$$

where $\bar{n}(\omega,T) = \exp(-\omega/2T)\operatorname{cosech}(\omega/2T)$ is the average number of field excitations at temperature T. In this way, effects due to the thermal noise are formally separated from the ones due to purely vacuum fluctuations. Because of the different frequency composition, the two kinds of fluctuations dominate on different time scales, the relative importance of vacuum to thermal contributions being determined by τ_β. This multiplicity of time scales is one of the factors

that makes the decoherence process quite complicated. It should be important to keep in mind that there is *no* generic decohering behavior, and in particular that the qubit dynamics depends crucially on both temperature and the details of the spectral function $I(\omega)$. We will comment further on this point later.

III. PULSED EVOLUTION OF QUANTUM COHERENCE

Let us now define a procedure aimed at modifying the decoherence properties discussed so far. We choose to implement it as a suitable perturbation acting on some observables $\{\mathcal{O}_i\}$ of system S. This is obtained by adding to Eq. (1) a time-dependent term $\Sigma_i \gamma_i(t)\mathcal{O}_i$, where the input functions $\{\gamma_i(t)\}$ are assumed to be programmable at will, e.g., a given schedule of time-varying magnetic fields in the case of a spin qubit. In control terminology, this realizes a so-called open-loop configuration [7]. Closed-loop (or feedback) configurations have also been proposed in recent years to manipulate decoherence in some quantum optical systems [32]. There are many possible choices for the control Hamiltonian. Leaving aside the problem of controllability of the system (1) at the abstract level, we make here a pragmatic choice, partly suggested by semiclassical considerations.

Suppose that the perturbation we add is able to induce spin-flip transitions. By inspection of the spin-bath interaction Hamiltonian H_{SB}, opposite contributions arise when the spin belongs to the down or up eigenstate. Since a relative minus sign will be present during time evolution, the effect of the H_{SB} coupling will eventually average out provided the spin is flipped rapidly enough. There are two reasons why a mechanism of this kind is expected to be possible. First, as we already mentioned, similar methods are routinely used in NMR experiments to get rid of the effects of some unwanted interactions [21]. For instance, in the so-called spin-flip narrowing, line broadening resulting from magnetic dipolar coupling is reduced by repetitively flipping the spins among distinct energy configurations. The major difference here is that the undesired decohering coupling is contributed by the infinitely many quantized degrees of freedom of the heat bath. The second reason is related to the finite response time τ_c of the environment itself. In general, if perturbations act on the system more rapidly than this fastest time scale present in the bath, we expect that memory effects would be relevant, and that this non-Markovian dynamics would eventually lead to modified decoherence properties.

Having anticipated the intuitive idea, we start by writing the total Hamiltonian as

$$H(t) = H_0 + H_{\mathrm{rf}}(\omega_0,t), \tag{18}$$

where $H_{\mathrm{rf}}(\omega_0,t)$ represents a monochromatic alternating magnetic field applied at resonance. At this level, the problem can be related to variants of the complete spin-boson model; none of them is exactly solvable [29]. The main simplification we introduce is to substitute the continuous-mode operation, where the actions of the bath and the controlling field are necessarily simultaneous, with a pulsed-mode operation analogous to classical bang-bang control. Then, under the working assumption that the typical decoherence time of the system and the duration of pulses define two widely different time scales, we solve the problem by setting

(i) $H_{SB}=0$ within each pulse; (ii) $H_{\text{rf}}=0$ between successive pulses. As usual, we invoke the rotating wave approximation (RWA) and only look at the corotating component of the rf field. The radio-frequency perturbation is assumed of the following form:

$$H_{\text{rf}}(\omega_0,t)=\sum_{n=1}^{n_P} V^{(n)}(t)\{\cos[\omega_0(t-t_P^{(n)})]\sigma_x$$

$$+\sin[\omega_0(t-t_P^{(n)})]\sigma_y\}, \qquad (19)$$

with $t_P^{(n)}=t_0+n\Delta t$, $n=1,\ldots,n_P$, and

$$V^{(n)}(t)=\begin{cases} V & t_P^{(n)}\leqslant t\leqslant t_P^{(n)}+\tau_P \\ 0 & \text{elsewhere.} \end{cases} \qquad (20)$$

Equations (19) and (20) schematize a sequence of n_P identical pulses, each of duration τ_P, applied at instants $t=t_P^{(n)}$. The separation Δt between pulses is assumed to be an input of the model. The amplitude of the field is equal to V during each pulse and will be further specified below together with τ_P. In order to depict the evolution associated to a given pulse sequence, it is convenient to think of the latter as formed by successive elementary cycles of spin flips, a complete cycle being able to return the spin back to the starting configuration. We begin by analyzing the time evolution during the first spin cycle.

A. Elementary spin-flip cycle

As in Sec. II, we exploit the interaction representation (3). Some instructive steps of an alternative derivation based on Heisenberg formalism are sketched in the Appendix. The interaction picture transformed Hamiltonian is now

$$\tilde{H}(t)=\tilde{H}_{SB}(t)+\tilde{H}_{\text{rf}}(t), \qquad (21)$$

where $\tilde{H}_{SB}(t)$ is given in Eq. (4), and $\tilde{H}_{\text{rf}}(t)$ is evaluated by using Eq. (19) and the properties of Pauli matrices. We get

$$\tilde{H}_{\text{rf}}(t)=\sum_{n=1}^{n_P} V^{(n)}(t)e^{i\omega_0(\sigma_z/2)t_P^{(n)}}\sigma_x e^{-i\omega_0(\sigma_z/2)t_P^{(n)}}. \qquad (22)$$

According to Eq. (22), the time dependence due to the rotating field is completely removed within each pulse. In fact, the interaction representation (3) on spin variables is identical, at resonance, with the description in the rotating frame associated to Eq. (19) [21]. The counter-rotating term that is omitted within the RWA is seen to be negligible at resonance. For the first spin cycle, $n_P=2$ and we have the following sequence: evolution under $\tilde{H}_{SB}(t)$ during $t_0\leqslant t\leqslant t_P^{(1)}$; pulse P_1 at time $t_P^{(1)}$; evolution under $\tilde{H}_{SB}(t)$ during $t_P^{(1)}+\tau_P\leqslant t\leqslant t_P^{(2)}$; pulse P_2 at time $t_P^{(2)}$. After a total time $t_1=t_0+2\Delta t+2\tau_P$, the first cycle is complete. In terms of evolution operators, we have

$$\tilde{U}_P(t_0,t_1)=\tilde{U}_{P_2}\tilde{U}_{P_1}[\tilde{U}_{P_1}^{-1}\tilde{U}_{\text{tot}}(t_P^{(1)}+\tau_P,t_P^{(2)})\tilde{U}_{P_1}]$$

$$\times[\tilde{U}_{\text{tot}}(t_0,t_P^{(1)})]. \qquad (23)$$

We can read the evolutions in the absence of rf field directly from Eq. (6), for instance,

$$\tilde{U}_{\text{tot}}(t_P^{(1)}+\tau_P,t_P^{(2)})=\exp\left\{\frac{\sigma_z}{2}\sum_k [b_k^\dagger e^{i\omega_k(t_0+\Delta t+\tau_P)}\xi_k(\Delta t)\right.$$

$$\left.-b_k e^{-i\omega_k(t_0+\Delta t+\tau_P)}\xi_k^*(\Delta t)]\right\}. \qquad (24)$$

