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Quantitative indicator for semiquantum chaos

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By generalizing to a mixed-state environment the treatment recently given to a model advanced by Cooper *et al.* [Phys. Rev. Lett. **72**, 1337 (1994)], we show that some characteristics of the so-called semiquantum chaos can be described by recourse to a special motion invariant of the problem, that thus becomes a chaos indicator. [S1050-2947(98)09109-4]

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I. INTRODUCTION

In recent years, the study of semiclassical models in which a few classical variables interact with quantum ones has received renewed impetus (see, for instance, Refs. [1–3] and references therein). These systems can be understood, in the $\hbar \rightarrow 0$ limit, either by reference to the effective potential approach [4,5] or by recourse to standard semiclassical treatments (WKB, for example). Such systems, characterized by the coexistence of both classical and quantum degrees of freedom, have been recently employed, for instance, by Bonilla and Guinea, who in such a fashion described measurement processes [6], and also by Pattanayak and Schieve, who studied quantum chaos by recourse to an appropriate, effective classical Hamiltonian [7].

In this paper we shall traverse the road paved by the above-cited works, and investigate the interaction, between a quantum system and a classical one, described by the Hamiltonian

$$\hat{H}=\tfrac{1}{2}\,[\hat{p}^2+P_A^2+\omega^2\hat{x}^2], \tag{1}$$

where $\hat{x}$ and $\hat{p}$ are quantum operators, $\omega^2=m^2+e^2A^2$, m is the mass, and A and P_A are *classical canonical conjugates variables*. This is, indeed, the Hamiltonian studied by Cooper *et al.* [1], that represents the zero-momentum part of the problem of pair production of charged mesons by a strong external electric field [1].

Interestingly enough, these authors encountered chaotic behavior (*semiquantum chaos*) associated with the workings of such a Hamiltonian, and concluded that one has to give up long-term forecasting for quantum-mechanical probabilities [1]. This work of Cooper *et al.* [1] was recently generalized to a fully quantum treatment of the concomitant problem in Ref. [8].

It is our intention here that of generalizing the treatment given in Ref. [1] from a pure-state environment to a mixed-state one, which entails working with density matrices. A discussion of the Gaussian density-matrix formalism can be found in Refs. [9,10]. Additionally, in this Brief Report we wish to show that an appropriately constructed invariant of the motion serves as a quantitative indicator of Cooper *et al.*'s semiquantum chaos [1].

The dynamical evolution (Heisenberg picture) of a quantal operator $\hat{O}$ is the canonical one $d\hat{O}/dt=(i/\hbar)[\hat{H},\hat{O}]$, and we assume that the classical degrees of freedom obey the deterministic classical equations of motion. Following standard procedures (see, for example, the illuminating discussion of Ref. [11]), the energy is taken to coincide with the quantum expectation value of the Hamiltonian, that in turn generates the temporal evolution of the classical variables. Consequently, the classical equations of motion to be used here are well-defined ones. If we take the classical variables to be the position A and the momentum P_A, we write

$$\frac{dA}{dt}=\{\langle\hat{H}\rangle,P_A\},\quad \frac{dP_A}{dt}=\{\langle\hat{H}\rangle,A\}. \tag{2}$$

It is of importance to point out that here one faces an easily solvable set of equations for the description of the time evolution of expectation values (EV's). Ours is a particular instance of that case in which the EV's of, say, q relevant operators are the focus of interest, *and* these operators close, under commutation, a partial Lie algebra with respect to the Hamiltonian $\hat{H}$ of the system. We have then a set of relations of the type [2]

$$[\hat{H}(t),\hat{O}_j]=i\hbar\sum_{i=1}^{q} g_{ij}(t)\hat{O}_i,\quad j=1,2,\ldots,q, \tag{3}$$

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where g_{ij} are the elements of a $q\times q$ matrix G. The generalized Ehrenfest theorem [12] here yields a set of first-order differential equations

$$\frac{d\langle\hat{O}_j\rangle}{dt}=-\sum_{i=1}^{q} g_{ij}(t)\langle\hat{O}_i\rangle,\quad j=1,2,\ldots,q, \tag{4}$$

for the temporal evolution of the EV's of our q relevant operators, which, in turn, will, *for our Hamiltonian* $\hat{H}$, depend on the classical ones through the g_{ji} elements of the matrix G [2]. The time evolution being canonical, all commutation relations are trivially conserved for all times [2].

