PHYSICAL REVIEW A VOLUME 58, NUMBER 3 SEPTEMBER 1998

Spectral properties of an injected laser

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(Received 21 January 1998)

The theory in which the laser is interpreted as a nonlinear amplifier and filter is applied to the calculation of its spectral density in the case of a homogeneous medium. The method is extended to an injected laser in which the source term includes both the spontaneous emission and the injected signal. Linewidth narrowing or widening, spectral amplification or attenuation are described, using simple semiclassical formulas. [S1050-2947(98)02609-2]

PACS number(s): 42.55.Ah, 42.55.Lt, 42.60.Da

I. INTRODUCTION

Up to now, the laser has essentially been understood as a quantum oscillator and studies of laser linewidths have also essentially been done in the frame of quantum optics [1–5]. A common feature of the traditional models is the use of Lamb's approximation of distributed losses together with the slowly varying envelope approximation (SVEA). The most popular methods that are used in theoretical studies on linewidths are based on Fokker-Planck [6] or Langevin [7,1] equations and the use of Wiener-Kintchine's theorem, which connects the time to the frequency domain. In Ref. [7], for instance, Langevin forces represent the different sources of noise in time-dependent equations of motion for the field, the population difference, and the polarization of the medium. These methods lead to a very small width of the laser line (in fact a δ Dirac distribution) and it is generally admitted that its measurable limit stems from quantum effects due to the phase diffusion of randomly emitted spontaneous photons. To these studies are linked those on noise effects [9], photon statistics [9,8,5], or squeezed light [10]. It should be noted that the formalism of quantum optics is not always used; for instance, the work done by Henry [11] brought a significant understanding of semiconductor laser linewidths by taking a more precise expression for the medium polarization than that obtained by using a simple adiabatic approximation. This method improved the formula previously established by Schawlow and Townes in 1958 [12] in their study of the laser considered below threshold as a narrow band amplifier of noise.

We have recently given [13] a semiclassical theory of the laser transition across the threshold that does not use the SVEA: the frequency domain is split into intervals small compared to the laser linewidth; a classical representation of the fields is used and the losses are localized onto the mirrors. This method allowed us to generalize the usual optical Airy function to the laser: this function was shown to be able to describe the spectral density $y(\nu)$ of the laser as well as its intensity Y, which is the integral of $y(\nu)$ over the frequency domain. The formula explicitly displays the three basic physical effects inside a laser, i.e., the spontaneous and the stimulated emission and the resonance effect of the Fabry-Pérot interferometer. A practical application was developed in the case of an inhomogeneous He-Ne gas laser in order to demonstrate that it was able to quantitatively describe the laser linewidth, below or above threshold. In this theory, the laser is again considered as a narrow-band amplifier of noise, but this time also above threshold. In this study the source of noise was the spontaneous emission only: field, population or pump noises were discarded. The associated spectral density of the spontaneous emission is flat inside the spectral domain of the laser line. Thus an idea to test the predictability capabilities of the laser Airy function is to change this source by injecting an external signal into the laser, i.e., by controlling the source. This external signal, being also a laser light, has a strong spectral variation. It is the aim of this paper to use the Airy function to describe how a master laser imposes its spectral lineshape onto a slave, mode-locked laser. We have previously proposed a method to study the response of injected lasers in the frame of the SVEA and we have experimentally verified its predictions in the case of a gas laser [14]. It has been known for a long time [15] that a master laser can decrease the linewidth of the slave. Again the problem of line shapes in the case of injection locking has been treated in the litterature using essentially quantum theories [16,17] together with Lamb's model of distributed losses and within the SVEA. One result brought by the Airy function is its ability to describe in a simple way this spectral transfer, and how it varies with the laser parameters. In the following we will first develop the theory of the Airy function for a single mode homogeneous laser. Then we will consider two identical lasers: the first plays the role of the master and the second is the slave. A very simple and practical method is described, which allows one to see the transfer of spectral (im)purity from the master onto the slave.

II. THE AIRY FUNCTION FOR A HOMOGENEOUS, SINGLE-MODE LASER

In Ref. [13], the laser Airy function was obtained and applied to the case of an inhomogeneous medium. In this paper, we would like to concentrate on the simpler case of a homogeneous, two-energy-level-medium. Discussions on optical properties of this active medium are thus suppressed. The same fundamental approximations as in Ref. [13] are

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taken: for instance, the field is classical and represents the cavity mean value, but here the source term is not explicitly calculated and is left as a parameter directly linked to the experiment.