Concerning the evolution operator associated to the generic jth pulse, this is found by exponentiating Eq. (22) ($n=j$):

$$\tilde{U}_{P_j}=\exp\{-iV\tau_P e^{i\omega_0(\sigma_z/2)t_P^{(j)}}\sigma_x e^{-i\omega_0(\sigma_z/2)t_P^{(j)}}\}$$

$$=e^{i\omega_0(\sigma_z/2)t_P^{(j)}}e^{-iV\tau_P\sigma_x}e^{-i\omega_0(\sigma_z/2)t_P^{(j)}}. \qquad (25)$$

In order to proceed, we have to say more about pulses. We require

$$\{\tilde{U}_{P_1},\sigma_z\}=0, \quad [\tilde{U}_{P_2}\tilde{U}_{P_1},\sigma_z]=0, \qquad (26)$$

$\{,\}$ and $[,]$ denoting anticommutator and commutator, respectively. These conditions imply, as expected from the intuitive explanation, that P_1,P_2 are π pulses, satisfying $2V\tau_P=\pm\pi$. To simplify things, we imagine that $\tilde{H}_{\text{rf}}(t)$ is large enough to produce (almost) instantaneous spin flips. Accordingly, $\tau_P\to 0$ henceforth and we have to deal with ideal "kicks" of infinite power. Putting things together, we find

$$\tilde{U}_{P_2}\tilde{U}_{P_1}=-e^{i\omega_0(\sigma_z/2)(t_1-t_0)}, \qquad (27)$$

$$\tilde{U}_{P_1}^{-1}\tilde{U}_{\text{tot}}(t_P^{(1)},t_P^{(2)})\tilde{U}_{P_1}=\exp\left\{-\frac{\sigma_z}{2}\sum_k [b_k^\dagger e^{i\omega_k(t_0+\Delta t)}\xi_k(\Delta t)\right.$$

$$\left.-b_k e^{-i\omega_k(t_0+\Delta t)}\xi_k^*(\Delta t)]\right\}, \qquad (28)$$

and we are therefore ready to write down the final result for the cycle evolution operator:

$$\tilde{U}_P(t_0,t_1)=\exp\left\{i\omega_0\frac{\sigma_z}{2}(t_1-t_0)+\frac{\sigma_z}{2}\sum_k [b_k^\dagger e^{i\omega_k t_0}\eta_k(\Delta t)\right.$$

$$\left.-b_k e^{-i\omega_k t_0}\eta_k^*(\Delta t)]\right\}, \qquad (29)$$

where, as before, c-number phase factors have been omitted and

$$\eta_k(\Delta t)=\xi_k(\Delta t)(e^{i\omega_k\Delta t}-1)=-\frac{2g_k}{\omega_k}(1-e^{i\omega_k\Delta t})^2. \qquad (30)$$

It is interesting to compare the evolution described by Eq. (29) with the one in the absence of pulses. By recalling Eq. (6), evaluated at time $t_1=t_0+2\Delta t$, we report two differ-

ences: a phase factor proportional to σ_z and the duration of the cycle; a combination $\eta_k(\Delta t) \propto [\xi_k(\Delta t)]^2$ in place of $\xi_k(2\Delta t)$. The first difference corresponds to the fact that, due to the pulses, the oscillation at the natural frequency ω_0 is lost once the evolution is transferred back to the Schrödinger picture [see Eq. (14)]. The second difference, as will be seen in a moment, is the signal that decoherence properties are modified.

B. Decoherence properties after a pulse sequence

The next step is to generalize the description to an arbitrary number N of elementary spin-flip cycles, the nth cycle ending at time

$$t_n = t_0 + 2n\Delta t, \quad n = 1, \ldots, N \quad (31)$$

and the number of π pulses involved in the sequence being $n_P = 2N$. This is straightforward since Eq. (29) enables us to write down the evolution operator for the nth cycle:

$$\tilde{U}_P(t_{n-1}, t_n) = \exp\left\{ i\omega_0 \frac{\sigma_z}{2} 2\Delta t + \frac{\sigma_z}{2} \right.$$
$$\times \sum_k [b_k^\dagger e^{i\omega_k t_{n-1}} \eta_k(\Delta t)$$
$$\left. - b_k e^{-i\omega_k t_{n-1}} \eta_k^*(\Delta t)] \right\}. \quad (32)$$

The time development corresponding to N cycles is then governed by the time-ordered finite product

$$\tilde{U}_P^{(N)}(t_0, \ldots, t_N) = \tilde{U}_P(t_{N-1}, t_N) \cdots \tilde{U}_P(t_1, t_2) \tilde{U}_P(t_0, t_1). \quad (33)$$

Note that $\tilde{U}_P^{(N)} \neq (\tilde{U}_P)^N$, the dependence on intermediate times $\{t_j\}$ in the sequence being introduced by the environment dynamics. A closed formula for $\tilde{U}_P^{(N)}(\{t_j\})$ can still be found quickly since, by neglecting state-independent global phase factors that are irrelevant to density matrix propagation, we are allowed to treat the factors in Eq. (33) as commuting operators. We get

$$\tilde{U}_P^{(N)}(t_0, \Delta t) = \exp\left\{ i\omega_0 \frac{\sigma_z}{2} 2N\Delta t + \frac{\sigma_z}{2} \right.$$
$$\times \sum_k [b_k^\dagger e^{i\omega_k t_0} \eta_k(N, \Delta t)$$
$$\left. - b_k e^{-i\omega_k t_0} \eta_k^*(N, \Delta t)] \right\}, \quad (34)$$

where

$$\eta_k(N, \Delta t) = \eta_k(\Delta t) \sum_{n=1}^{N} e^{2i(n-1)\omega_k \Delta t}, \quad (35)$$

and definition (31) has been exploited. Of course, the results of Sec. III A are recovered for $N=1$. A more interesting check can be done by relating the evolution (34) to the cor-

responding propagator in the absence of pulses. The trick is to switch back the minus sign in the definition (30) of $\eta_k(\Delta t)$, that has been recognized as the key dynamical effect due to the pulsing procedure. Since $\xi_k(\Delta t)(e^{i\omega_k \Delta t} + 1) = \xi_k(2\Delta t)$, we are left with

$$\xi_k(2\Delta t) \sum_{n=1}^{N} e^{2i(n-1)\omega_k \Delta t} = \xi_k(2N\Delta t) = \xi_k(t_N - t_0) \quad (36)$$

in place of Eq. (35). Therefore this procedure gives

$$\tilde{U}_{\text{tot}}(t_0, t_N) = \exp\left\{ \frac{\sigma_z}{2} \sum_k [b_k^\dagger e^{i\omega_k t_0} \xi_k(t_N - t_0) \right.$$
$$\left. - b_k e^{-i\omega_k t_0} \xi_k^*(t_N - t_0)] \right\}, \quad (37)$$

which does agree with the direct evaluation based on Eq. (5). It may be worth to make the connection with unperturbed evolution more explicit, which is done by writing

$$\eta_k(N, \Delta t) = \xi_k(2N\Delta t) - 2\xi_k(\Delta t) \sum_{n=1}^{N} e^{2i(n-1)\omega_k \Delta t}, \quad (38)$$

where the contribution due to the pulse sequence shows up in the form of a typical interference factor.