II. EQUATIONS OF MOTION AND THE INVARIANT *I*

Following Ref. [1], we recast things in terms of adimensional variables $(\hbar=1)$, i.e., $\hat{x}'=m^{1/2}\hat{x}$, $\hat{p}'=\hat{p}/m^{1/2}$, $A'=m^{1/2}A$, and $P_A'=P_A/m^{1/2}$, together with $t'=mt$, $e'=e/m^{3/2}$, and $\omega'=\omega/m$. For the sake of notational simplicity, we afterwards exchange primed and unprimed quantities. The set of operators $\{\hat{x}^2,\hat{p}^2,\hat{L}=\hat{x}\hat{p}+\hat{p}\hat{x}\}$, closes, under commutation, a partial semialgebra of the above-mentioned kind, so that we are led to

$$\frac{d\langle\hat{x}^2\rangle}{dt}=\langle\hat{L}\rangle, \tag{5a}$$

$$\frac{d\langle\hat{p}^2\rangle}{dt}=-\omega^2\langle\hat{L}\rangle, \tag{5b}$$

$$\frac{d\langle\hat{L}\rangle}{dt}=2(\langle\hat{p}^2\rangle-\omega^2\langle\hat{x}^2\rangle), \tag{5c}$$

$$\frac{dA}{dt}=P_A, \tag{5d}$$

$$\frac{dP_A}{dt}=-e^2A\langle\hat{x}^2\rangle, \tag{5e}$$

with

$$\omega^2=1+e^2A^2. \tag{6}$$

One appreciates here the fact that Eqs. (5) constitute an autonomous system of nonlinear coupled equations that governs the time evolution of our relevant variables. Of course, studying the dynamics governed by Eqs. (5) entails working in a mixed-state environment, as only expectation values are involved and one never deals in *direct* fashion with the density matrix, so there is no way of imposing an idempotency requirement. We shall now pay special attention to the quantities

$$I=\langle\hat{x}^2\rangle\langle\hat{p}^2\rangle-\frac{\langle\hat{L}\rangle^2}{4}, \tag{7a}$$

$$E=\tfrac{1}{2}[\langle\hat{p}^2\rangle+P_A^2+\omega^2\langle\hat{x}^2\rangle] \tag{7b}$$

(total energy per unit mass), that happen to be *invariants of the motion* [3]. The former will play a leading role in our considerations. Notice that Eq. (7a) allows for the elimination of one of our quantum variables, as, for instance,

$$\langle\hat{p}^2\rangle=\frac{1}{4\langle\hat{x}^2\rangle}(\langle\hat{L}\rangle^2+4I), \tag{8}$$

which is to be inserted into Eq. (5). For the sake of facilitating comparisons with Ref. [1], it is convenient to use Cooper *et al.*'s notation, i.e., to set $G(t)=\langle\hat{x}^2\rangle(t)$, so that our equations of motion become

$$\frac{1}{2}\left(\frac{\ddot{G}}{G}\right)-\frac{1}{4}\left(\frac{\dot{G}}{G}\right)^2-\frac{I}{G^2}+\omega^2=0, \tag{9a}$$

$$\ddot{A}+e^2GA=0, \tag{9b}$$

which closely resemble the equations of motion of Ref. [1]. Indeed, the above relations become *identical* to their counterparts in Ref. [1], for the *particular case* $I=\frac{1}{4}$. The invariant I will play a leading part in what follows, so that it is of importance to ascertain its significance.

Consider the quantum correlations [2] $\langle\hat{K}_{11}\rangle=\Delta x^2=\langle\hat{x}^2\rangle-\langle\hat{x}\rangle^2$, $\langle\hat{K}_{22}\rangle=\Delta p^2=\langle\hat{p}^2\rangle-\langle\hat{p}\rangle^2$, and $\langle\hat{K}_{12}\rangle=\langle\hat{L}\rangle-2\langle\hat{x}\rangle\langle\hat{p}\rangle$. In terms of them, Heisenberg's uncertainty principle can be cast in the fashion [12]

$$\langle\hat{K}_{11}\rangle\langle\hat{K}_{22}\rangle-\frac{\langle K_{12}\rangle^2}{4}\geqslant\frac{1}{4}. \tag{10}$$