The generalized Airy function for the laser [13] describes the *response* $\mathcal{E}_{\rm cav}$ of the *system* ''cavity+active medium'' to its *excitation* by the spontaneous emission $\mathcal{E}_{\rm sp}$ in the frequency domain. This response gives the complex optical Airy function:

$$\mathcal{E}_{\rm cav}=\frac{\mathcal{E}_{\rm sp}}{1-e^{-L+g-i\phi}}. \tag{2.1}$$

This expression links the spectral mean field density $\mathcal{E}_{\rm cav}$ inside the cavity to the spectral mean-field density $\mathcal{E}_{\rm sp}$ corresponding to the spontaneous emission inside the geometrical mode for each frequency ν. It can be demonstrated from first principles using Maxwell equations together with boundary conditions on the mirrors in the frequency domain. If a frequency slice narrow enough around ν is considered, $\mathcal{E}_{\rm cav}$ and $\mathcal{E}_{\rm sp}$ can represent fields instead of field densities. This is usually the case when passive interferometers are probed using fields whose spectral widths are narrow as compared to the width of their Airy function. In Eq. (2.1), $\phi=\phi(\nu)$ is the cumulated round-trip phase. The key in choosing the width of the frequency slice is the possibility of defining a meaningful value for ϕ. One should note that the fundamental property of a Fabry-Pérot interferometer, i.e., the dependence of ϕ upon frequency is simply lost in the SVEA. In Eq. (2.1), $g=g(\nu)$ is the saturated gain and losses are represented by L (one writes $r_1r_2=e^{-L}$ where r_1 and r_2 are the reflectances on the mirrors). We will not discuss here the properties linked to the phase of the field: Taking the modulus square of Eq. (2.1), one obtains the power spectral density. Together with the intensity, this density describes one of the most important properties of light, its Fourier transform being the second-order correlation function. One should be aware that the complex Airy function contains also the physics associated to the random phase appearing in $\mathcal{E}_{\rm sp}$ or in $\mathcal{E}_{\rm cav}$ and the transformation of light statistics by the laser. We leave this subject to another study.

The laser spectral density looks like that which is usually written for a Fabry-Pérot interferometer :

$$y=\frac{S}{[1-e^{-L+g}]^2+4e^{-L+g}\sin^2(\phi/2)}. \tag{2.2}$$

Here $y=y(\nu)$ and $S=S(\nu)$ stand respectively for the spectral densities of the geometrical mean intensity of the internal laser light and of the source. S and y are both normalized by the saturating intensity I_s, which characterizes the medium divided by $c/2d$, the cavity F.S.R. In the case of an empty cavity, the gain $g=0$ and the source S correspond to the externally launched field. In the following we will consider a single-mode laser resonance at (central) angular frequency ω_0 (one uses the usual relation $\omega=2\pi\nu$). The laser line is very narrow as compared to the spontaneous emission line. In the vicinity of ω_0, it is thus legitimate to neglect the variation of the source spectral density S and of the gain g, which become frequency independant when low-frequency noise is discarded. As the saturated index n appears inside an angle (one has $\phi=2\pi\nu 2nd/c$, where d is the geometrical length of the laser), one should take its variation into account. Equation (2.2) imposes always $g<L$ by contrast to the commonly admitted $g=L$ in the stationary regime. It shows that the laser can be interpreted as a nonlinear filter and amplifier acting on the source S.

Let us first consider the simple model of a single-mode, solitary laser with a two-level, homogeneous medium whose linewidth is $2\gamma_{lu}$. This medium is characterized by a mean complex polarizability α, which is written as

$$\alpha=\frac{\mu^2}{\hbar\gamma_{lu}}\frac{\mathcal{L}_\omega D_0}{1+\mathcal{L}_{\omega_q}Y}\left[\frac{\omega-\omega_0}{\gamma_l u}+i\right]. \tag{2.3}$$

Here $\mathcal{L}_\omega$ is the normalized Lorentzian:

$$\mathcal{L}_\omega=\frac{\gamma_{lu}^2}{(\omega-\omega_0)^2+\gamma_{lu}^2}. \tag{2.4}$$