The decoherence properties corresponding to the pulsed qubit plus bath evolution (34) are derived through the steps outlined in Sec. II for the unperturbed case. Incidentally, we still expect that spin populations are unchanged after a sequence of N complete spin cycles, while qubit coherence at final time $t_N = t_0 + 2N\Delta t$ is given by

$$\tilde{\rho}_{01}(t_N) = \langle 0| \text{Tr}_B\{ \tilde{U}_P^{(N)}(t_0, \Delta t) \tilde{\rho}_{\text{tot}}(t_0) \tilde{U}_P^{(N)\dagger}(t_0, \Delta t) \} |1\rangle. \quad (39)$$

The result is

$$\tilde{\rho}_{01}(t_N) = e^{-i\omega_0(t_N - t_0) - \Gamma_P(N, \Delta t)} \tilde{\rho}_{01}(t_0), \quad (40)$$

where

$$\Gamma_P(N, \Delta t) = \sum_k \frac{|\eta_k(N, \Delta t)|^2}{2} \coth\left(\frac{\omega_k}{2T} \right). \quad (41)$$

Comparing with Eq. (13), the mathematical prescription for decoherence in the presence of pulses looks very simple: just use $\eta_k(N, \Delta t)$ instead of $\xi_k(t_N - t_0)$ for each mode of the bath. However, the final effect leading to Eq. (41) is not easy to figure out and it is useful to compare first the decoherence due to a single mode with frequency ω with and without pulses, respectively. Apart from identical time-independent factors, we have to consider

$$|\eta(N, \omega\Delta t)|^2 = 4(1 - \cos \omega\Delta t)^2$$
$$\times \left(N + \sum_{n=0}^{N-1} 2n \cos[2(N-n)\omega\Delta t] \right), \quad (42)$$

vs

$$|\xi(N,\omega\Delta t)|^2=2(1-\cos 2N\omega\Delta t)=2[1-\cos \omega(t_N-t_0)], \tag{43}$$

where Eqs. (7) and (35) have been used and an identical proportionality factor is understood. The unperturbed contribution $|\xi|^2$ simply oscillates between values 0,4 with a period (π/N). The function $|\eta|^2$, instead, is strongly oscillating for increasing N, developing $2(N-1)$ local minima and a sharply peaked absolute maximum at $\omega\Delta t=\pi$. Constructive interference is highest at the maximum, leading to a value $|\eta|^2_{max}=16N^2$ that can be very large, while destructive interference strongly damps the function for $\omega\Delta t<\pi/2$, which is a zero for both $|\eta|^2$ and $|\xi|^2$ for any N. One can show that

$$|\eta(N,\omega\Delta t)|^2\leq|\xi(N,\omega\Delta t)|^2 \quad \text{on } [0,\pi/2] \text{ for any } N. \tag{44}$$

Back to decoherence properties, condition (44) means that, for a mode at frequency ω, a finite region $\omega\Delta t\lesssim\pi/2$ exists, where the contribution to decoherence is *smaller* in the presence of pulses. Since the ''correcting region'' is entered for small Δt values, this effect takes place in the regime of rapid flipping we expected. Since, moreover, smaller Δt values require longer pulse sequences in order to evolve the system over the same interval, it is useful to consider an interesting limiting case.

C. Limit of continuous flipping and suppression of decoherence

Let us study an idealized situation represented by the following mathematical limit:

$$\Delta t\rightarrow 0,$$

$$N\rightarrow\infty,$$

$$2N\Delta t=t_N-t_0. \tag{45}$$

It is convenient to rewrite the decoherence function (41) by exploiting Eq. (38) to separate formally the unperturbed and the interference contributions. Thus

$$\Gamma_P(N,\Delta t)=\sum_k \frac{|\xi_k(2N\Delta t)|^2}{2}\coth\left(\frac{\omega_k}{2T}\right)|1-f_k(N,\Delta t)|^2, \tag{46}$$

where

$$f_k(N,\Delta t)=2\frac{\xi_k(\Delta t)}{\xi_k(2N\Delta t)}\sum_{n=1}^N e^{2i(n-1)\omega_k\Delta t}. \tag{47}$$

In this way, $\Gamma_0(t_N-t_0)$ is recovered by setting $f_k=0$ for each mode, see Eq. (13). We evaluate the asymptotic limit of $f_k(N,\Delta t)$ as follows:

$$\lim_{\Delta t\rightarrow 0} f_k(N,\Delta t)=\frac{e^{-i\omega_k t_0}}{1-e^{i\omega_k(t_N-t_0)}}$$

$$\times\lim_{\Delta t\rightarrow 0}\frac{(1-e^{i\omega_k\Delta t})}{\Delta t}\sum_{n=1}^N 2\Delta t e^{i\omega_k t_{n-1}}$$

$$=\frac{e^{-i\omega_k t_0}}{1-e^{i\omega_k(t_N-t_0)}}\lim_{\Delta t\rightarrow 0}\frac{(1-e^{i\omega_k\Delta t})}{\Delta t}\int_{t_0}^{t_N}ds\, e^{i\omega_k s}$$

$$=\lim_{\Delta t\rightarrow 0}\left[\frac{\sin \omega_k\Delta t}{\omega_k\Delta t}+i\frac{1-\cos \omega_k\Delta t}{\omega_k\Delta t}\right]=1. \tag{48}$$

If this result holds for an arbitrary field mode, then the implications for the decoherence properties are transparent:

$$\lim_{\Delta t\rightarrow 0}\Gamma_P(N,\Delta t)=0, \tag{49}$$

i.e., in the limit of continuous flipping, decoherence is completely and exactly eliminated for any temperature and any spectral density function.

Obviously, there is no hope that a continuous limit of this kind would ever be attained in practice. However, situation (45) should be approached if Δt is made small compared to the fastest characteristic time present in the dynamics of the system. From the considerations of Sec. II, the environment correlation time τ_c certainly provides a lower bound since there is no spectral content of the environmental noise at frequency higher than ω_c. Hence, we expect that a sufficient condition in order to meet Eq. (45) is

$$\omega_c\Delta t\lesssim 1. \tag{50}$$

The question of whether we can do better than this may not be completely obvious, since time scales different from τ_c are also involved in the decoherence process. In what follows, we try to understand this point both by presenting a physical explanation for the decoherence suppression and by analyzing some specific situations.