We gather from a comparison of Eq. (10) to the form I given in the first of Eqs. (7), that I values are intimately related to the uncertainty principle. Indeed, minimization of I with the constraint that Eq. (10) adopts its minimum possible value $(\frac{1}{4})$, leads to the value $I=\frac{1}{4}$, investigated by Cooper *et al.* [1]. The case $I=\frac{1}{4}$ (*minimum uncertainty*) corresponds to a pure state with $\langle\hat{N}\rangle=0$. This assertion is immediately verified by writing I in terms of the second quantization operators a and $a^\dagger$ [cf. the ansatz (5) of Ref. [1]], which leads to

$$I=\langle\hat{N}+1/2\rangle^2-\langle\hat{a}^2\rangle\langle(\hat{a}^\dagger)^2\rangle, \tag{11}$$

where $\hat{N}=a^\dagger a$ stands for the number operator, and the Wronskian condition of Eq. (6) in Ref. [1] has been employed. For the initial conditions $\langle\hat{N}\rangle=n$ and $\langle\hat{a}^2\rangle=0$ [or $\langle(\hat{a}^\dagger)^2\rangle=0$], Eq. (11) adopts the appearance $I=(n+1/2)^2$, which shows that I grows in quadratic fashion with the mean number of phonons n.

The natural question to be asked is, thus, what happens for *other* (larger) I values? Providing an answer is the *leitmotif* of the present Brief Report.

I is bounded from below, but possesses an (energy-dependent) upper bound as well, as it is easily seen from Eqs. (7) by considering A, $\langle\hat{L}\rangle$, and $\langle\hat{x}^2\rangle$ as independent variables. One concludes that I varies within the range

$$\frac{1}{4}\leqslant I\leqslant\frac{E^2}{\omega^2}-\frac{\langle\hat{L}\rangle^2}{4}. \tag{12}$$

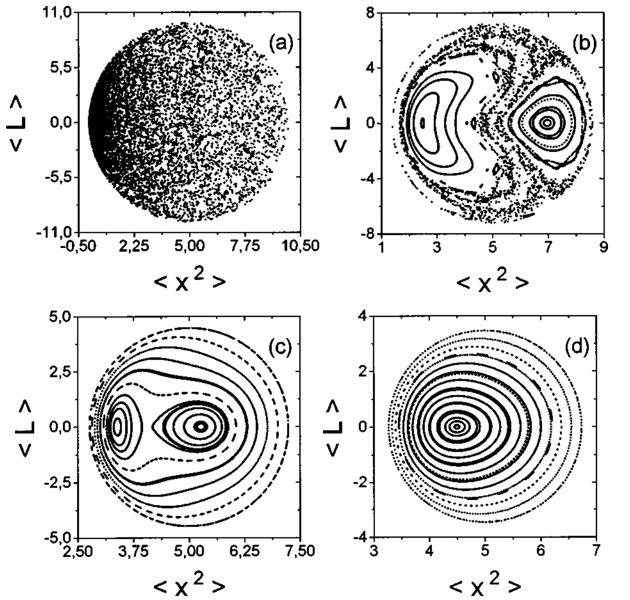

FIG. 1. Poincaré surfaces of section: $\langle \hat{L} \rangle$ vs $\langle \hat{x}^2 \rangle$, for $E=5$, $A(t=0)=0$, and $e=m=1$. I takes the values (a) 0.25 (chaotic regime, minimum uncertainty), (b) 12 (transition region), (c) 20 (regular regime starts becoming evident), and (d) 22 (regular regime). See text for the units of these quantities.

The dynamics described by the set of Eqs. (5) is suitably restricted by the existence of the invariants of the motion (7) to a submanifold of dimension 3, on account of the inequality (12).

Now, the lowest I value investigated by Cooper *et al.* [1] corresponds to a situation that they showed is characterized by semiquantum chaos. One may wonder whether by augmenting I one may not be able to observe a gradual vanishing of the chaotic features. If so, this should be of interest, because the invariant I will provide one with a signature of semiquantum chaos.

Accordingly, the set formed by all possible initial conditions (ic's) becomes subdivided into ic subsets *compatible* with specified fixed values of the invariants I and E. The idea is then to (i) fix an E value and then choose different I values in the pertinent range (12), that determines the phase space segment of interest; (ii) select an initial value for A (which fixes ω), and then choose suitable initial values for $\langle L \rangle$ and $\langle \hat{x}^2 \rangle$; and (iii) let the system evolve and draw graphs[1] $\langle \hat{L} \rangle$ versus $\langle \hat{x}^2 \rangle$.