ω_0 is the central angular frequency of the emission line of the active medium. ω_q is the central laser frequency: here $\omega_q=\omega_0$ and $\mathcal{L}_{\omega_q}=1$. D_0 is the population difference and μ the dipolar moment of the transition. Y is the laser intensity (one has $Y=\int y\,d\nu$) normalized by the saturating intensity I_s. The round-trip saturated mean gain is written as

$$g=\frac{\omega}{c}2d\,\frac{\alpha^i}{2\epsilon_0}, \tag{2.5}$$

where the superscript i stands for the imaginary part of α. One can normalize the gain at threshold, which corresponds to the value of D_0 such as $L=g$. This normalized gain is thus written as

$$g=rL\,\frac{\mathcal{L}_\omega}{1+Y}. \tag{2.6}$$

We have adopted the traditional notation where r stands for the small signal gain at threshold: $r=1$ when $g=L$ for $Y=0$ and $\omega=\omega_0$. The cumulated round-trip phase is defined for each frequency ν (or angular frequency ω). It depends upon the saturated index n (or the real part α^r of the polarizability) of the medium and is written as

$$\phi=\frac{\omega}{c}2dn=\frac{\omega}{c}2d\left[1+\frac{\alpha^r}{2\epsilon_0}\right]=\frac{\omega}{c}2d+\frac{\omega-\omega_0}{\gamma_{lu}}g. \tag{2.7}$$

As already stated above, the laser line is very narrow in comparison to the emission line and the variations of the gain, as well as the spontaneous emission, can be neglected inside its frequency range. In this case, the saturated gain is given by $g=g_0/(1+Y)$, where $g_0=rL$ is the small signal gain. As the source term is proportional to the population of the upper level, one can write $S=Kr/(1+Y)$ where K is a constant that contains the geometrical and spectroscopic properties of the spontaneous emission inside the laser mode [13].

Let us consider now Eq. (2.2) and let us write the phase variation when the frequency varies around ν_0. One writes

$$\phi-2q\pi=\frac{\omega-\omega_0}{c}2d+\frac{\omega-\omega_0}{\gamma_{lu}}g, \tag{2.8}$$

where q is an integer that is the order of the lasing mode. Introducing the normalized frequency detuning $x-x_0=[(\omega-\omega_0)/c]2d$ and the phase-amplitude coupling factor $A=1+gc/2d\gamma_{lu}$, one obtains the simple expression

$$\phi-2q\pi=A(x-x_0). \tag{2.9}$$

Around the laser resonance at ν_0, the phase variation remains very small; one writes

$$\sin[\phi/2-q\pi]\simeq\phi/2-q\pi, \tag{2.10}$$

and thus

$$4\sin^2\phi/2\simeq A^2(x-x_0)^2. \tag{2.11}$$

The spectral density of this laser is thus described by

$$y=\frac{S}{[1-e^{-L+g_0/(1+Y)}]^2+e^{-L+g_0/(1+Y)}A^2(x-x_0)^2}. \tag{2.12}$$

y becomes essentially Lorentzian shaped:

$$y=\frac{S}{e^{-L+g_0/(1+Y)}A^2}\frac{1}{\Gamma^2+(x-x_0)^2}, \tag{2.13}$$

where Γ is the half width at half maximum in the angular frequency domain, normalized by $c/2d$:

$$\Gamma=\frac{1-e^{-L+g_0/(1+Y)}}{Ae^{[-L+g_0/(1+Y)]/2}}. \tag{2.14}$$

Integrating y over x gives an equation for Y:

$$Y=\frac{S}{A^2e^{-L+g_0/(1+Y)}}\frac{\pi}{\Gamma}. \tag{2.15}$$

This equation can be numerically solved and the results given below have been obtained using the proper Newton method adapted to the different cases: below (where $Y\sim0$), around (where $Y\ll1$), or above (where $Y\sim r-1$) threshold. However, it is interesting to have an approximate analytical expression for Y above threshold. Such an approximate expression can be used to initiate the Newton loop in numerical calculations. It will be interesting to compare with the usual Shawlow-Townes formula.

Above threshold ($r>1$), the source S becomes a very small quantity as compared to Y and the physically acceptable solution for Y is certainly very close to the usual solution Y_L given by the equation $g=L$ (gain=losses), i.e., $Y_L=r-1$. One can thus write a first-order development in the small term δY such as

$$Y=Y_L+\delta Y \tag{2.16}$$

and develop Eq. (2.15) in Y around Y_L. One finds at first order in S:

$$Y=Y_L+\frac{r^2}{(r-1)}\frac{\pi S}{g_0A} \tag{2.17}$$

and

$$\Gamma=\frac{\pi S}{A^2Y_L}. \tag{2.18}$$

In these equations, the intensity that appears in A and S is Y_L. As already remarked in Ref. [13] the expression (2.18) for Γ is more precise but has the same structure as that previously given by Shawlow and Townes [12]. The variations of the intensity Y and the linewidth versus r give the laser characteristic curves: we will use them in Figs. 1 and 3 when r varies between 0 and 4. When drawn in logarithmic units, these curves clearly show the thresholds for amplification (at $r=0$) and for laser action (at $r=1$). One notes that the Airy function, Eq. (2.2), can also be used when $r<0$. It would describe the spectral density associated to an absorbing source (a candle, for instance) inside a Fabry-Pérot interferometer.