D. Physical interpretation

People familiar with quantum Zeno effect [20] may have found some similarities with the behavior we are discussing. In particular, the basic mathematical modification associated with pulses is a function $|\eta_k|^2$, which is $O(\omega_k^4\Delta t^4)$ for short time intervals, compared to $|\xi_k|^2=O(\omega_k^2\Delta t^2)$. Moreover, Eq. (45) is formally the same continuous limit involved in many quantum Zeno related proposals, notably the one by Cook [33]. In both cases, a preexisting dynamics—the qubit plus bath evolution here, a stimulated two-level transition in Cook's scheme—is modified through a pulsing procedure. Pulses, respectively, represent spin-flip interactions, short enough that the action of the bath is made negligible, or measurement pulses, long enough that the coupling to the external environment (the measuring apparatus) is made appreciable. A dynamical inhibition phenomenon occurs when pulses become sufficiently frequent. One can say that two opposite configurations are realized for decoherence: with continuous measurements, the interaction with the bath is always ''on'' and internal dynamics becomes frozen; in the

limit of continuous flipping, it is the two-level controlled dynamics that dominates, and the interaction with the bath tends to be always ''off,'' as indicated by Eq. (49). However, the analogy stops from a more physical point of view.

A more interesting interpretation of the decoherence suppression (49) can be obtained by connecting it to effects already observed in magnetic resonance experiments, like spin echoes, solid echoes, or spin-flip narrowing [21]. All of these are basically time-reversal experiments. In order to capture the basic physical mechanism of our model, let us go back to the elementaty spin cycle (Sec. III A) and look more carefully at the evolution operator (23). This is made of two pieces: a free evolution during the first Δt, followed by an evolution governed by Eq. (28) during the second interval Δt. It is this transformed operator that *simulates* the effect of a time reversal. In fact, backward propagation during the second part of the cycle would correspond to

$$\tilde{U}_{tot}(t_0 + \Delta t, t_0) = \exp\left\{ -\frac{\sigma_z}{2}\sum_k [b_k^\dagger e^{i\omega_k t_0}\xi_k(\Delta t) \right.$$
$$\left. - b_k e^{-i\omega_k t_0}\xi_k^*(\Delta t)] \right\}. \quad (51)$$

By comparison with Eq. (28), forward propagation in the presence of rf kicks only differs for an additional term $e^{i\omega_k \Delta t}$ affecting each mode. This phase factor, due to the dynamics of the bath oscillators, is ultimately responsible for the decoherence properties in the pulsed evolution of the system. If not for this phase difference between Eqs. (28) and (51), we would have $\eta_k(\Delta t) = 0$ and no decoherence. Instead, reversal is approximate since the bath restarts at time $t_0 + \Delta t$ after the first pulse with a dephased initial condition (see also the Appendix). However, if $e^{i\omega_k \Delta t} \approx 1$ for each mode, then the couple of kicks produces an exact time reversal and, by iteration, we arrive at Eq. (49). Equivalently, if the bath Hamiltonian can be considered as a constant, then the total Hamiltonian (1) acquires a minus sign and the system retraces the previous evolution. The validity of this condition depends on the time scale we are considering. For a single mode of frequency ω, the time needed to produce appreciable dephasing is $\tau \approx \omega^{-1}$, so we expect decoherence correction for $\tau/\Delta t \gtrsim 1$. This is in agreement with both the interpretation of Eq. (44) and the semiclassical NMR argument [21], where motional effects are predicted for τ comparable to the mean time spent in a given spin configuration (here Δt). For the whole environment, the correlation time τ_c is the minimum time scale over which the dynamics is approximately unchanged, and the same reasoning leads to a physical explanation of the rapid flipping condition (50).

As a final remark, we point out that the mechanism accomplishing time reversal in our model is purely *macroscopic* in the sense that no reference is made to the dynamical state of the system. This is different from the familiar case of the Maxwell demon, that effectively reverses a dynamical evolution by operating over some microscopic variables (like velocities) at a given instant. Rather, the reversal is obtained by changing the sign of the system Hamiltonian through the action of suitable external fields (control). A different kind of demon, the so-called Loschmidt demon, has been introduced by some authors to account for this behavior

[34]. In this terminology, the spin-flip procedure realizes in principle a Loschmidt demon for a decohering qubit.

IV. ANALYSIS AND EXAMPLES

In this section we try to give some semiquantitative picture of the decoherence mechanisms discussed so far. We focus our attention on a representative class of reservoirs, corresponding to so-called Ohmic environments. The appropriate spectral density is given by Eq. (16) with $n=1$. The time dependence of the decoherence is summarized by the following expression:

$$\Gamma(t) = \alpha \int_0^\infty d\omega e^{-\omega/\omega_c}[2\bar{n}(\omega,T) + 1]\frac{1-\cos\omega t}{\omega}$$
$$\times |1 - f(\omega, N, \Delta t)|^2, \quad (52)$$

where we assume $t_0 = 0$. Equation (52) reproduces the unperturbed behavior of Sec. II when $f=0$, in which case $\Gamma(t) = \Gamma_0(t)$, Eq. (13). In the presence of N spin-flip cycles, Eq. (52) is found from the continuous limit of $\Gamma_P(N,\Delta t)$ in Eq. (41), with $f(\omega,N,\Delta t)$ derived from Eq. (47). According to Sec. III, in this case we are interested in decoherence after a complete pulse sequence, i.e., $t = t_N = 2N\Delta t$. For a fixed strength α of the system-reservoir coupling, the properties of the environment enter Eq. (52) with two parameters, $\omega_c \sim \tau_c^{-1}$, $T \sim \tau_\beta^{-1}$.

Let us first analyze decoherence in the absence of any correction, $f=0$. Qualitatively different behaviors arise depending on the relationship between the cutoff frequency ω_c and the thermal frequency $\omega_\beta = T$. Typically, two extreme situations are considered.

(i) $\omega_c \ll \omega_\beta$: high-temperature limit or *classical* environment. Decoherence dynamics is relatively easy to describe in this regime since, due to the exponential dependence on the cutoff, ω_c is actually the only characteristic frequency accessible to the system. The environment looks classical in the sense that its quantized structure cannot be appreciated compared to the thermal quantum ω_β. Accordingly, thermal fluctuations always dominate over vacuum ones and, after a short transient $O(\tau_\beta)$ where decoherence is almost ineffective, dynamics becomes very fast and coherence is lost completely after a time comparable to τ_c. In general, one expects that in this limit equivalent results are obtained by assuming a heat bath of classical harmonic oscillators. A mixed quantum-classical derivation for a two-state open system is given, for instance, in [35].