One finds out that the global[2] degree of ''chaoticity'' (GCD) decreases as I is augmented. Indeed, regular orbits are to be found when I approaches its maximum allowed value ($I_{\max}=E^2$). On the other hand, if we proceed to fix an I value and then choose different E-values in the appropriate range, then the GCD grows with E (as expected).

Changes become noticeable around a ''signal point''

$$I_{\mathcal{P}}=\frac{E^2+1/4}{2} \tag{13}$$

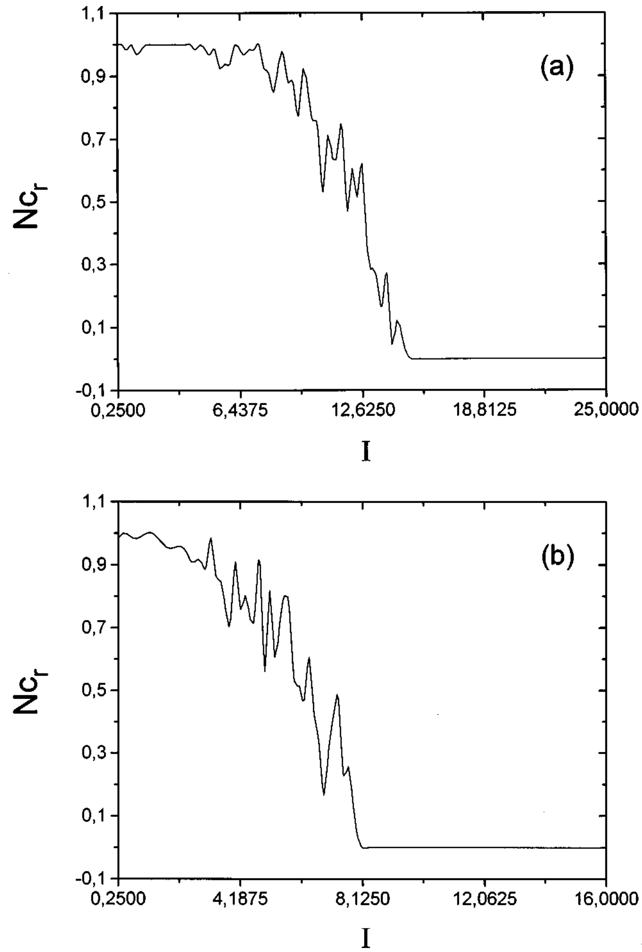

FIG. 2. (a) Relative number of chaotic orbits vs I, as determined by the mutual information criteria given in the second paragraph below Eqs. (16). To this effect the initial condition $\langle \hat{x}^2 \rangle (t=0)$ is subjected to a small shift ($\epsilon=\Delta\langle \hat{x}^2 \rangle=10^{-4}$). We take, as in Fig. 1, $E=5$, $A(t=0)=0$, and $e=m=1$, but here for 100 different values of I. For each I value, we average over 132 orbits, corresponding to distinct, randomly selected initial values ($\langle \hat{L} \rangle$, $\langle \hat{x}^2 \rangle$). (b) Relative number of chaotic orbits vs I, as in Fig. 2(a), but for $E=4$.

(with error bars of the order of a 15%). Roughly speaking, one finds chaos for $I \leqslant I_{\mathcal{P}}$, and ''regular orbits'' for $I > I_{\mathcal{P}}$.

Some relevant Poincaré surfaces (section cuts with $A=0$) are depicted in Fig. 1. All our orbits are enclosed within the region circumscribed by the curve

$$(\langle \hat{x}^2 \rangle - E)^2 + \left(\frac{\langle \hat{L} \rangle}{2} \right)^2 = E^2 - I, \tag{14}$$

a result easily obtained from Eqs. (7). This curve (with $A=0$ and $P_A=0$) represents a stable periodic solution for the system of equations (5). The transition process from ''regular orbits'' to chaotic ones can be clearly appreciated.