III. THE INJECTED LASER

Let us now consider a second laser injected by a master [14]. For simplicity we assume both lasers to be identical and tuned to the central frequency ν_0. We prefer to focus on a typical well-defined case: The generalization is obvious. Inside the injected laser, the source term includes both the spontaneous emission as before and the injected field whose spectral density is ηy_1 where η is a given constant. Again we insist on the fact that we do not consider any other source of noise than the spontaneous emission. We obtain the equation for the spectral density y_2:

$$y_2=\frac{\eta y_1+S_2}{e^{-L+g_0/(1+Y_2)}}\frac{1}{\Gamma_2^2+(x-x_0)^2}. \tag{3.1}$$

In this equation indices 1 and 2 respectively refer to the first (master) and second (slave) laser with the same meaning for the symbols as given before [however, Γ_2 is no longer the spectral width of y_2, it stands for an abreviation analogous to Eq. (2.14)]. Integrating y_2 in Eq. (3.1) over x gives an equation for the total intensity Y_2:

$$Y_2=\frac{1}{A_2^2e^{-L_2+g_0/(1+Y_2)}}\frac{1}{\Gamma_2}\left[\pi S_2+\frac{\eta Y_1}{\Gamma_1+\Gamma_2}\right]. \tag{3.2}$$

Equations (2.12) and (2.15) as well as (3.1) and (3.2) represent the basic laser properties.

Equation (3.2) can be numerically solved (using again a Newton method) and expression (3.1) for y_2 can be used to draw spectral profiles and to deduce the spectral width of the injected laser. Below we give two examples where both lasers are identical but have a different gain: The characteristic curves for each laser alone are first computed with losses corresponding to $L=\ln[1/r_1r_2]$ with $r_1r_2=0.81$. The source constant K in S can be quantitatively computed [13] but here K is simply taken to give a (non-normalized) linewidth corresponding to 20 kHz for $r=4$; the free spectral range is 1.5

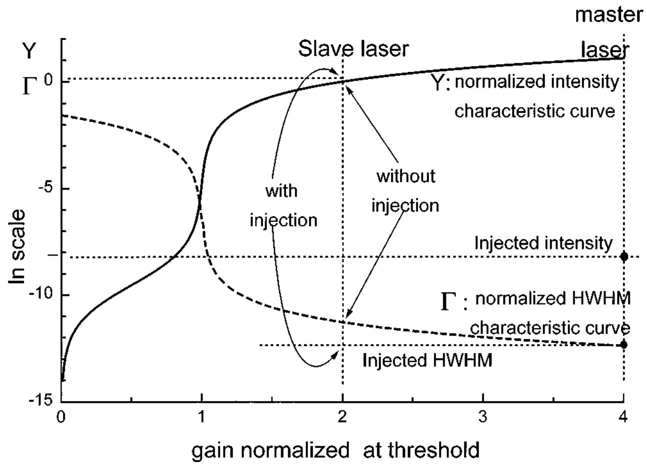

FIG. 1. Laser characteristic curves as computed from Eqs. (2.12) and (2.15) and data given in text. Full line: normalized intensity; dashed line: half width at half maximum. Note the ln scale, which allows the representation of the variations of intensity and HWHM over several orders of magnitude when the gain varies from 0 to 4. Working points of the master laser correspond to $r=4$. Working points of the slave laser correspond to $r=2$ and are shown with and without injection. One observes only a small increase of the intensity of the laser under injection but a large decrease of its HWHM down to that of the master.