(ii) $\omega_c \gg \omega_\beta$: low-temperature limit or *quantum* environment. In this limit a more complex interplay between thermal and vacuum effects arises. Thermal fluctuations are only effective for $t > \tau_\beta$ and, due to the exponential suppression of $\bar{n}(\omega,T)$, they are almost totally contributed by low-frequency modes $\omega \lesssim \omega_\beta$. The effects of vacuum fluctuations dominate on an intermediate region $\tau_c < t < \tau_\beta$, a non-vanishing contribution remaining, however, at longer time scales $t > \tau_\beta$. The frequency composition of the fluctuations is now less clear: while modes below the thermal threshold are still responsible for the long-time dynamics $t > \tau_\beta$, frequencies in the range up to ω_c mostly contribute for $t < \tau_\beta$ but are also present beyond τ_β. As a consequence, even at
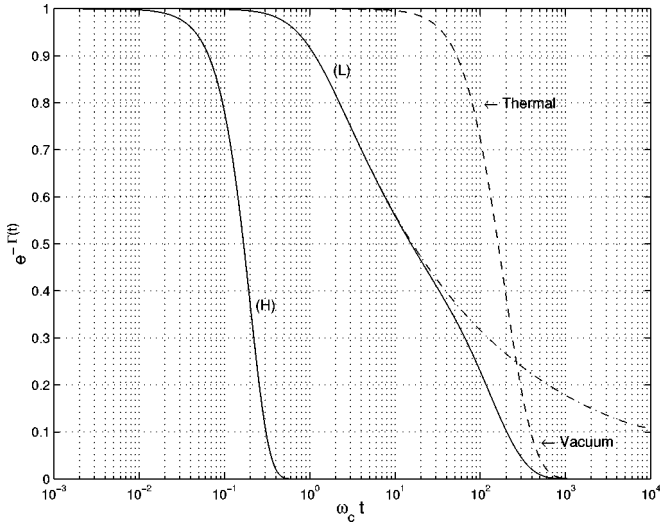
FIG. 1. Qubit decoherence as a function of time for an Ohmic environment, Eq. (52) ($f=0$). Time is in units of T^{-1} and the values $\omega_c=100$, $\alpha=0.25$ have been chosen. High- and low-temperature behaviors are shown, (H) $\omega_c/T=10^{-2}$ and (L) $\omega_c/T=10^2$, respectively. The contributions arising from the separate integration of thermal and vacuum fluctuations are displayed in the latter case, $e^{-\Gamma(t)}=e^{-\Gamma_{th}(t)}e^{-\Gamma_v(t)}$.

thermal time scales $t>\tau_\beta \gg \tau_c$, high-frequency modes contribute appreciably to the decoherence process and characteristic times $O(\tau_c)$ are still relevant in the underlying dynamics.

Typical decoherence curves $e^{-\Gamma(t)}$ for the Ohmic environment are found by numerical integration of Eq. (52) and are shown in Fig. 1 for two choices corresponding to high- and low-temperature limit, $\omega_c/T=10^{-2}$ and $\omega_c/T=10^2$, respectively. The partial contributions due to thermal and vacuum fluctuations are indicated separately where possible. In the low-temperature case, a quiet ($t<\tau_c$), a quantum ($\tau_c<t<\tau_\beta$), and a thermal ($t>\tau_\beta$) regime are easily identified in the process, as indicated above and discussed in more detail by many authors [2,3,29]. In both configurations, the qubit coherence decays exponentially fast once the thermal regime is well established,

$$e^{-\Gamma(t)}\approx e^{-t/t_{th}}, \tag{53}$$

for a suitable time constant t_{th}. In models where the decoherence rate is constant in time, Eq. (53) is usually assumed as the definition of a typical decoherence time, $t_{dec}=t_{th}$. In our case, since the whole behavior of $\Gamma(t)$ is required for a complete knowledge of the decoherence dynamics, the definition of a characteristic time for loss of unitarity is less clear. Oversimplifying things, the situation can be summarized as follows: for both the high- and the low-temperature regimes, a characteristic time exists, indicating the departure of coherence from unity. This time is determined by the shortest between the two time scales τ_c, τ_β. Once this transient is over, the duration of the actual decoherence process is at least comparable to τ_c in the classical environment, and to τ_β in the quantum one. If this information is used as an estimate for a characteristic decoherence time, we are led to $t_{dec}\approx O(\tau_c/\alpha)$ for the high-temperature limit and $t_{dec}\approx O(\tau_\beta/\alpha)$ for the low-temperature limit, respectively. For
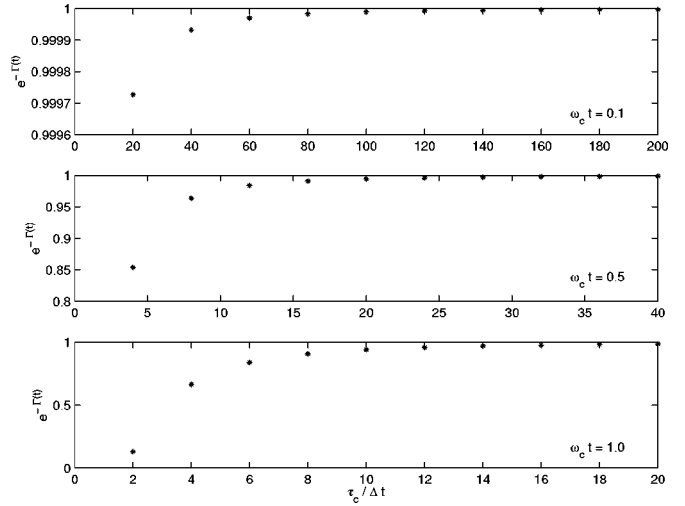

FIG. 2. Qubit decoherence in the presence of rf pulses for the high-temperature configuration, $\omega_c/T=10^{-2}$. For a fixed time, each point corresponds to a number N of cycles, $N=1,\ldots,N_{max}=10$. The results from Eq. (52) are plotted as a function of the normalized pulse frequency $\tau_c/\Delta t$. The unperturbed values of decoherence are read from Fig. 1(H).

identical values of α and ω_c, decoherence occurs extremely faster in the former case, as expected on intuitive grounds.