III. A MORE QUANTITATIVE CRITERION

In order to discriminate between chaotic and regular orbits, it is advisable at this point to make use of a more quantitative technique, Lyapunov's exponent approach constituting the paramount alternative. However, we found it

[1] Poincaré's surfaces of section.

[2] The meaning of ''global'' is given in the text below Eqs. (16).

convenient here, for simplicity's sake, to employ the indicator of chaos recently advanced in Ref. [13]. Consider two vectors $\mathbf{y}_1(t)$, $\mathbf{y}_2(t)$ in $\mathcal{R}^N$, characteristic of a given orbit, but corresponding to slightly different initial conditions ''1'' and ''2.'' We assume that the difference $(2-1)$ is measured by some suitable small quantity $\epsilon=|\mathbf{y}_2(0)-\mathbf{y}_1(0)|$. The orbits' lengths traversed during an appropriate time T are given by $L_k=\int_0^T dt|d\mathbf{y}_\mathbf{k}/dt|$, $k=1$ and 2, and we denote by $L(1,2)$ the total length of the parametric curve $[\mathbf{y}_1(t),\mathbf{y}_2(t)]$. Consider now the normalized quantities

$$p_k(t)=\frac{1}{L_k}\left|\frac{d\mathbf{y}_\mathbf{k}}{dt}\right|, \quad k=1,2, \tag{15a}$$

$$p(1,2)(t)=\frac{1}{L(1,2)}\left(\frac{d\mathbf{y}_1^2}{dt}+\frac{d\mathbf{y}_2^2}{dt}\right)^{1/2}. \tag{15b}$$

A fundamental tenet of information theory establishes that suitable information measures (IM's) can be associated with these $\{p_k\}$. Following Fraser and Swinney [14], the authors of Ref. [13] made use of the three Shannon IM's (computed up to time T)

$$S_k=-\int_0^T dt\; p_k \ln(p_k), \quad k=1,2, \tag{16a}$$

$$S(1,2)=-\int_0^T dt\; p(1,2)\ln p(1,2), \tag{16b}$$

$$I(1,2)=S(1,2)-\tfrac{1}{2}(S_1+S_2). \tag{16c}$$

The ''mutual information'' $I(1,2)$ can discriminate between regular and chaotic orbits, as demonstrated in Ref. [13] with reference to orbits pertaining to the celebrated Hénon-Heiles potential [15].

After comparing it to other chaos indicators, it was shown in Ref. [13] that $I(1,2)$ constitutes a rather efficient one. One finds that [13] $I(1,2)\ll\epsilon$ gives a ''regular'' regime, and $I(1,2)\gg\epsilon$ yields a ''chaotic'' one. We have computed $I(1,2)$, in the case of our system, for different values of the relevant parameters. Figure 2 illustrates some typical results. We took $\epsilon\approx 10^{-4}$. As global properties are of interest here, we have performed ''phase-space averages'' (mean values over all initial conditions compatible with given values of E and I). Averages over $A(t=0)$ are not performed, so as to be able to compare the results of Fig. 2(a) to these of Fig. 1. In any case, our results are not very sensitive to the $A(t=0)$-value. Figure 2(b) is similar to Fig. 2(a), but a different value of the energy (here $E=4$) is used. Comparison of Figs. 2(a) and 2(b) illustrates the fact that the location of the signal point strongly depends upon the E value.

''Chaoticity'' diminishes as I grows toward its maximum value $I_{\max}=E^2$, where the system of Eqs. (5) attains its only (unique) fixed point ($\langle\hat{x}^2\rangle=I^{1/2}$, $\langle\hat{p}^2\rangle=I^{1/2}$, $\langle\hat{L}\rangle=0$, $A=0$, and $P_A=0$), which is stable. The above-mentioned stable periodic solution goes over to the fixed point for $I=I_{\max}$. The signal points, located at $I\equiv 14.5$ [Fig. 2(a)] and $I\equiv 8.125$ [Fig. 2(b)], indicate the beginning of the ''chaos-free'' zone. We see that a transition region between a zone that exhibits chaos and one free of it can clearly be appreciated.

IV. CONCLUSIONS

Based upon the semiclassical treatment of Cooper *et al.* [1], we have shown here that, in their model, the chaos to chaos-free transition regime can be investigated by recourse to one of the model's invariants, namely, I. This transition region becomes delineated in nitid fashion, I values seemingly yielding the milestones of a route that traverses the road toward chaos.

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