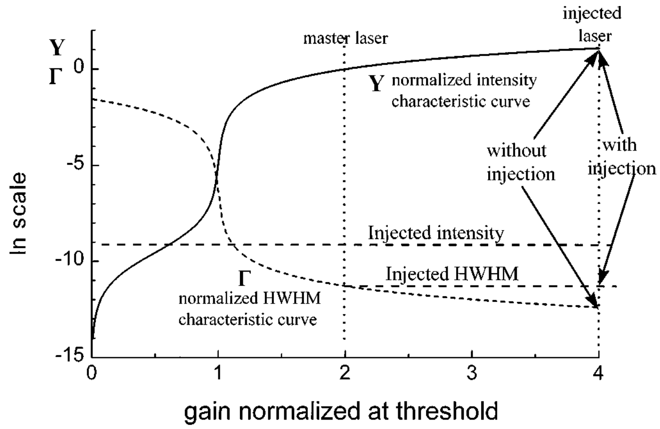

FIG. 3. Same as Fig. 1 with the roles of master and slave lasers inverted. Here the master has a gain $r=2$ and the slave $r=4$. Again the total intensity of the slave does not change very much under injection while its HWHM increases up to the value of that of the slave.

GHz. Here we deliberately take the case where the injected signal is much bigger ($\simeq 100$ times) than the natural source term.

In the first case, the master has a gain $r=4$, a half-linewidth (HWHM) $\Gamma_1=4.2\times10^{-6}$ and an intensity $Y_1 \simeq 3.\Gamma_1$ is expressed in rad/s normalized by $c/2d$. This signal is divided by 10^4 and injected into the slave whose gain is $r=2$. Figure 1 shows the working points of both lasers on their characteristic curves. Under injection the HWHM of the slave decreases from $\Gamma_2=1.25\times10^{-5}$ down to Γ_1 and its intensity Y_2 increases from 1 to 1.16. The spectral densities of the solitary and injected laser are displayed on Fig. 2. Together with the line narrowing, one observes a spectacular increase of the signal between the points noted *A* and *B*. This increase is obtained at the expense of a decrease outside of this range. This decrease has been experimentally observed [17]. In the second case, the roles of both lasers are inverted: The master has a gain $r=2$, a linewidth $\Gamma_1=1.25\times10^{-5}$, and an intensity $Y_1\simeq1$. This signal is again divided by 10^4 and injected into the slave whose gain is $r=4$. Figure 3 shows the working points of both lasers together with the same characteristic curves as in Fig. 1. Under injection the HWHM of the slave increases from 4.2×10^{-6} to 1.25×10^{-5} and its intensity from 3 to 3.11. Figure 4 displays the spectral profiles of the solitary and injected laser. The situation is the converse of the preceding case: here the solitary laser is much more intense in the central region between *A* and *B*. This result shows that injection locking of a laser onto another means far more than a simple slaving frequency effect, i.e., a slaving onto the spectral properties of the master, this includes the line widening effects.

IV. CONCLUSION

The conclusion that can be drawn from these results is that the master laser imposes its spectral distribution to the slaved laser [18]. The laser alone works with a very small source term [term *S* in Eq. (2.12)] and the intensity adjusts

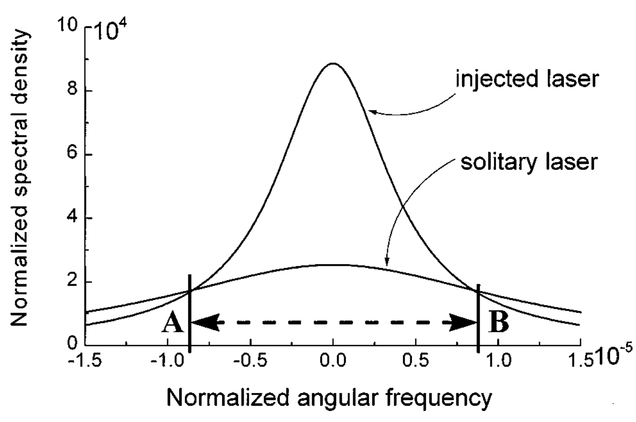

FIG. 2. Spectral lines of the laser working at $r=2$ in Fig. 1. As seen in Fig. 1, the total intensity does not change very much when the laser is injected or not. However, its spectral distribution is completely modified by injection: it considerably increases inside the range between *A* and *B* and decreases outside.