We now come back to examine how decoherence is improved in the presence of spin-flip cycles. With the interference contribution f restored in Eq. (52), we have calculated numerically the decoherence values obtained when a fixed time interval t is divided in an increasing number of cycles of duration $2\Delta t$, i.e.,

$$\Delta t=\frac{t}{2N}, \quad N=1,2,\ldots,N_{max}. \tag{54}$$

The behavior of $e^{-\Gamma(t)}$ as a function of the pulse frequency $1/\Delta t$ has been studied, and the procedure repeated for different representative times. The results for the high- and low-temperature reservoirs considered above are shown in Figs. 2 and 3, respectively. The unperturbed values of decoherence at the appropriate times can be read from Fig. 1. We see that, as predicted by Eq. (49), decoherence correction starts as soon as the region $\tau_c/\Delta t \gtrsim 1$ is entered. For times short enough that $\omega_c t<2N$, this may even be accomplished with a single cycle. In general, once this condition is fulfilled, no further reduction of Δt is demanded to evolve the system in a decoherence-free way. However, a warning also emerges from Fig. 3: if flipping is not frequent enough, not only does the correction effect disappear, but decoherence can actually be made worse compared to the one in the absence of pulses. The explanation of this behavior is rooted into the interference mechanism that builds up the correction factor (30): in a sense, decoherence can be subtracted almost completely from the frequency range $\omega\Delta t\lesssim 1$ only at the expense of enhancing the decoherence contributions from modes outside that region. This intrinsic feature of the model is also relevant to understand why, despite the differences existing between the high- and low-temperature decoherence properties, the same condition $\omega_c\Delta t\lesssim 1$ is required to prevent both thermal and vacuum noises. In fact, this condition comes quite
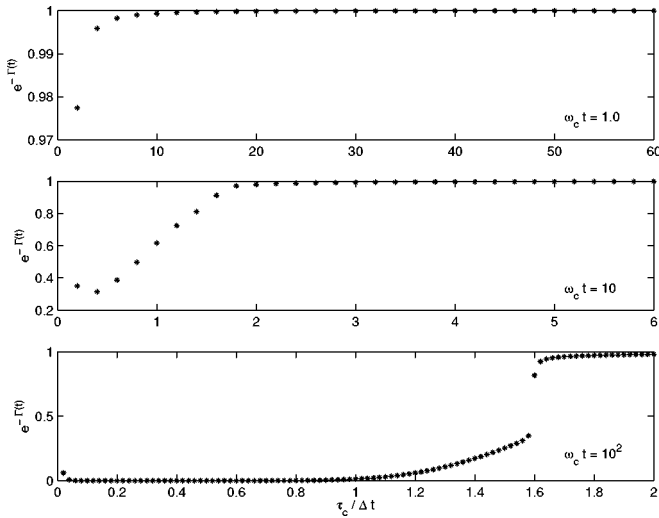
FIG. 3. Same as in Fig. 2 for the low-temperature configuration, $\omega_c/T = 10^2$. The maximum number of spin cycles is equal to $N_{\max} = 30$ in the simulations at $\omega_c t = 1.0, 10$, while $N_{\max} = 100$ at $\omega_c t = 10^2$. The unperturbed values of decoherence are read from Fig. 1(L).

natural for a classical environment, but one might at first wonder why a weaker condition $\omega_\beta \Delta t \lesssim 1$ would not suffice for the quantum case, even if the frequencies $\omega \lesssim \omega_\beta$ contain the fraction that mostly contributes at times of the order of t_{dec}. The reason for this failure is the presence of vacuum fluctuations. By satisfying condition $\omega_\beta \Delta t \lesssim 1$, we do get rid of thermal dephasing, but we do not completely correct vacuum noise until $\omega_c \Delta t \lesssim 1$. Precisely, we are missing modes of intermediate frequency $\omega_\beta \lesssim \omega \lesssim \omega_c$ that, although of minor importance at long times, may introduce amplified decoherence contributions if not properly corrected.

We conclude with a few comments on the relevance of our procedure for quantum information processing. While the idealized character of the model prevents us from a quantitative discussion of implementation criteria, we can compare with the principles underlying current quantum error-correction proposals. Essentially, these are schemes to encode redundantly information in such a way that it can be restored also when errors due to external sources have occurred. The syndrome-identification and the error-correction stages may be regarded as a feedback configuration: suitable measurement protocols are required both in conventional schemes [5,8–15] and in alternate techniques based on the quantum Zeno effect [16,17], while conditional logic is exploited in the coherence-preserving routines proposed in [18]. In any case, error-correction methods have the effect of reducing the error rate per unit time and, in order to be effective, they must be repeated at time intervals Δt shorter than the typical decoherence time of the system, i.e., $t_{\mathrm{dec}}/\Delta t \gtrsim 1$. In comparison to quantum error-correction schemes, our procedure exhibits two fundamental differences: no ancillary bits are required to store the information; no measurements are performed. In principle, these could be advantageous features, since encoding would be more efficient and, by avoiding measurements, no slow down of the computational speed would be introduced. In addition, rather than reducing the error rate, this method would suppress completely the error source provided the appropriate condi-

tion $\tau_c/\Delta t \gtrsim 1$ is fulfilled. From a more practical perspective, it is the accessibility of this rapid flipping limit, demanding fast and short pulses, $\tau_c/\Delta t \gtrsim 1$ and $\tau_P/\Delta t \ll 1$, that determines the viability of the procedure itself. If such requirements can be satisfied, our method might be valuable in configurations where t_{dec} tends to be shorter than τ_c [36] or, even in case t_{dec} is longer compared to τ_c, for systems where tipping the state is easier than exploiting conventional error-correction protocols. While the existence of an interaction able to implement a NOT gate by inverting the state is demanded for any two-level system relevant to quantum computation, both the present technological capabilities for applying π pulses and the relevant environmental cutoff frequencies depend considerably on the specific physical system and the mechanism responsible for decoherence. Some important time scales for various prospective qubits can be found in [1].

V. CONCLUSIONS

Our work demonstrates the possibility to modify the evolution of a quantum open system through the application of an external controllable interaction. A prototype situation involving a two-level system coupled to a quantized reservoir in thermal equilibrium has been worked out in detail and dynamical suppression of quantum decoherence has been evidenced. From the perspective of quantum information, the analysis suggests a different direction compared to conventional quantum error-correction techniques, based on the idea of forcing the system into a dynamics that disturbs the decoherence process. Our present study for a specific example brings up, among other issues, the question of whether similar decoherence correction mechanisms would be operating under more general conditions, including either different open system dynamics, or different control configurations, or both. In particular, an interesting possibility could emerge from examining decoherence properties within a fully quantum mechanical description where the control degrees of freedom are explicitly included and the system is driven by a quantum controller as recently proposed in [7].

APPENDIX: HEISENBERG REPRESENTATION

Compared to the interaction picture, the Heisenberg representation has two advantages: first, it does not require preliminary transformations on the state vector; second, it gives to a certain extent a more intuitive description of the spin motion. In this appendix, we outline the evolution of the qubit coherence, by restricting ourselves to the first elementary spin-flip cycle. In the Heisenberg picture, the relevant information is contained in

$$\sigma_+(t) = \frac{1}{2}[\sigma_x(t) + i\sigma_y(t)], \tag{A1}$$

since, by averaging over the quantum state, $\langle\sigma_+(t)\rangle = \rho_{01}(t)$. As in Sec. III, we evaluate the qubit dynamics under the separate action of the spin-bath Hamiltonian H_{SB} and the radio-frequency perturbation H_{rf}. The description of a π pulse turns out to be extremely simple in the Heisenberg representation. Nothing happens to the bath operators $b_k, b_k^\dagger$ in the limit of instantaneous pulses, while spin dynamics is governed by the equations

$$\dot\sigma_\alpha = -i[\sigma_\alpha, (H_S + H_{\mathrm{rf}}(\omega_0, t))], \quad \alpha = x, y, z \tag{A2}$$

with $H_{\mathrm{rf}}(\omega_0, t)$ given in Eq. (19). By denoting with $t_P^{-(+)}$ the instants immediately before (after) a pulse, respectively, a very simple result is found:

$$\sigma_z(t_P^+) = -\sigma_z(t_P^-),$$
$$\sigma_+(t_P^+) = [\sigma_+(t_P^-)]^\dagger. \tag{A3}$$

The action on σ_z corresponds, in particular, to the pictorial spin-flip effect operated by a π pulse. Now denote as $G_{\mathrm{tot}}(t_i, t_j)$ the operator evolving coherence from t_i to t_j in the absence of rf pulses, i.e.,

$$\sigma_+(t_j) = G_{\mathrm{tot}}(t_i, t_j)\sigma_+(t_i). \tag{A4}$$

Then, using relation (A3) for σ_+ twice, we find the following representation for the coherence evolution during the first complete cycle:

$$\sigma_+(t_0 + 2\Delta t) = G_{\mathrm{tot}}(t_0, t_0 + \Delta t)\sigma_+(t_0)$$
$$\times G_{\mathrm{tot}}^\dagger(t_0 + \Delta t, t_0 + 2\Delta t), \tag{A5}$$

to be compared with

$$\sigma_+(t_0 + 2\Delta t) = G_{\mathrm{tot}}(t_0 + \Delta t, t_0 + 2\Delta t)G_{\mathrm{tot}}(t_0, t_0 + \Delta t)\sigma_+(t_0) \tag{A6}$$

in the absence of pulses. Even before knowing the explicit form of G_{tot}, we see from Eqs. (A5) and (A6) that the presence of a time-reversed evolution during the second interval Δt of the cycle is already enucleated at this stage.

In order to evaluate the propagator G_{tot}, the Heisenberg equations for the coupled spin plus bath motion have to be solved. From Hamiltonian (1) we get

$$\dot b_k = -i\omega_k b_k - ig_k\sigma_z,$$
$$\dot b_k^\dagger = +i\omega_k b_k^\dagger + ig_k^*\sigma_z,$$
$$\dot\sigma_+ = i\omega_0\sigma_+ + 2i\sum_k (g_k b_k^\dagger + g_k^* b_k)\sigma_+, \tag{A7}$$
$$\dot\sigma_z = 0.$$

Since instantaneous pulses introduce discontinuous changes in operators, the propagators $G_{\mathrm{tot}}(t_0, t_0 + \Delta t)$, $G_{\mathrm{tot}}(t_0$

$+\Delta t, t_0 + 2\Delta t)$ have to be considered separately, by solving Eq. (A7) with initial conditions at $t = t_0$, $t = t_P^+ = t_0 + \Delta t$, respectively.

(i) $t_0 \mapsto t_0 + \Delta t$: Since $\sigma_z(t) = \sigma_z(t_0)$, the equations for the bath variables are completely solved by

$$b_k(t) = e^{-i\omega_k(t-t_0)}b_k(t_0) - \sigma_z(t_0)\frac{g_k}{\omega_k}(1 - e^{-i\omega_k(t-t_0)}), \tag{A8}$$

and $b_k^\dagger(t) = [b_k(t)]^\dagger$. These solutions should be inserted in the expression for $\sigma_+(t)$:

$$\sigma_+(t) = T\exp\left\{i\int_{t_0}^t ds\{\omega_0 + 2[g_k b_k^\dagger(s) + g_k^* b_k(s)]\}\right\}\sigma_+(t_0). \tag{A9}$$

The time-ordered exponential can be evaluated exactly and the following result is found for the first propagator:

$$G_{\mathrm{tot}}(t_0, t_0 + \Delta t) = \exp\left\{i\omega_0\Delta t + 4i\frac{|g_k|^2}{\omega_k}\left(\Delta t - \frac{\sin\omega_k\Delta t}{\omega_k}\right)\right.$$
$$\left.\times[1 - \sigma_z(t_0)]\right\}\exp\left\{-\sum_k [b_k^\dagger(t_0)\xi_k(\Delta t)\right.$$
$$\left. - b_k(t_0)\xi_k^*(\Delta t)]\right\}, \tag{A10}$$

where the same notation for $\xi_k(\Delta t)$ has been used, Eq. (7).

(ii) $t_0 + \Delta t \mapsto t_0 + 2\Delta t$: By exploiting Eq. (A10), we can immediately write down the expression for the propagator $G_{\mathrm{tot}}^\dagger(t_0 + \Delta t, t_0 + 2\Delta t)$ in terms of the new initial condition at $t = t_P^+$:

$$G_{\mathrm{tot}}^\dagger(t_0 + \Delta t, t_0 + 2\Delta t) = \exp\left\{-i\omega_0\Delta t - 4i\frac{|g_k|^2}{\omega_k}\right.$$
$$\left.\times\left(\Delta t - \frac{\sin\omega_k\Delta t}{\omega_k}\right)[1 - \sigma_z(t_P^+)]\right\}$$
$$\times\exp\left\{+\sum_k [b_k^\dagger(t_P^+)\xi_k(\Delta t)\right.$$
$$\left. - b_k(t_P^+)\xi_k^*(\Delta t)]\right\}. \tag{A11}$$

Now everything can be evaluated with respect to the initial time of the cycle. We exploit Eq. (A3) for $\sigma_z(t_P^+)$ and, since $b_k(t_P^+) = b_k(t_P^-) = b_k(t_0 + \Delta t)$, bath operators evolve as

$$b_k(t_0 + \Delta t) = e^{-i\omega_k\Delta t}b_k(t_0) - \sigma_z(t_0)\frac{g_k}{\omega_k}(1 - e^{-i\omega_k\Delta t}), \tag{A12}$$

and Hermitian conjugate. As we discussed in the text, the presence in Eq. (A12) of an initial condition dephased by $e^{\pm i\omega_k\Delta t}$ for each environmental mode is the ultimate source of decoherence. We arrive at the following expression for the cycle evolution:

$$\sigma_+(t_0+2\Delta t)=\exp\Big\{ i\varphi_1(\Delta t)[1-\sigma_z(t_0)]$$

$$+\sum_k [b_k^\dagger(t_0)\,\eta_k(\Delta t)-b_k(t_0)\,\eta_k^*(\Delta t)]$$

$$+i\varphi_2(\Delta t)\Big\}\sigma_+(t_0)\exp\{i\sigma_z(t_0)\varphi_2(\Delta t)$$

$$-i\varphi_1(\Delta t)[1+\sigma_z(t_0)]\},\qquad (A13)$$

where

$$\varphi_1(\Delta t)=4\sum_k \frac{|g_k|^2}{\omega_k}\Big(\Delta t-\frac{\sin \omega_k\Delta t}{\omega_k}\Big),$$

$$\varphi_2(\Delta t)=8\sum_k \frac{|g_k|^2}{\omega_k^2}\sin \omega_k\Delta t(1-\cos \omega_k\Delta t). \quad (A14)$$

The final step is to calculate the coherence evolution as

$$\rho_{01}(t_0+2\Delta t)=\langle\sigma_+(t_0+2\Delta t)\rangle$$

$$=\sum_{j=0,1}\langle j|\mathrm{Tr}_B\{(\rho_B\otimes\rho_S)\sigma_+(t_0+2\Delta t)\}|j\rangle.$$

$$(A15)$$

By inserting Eqs. (A13) and (A14), phase factors drop out and we find

$$\rho_{01}(t_0+2\Delta t)=\rho_{01}(t_0)e^{-\Gamma_P(N=1,\Delta t)},\qquad (A16)$$

in agreement with the result found from Eqs. (40) and (41) in the Schrödinger representation. The procedure can be generalized to arbitrary N and the complete expression $\Gamma_P(N,\Delta t)$ is thereby recovered.