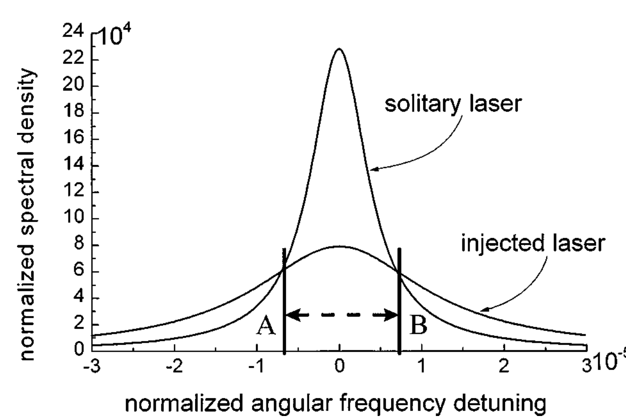

FIG. 4. Spectral lines of the laser working at $r=4$ as shown in Fig. 3. Here the spectral distribution considerably decreases under injection in the range between *A* and *B* and increases outside.

itself for the gain to *almost* compensate for the losses, which *almost* cancels the denominator in Eq. (2.12). This is no longer realized in the injected laser whose properties are essentially fixed by those of the source. This result is not surprising and we acknowledge the fact that many physicists having studied the subject have already intuitively understood what happens if one injection locks a single-mode laser onto a noisy signal.

The results presented here reinforce the interpretation of the laser as a nonlinear amplifier and filter whose Airy function is fundamental to explain basic properties. It should be noted that this interpretation is not opposite to the usual one, which states that the laser linewidth results from the wandering of the phase, which finds no reference to lock on. The Airy function thus fixes only a rate to this wandering. When the laser is locked, a phase reference together with a rate of change (i.e., a linewidth) is given to it. One should note that a frequency locking is always accompanied by a phase locking but one can have a phase locking, or a transfer of spectral width, without having any frequency locking. An experiment has been made in our laboratory in order to verify this point of view [19].

The Airy function is more precise than the slowly varying envelope approximation and can be used to study various phenomena such as line shapes asymmetries, bistabilities in injection-locking domains, or an analysis of a given laser lineshape using another laser. It bridges also the gap with the description of optical passive dispersion or absorption bistabilities. The effects of an amplitude classical noise can then easily be added into the expression of the source and the complex gain. The transformation of the phase probability distribution function of the source by the complex Airy function for the field can be done: It gives a complementary physical interpretation to the transformation of Gaussian into Poisson statistics associated to the electromagnetic field. Finally this semiclassical study gives also a good basis for a further understanding of optical quantum phenomena [20].

[1] M. Sargent, M. O. Scully, and W. Lamb, Jr., in *Laser Physics* (Addison-Wesley, New York, 1974).

[2] K. Ujihara, IEEE J. Quantum Electron. **20**, 814 (1984).

[3] P. Goldberg, P. W. Milonni, and B. Sundaram, Phys. Rev. A **44**, 1969 (1991); **44**, 4556 (1991).

[4] S. Prasad, Phys. Rev. A **46**, 1540 (1992).

[5] L. Mandel and E. Wolf, *Optical Coherence and Quantum Optics* (Cambridge University Press, New York, 1995).

[6] R. D. Hempstead and M. Lax, Phys. Rev. **161**, 350 (1967).

[7] M. I. Kolobov, L. Davidovitch, E. Giacobino, and C. Fabre, Phys. Rev. A **47**, 1431 (1993).

[8] F. T. Arecchi and V. Degiorgio, Phys. Rev. A **3**, 1108 (1971).

[9] H. A. Hauss, J. Opt. Soc. Am. B **12**, 2019 (1995).

[10] See special issue on squeezed light: J. Opt. Soc. Am. B **4**, 1450 (1987).

[11] C. H. Henry, IEEE J. Quantum Electron. **18**, 259 (1982); **19**, 1391 (1983).

[12] A. L. Schawlow and C. H. Townes, Phys. Rev. **112**, 1940 (1958).

[13] G. M. Stéphan, Phys. Rev. A **55**, 1371 (1997).

[14] P. Even, K. Ait-Ameur, and G. M. Stéphan, Phys. Rev. A **55**, 1441 (1997).

[15] L. E. Erickson and A. Szabo, Appl. Phys. Lett. **18**, 443 (1971).

[16] H. A. Haus and Y. Yamamoto, Phys. Rev. A **29**, 1261 (1984).

[17] S. Inoue, S. Machida, Y. Yamamoto, and H. Ohzu, Phys. Rev. A **48**, 2230 (1993).

[18] Since the completion of the present work, we have experimentally verified the transfer of the spectral width of the master laser onto the slave. Semiconductor lasers have been used [M. Bondiou, R. Gabet, P. Besnard, and G. M. Stéphan (unpublished)].

[19] M. Bondiou, P. Besnard, M. L. Chares, R. Gabet, and G. M. Stéphan (unpublished).

[20] M. J. Freeman, D. C. Kilper, D. G. Steel, R. Graig, and D. R. Scifres, Opt. Lett. **20**, 183 (1995).