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[1] For reviews, see D. P. DiVincenzo, Science **270**, 255 (1995); P. W. Shor *et al.*, *ibid.* **270**, 1633 (1995); A. Ekert and R. Josza, Rev. Mod. Phys. **68**, 733 (1996).

[2] W. G. Unruh, Phys. Rev. A **51**, 992 (1995).

[3] G. M. Palma, K.-A. Suominen, and A. K. Ekert, Proc. R. Soc. London, Ser. A **452**, 567 (1996).

[4] P. Zanardi and M. Rasetti, Phys. Rev. Lett. **79**, 3306 (1997), and references therein.

[5] P. W. Shor, Phys. Rev. A **52**, R2493 (1995); A. Ekert and C. Macchiavello, Phys. Rev. Lett. **77**, 2585 (1996).

[6] A. G. Butkovskiy and Yu. I. Samoilenko, *Control of Quantum Mechanical Processes and Systems* (Kluwer, Dordrecht, 1990).

[7] S. Lloyd, LANL e-print quant-ph/9703042.

[8] A. M. Steane, Phys. Rev. Lett. **77**, 793 (1996); Proc. R. Soc. London, Ser. A **452**, 2551 (1996); Phys. Rev. Lett. **78**, 2252 (1997).

[9] A. R. Calderbank and P. W. Shor, Phys. Rev. A **54**, 1098 (1996); A. R. Calderbank *et al.*, Phys. Rev. Lett. **78**, 405 (1997).

[10] R. Laflamme *et al.*, Phys. Rev. Lett. **7**, 198 (1996); W. H. Zurek and R. Laflamme, *ibid.* **77**, 4683 (1996); E. Knill and T. Laflamme, Phys. Rev. A **55**, 900 (1997).

[11] C. H. Bennett *et al.*, Phys. Rev. A **54**, 3824 (1996); C. H. Bennett *et al.*, Phys. Rev. Lett. **76**, 722 (1996); C. H. Bennett, D. P. DiVincenzo, and J. A. Smolin, *ibid.* **78**, 3217 (1997).

[12] D. Gottesman, Phys. Rev. A **54**, 1862 (1996).

[13] J. I. Cirac *et al.*, Science **273**, 1207 (1996); J. I. Cirac *et al.*, Phys. Rev. Lett. **78**, 3221 (1997).

[14] D. P. DiVincenzo and P. W. Shor, Phys. Rev. Lett. **77**, 3260 (1996).

[15] B. Schumacher, Phys. Rev. A **54**, 2614 (1996); B. Schumacher and M. A. Nielsen, *ibid.* **54**, 2629 (1996).

[16] L. Vaidman, L. Goldenberg, and S. Wiesner, Phys. Rev. A **54**, R1745 (1996).

[17] L. M. Duan and G. C. Guo, Phys. Rev. Lett. **79**, 1953 (1997); Phys. Rev. A **57**, 2399 (1998).

[18] S. Lloyd and J.-J. E. Slotine, Phys. Rev. Lett. **80**, 4088 (1998).

[19] A ''bang-bang'' control is a piecewise constant control with values in the extreme points of the admissible control range, i.e., loosely speaking, a control which at all times utilizes all the control available. See, for instance, Y. Takahashi, M. J. Rabins, and D. M. Auslander, *Control and Dynamic Systems* (Addison-Wesley, Reading, MA, 1970); J. Macki and A. Strauss, *Introduction to Optimal Control Theory* (Springer-Verlag, New York, 1982); L. Lapidus and R. Luus, *Optimal Control of Engineering Processes* (Blaisdell Publishing, Waltham, MA, 1967).

[20] For reviews, see C. Presilla, R. Onofrio, and U. Tambini, Ann. Phys. (N.Y.) **248**, 95 (1996); D. Home and M. A. B. Whitaker, *ibid.* **258**, 237 (1997).

[21] C. P. Slichter, *Principles of Magnetic Resonance*, 3rd ed. (Springer-Verlag, New York, 1990).

[22] E. L. Hahn, Phys. Rev. **80**, 580 (1950).

[23] M. Gell-Mann and J. Hartle, in *Complexity, Entropy, and the Physics of Information*, edited by W. H. Zurek, Santa Fe Institute Studies in the Sciences of Complexity VIII (Addison-Wesley, Redwood City, 1990); Phys. Rev. D **47**, 3345 (1993), and (unpublished).

[24] W. H. Zurek, Phys. Today **44**, 36 (1991); Phys. Rev. D **24**, 1516 (1981); **28**, 1516 (1981).

[25] R. Griffiths, J. Stat. Phys. **36**, 219 (1984).

[26] R. Omnes, J. Stat. Phys. **53**, 893 (1988); **53**, 933 (1988); **53**, 957 (1988).

[27] N. A. Gershenfeld and I. L. Chuang, Science **275**, 350 (1997).

[28] D. G. Cory, M. P. Price, and T. F. Havel, Physica D (to be published); D. G. Cory *et al.*, LANL e-print quant-ph/9802018.

[29] The spin-boson model has had a long history of use in the effort of understanding the tunneling problem in the presence of dissipation. In the Hamiltonian (1), the tunneling matrix element is zero. See A. J. Leggett *et al.*, Rev. Mod. Phys. **59**, 1 (1987).

[30] M. Hillery *et al.*, Phys. Rep. **106**, 121 (1984).

[31] See, for instance, B. L. Hu, J. P. Paz, and Y. Zhang, Phys. Rev. D **45**, 2843 (1992).

[32] D. Vitali, P. Tombesi, and G. J. Milburn, Phys. Rev. Lett. **79**, 2442 (1997), and references therein.

[33] R. Cook, Phys. Scr. **T21**, 49 (1988).

[34] W.-K. Rhim, A. Pines, and J. S. Waugh, Phys. Rev. Lett. **25**, 218 (1970); Phys. Rev. B **3**, 684 (1971).

[35] M. Grigorescu, LANL e-print quant-ph/9709033.

[36] This may happen in the high-temperature and/or strong-coupling limit, or for different classes of environment not considered here. See [3] for supra-Ohmic qubit decoherence.