Solitary waves in clouds of Bose-Einstein condensed atoms

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We consider the conditions under which solitary waves can exist in elongated clouds of Bose-Einstein condensed atoms. General expressions are derived for the velocity, characteristic size, and spatial profile of solitary waves, and the low- and high-density limits are examined. [S1050-2947(98)06709-2]

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Clouds of Bose-Einstein condensed atoms in elongated traps provide excellent conditions for investigating the propagation of essentially one-dimensional sound pulses under [1]. In previous work, propagation of pulses in such traps was considered in the Thomas-Fermi approximation, and it was demonstrated that pulses propagate at a speed of $(\bar{n}U_0/m)^{1/2}$ in the linear regime. Here, $U_0 = 4\pi\hbar^2 a_{sc}/m$ is the effective two-body interaction matrix element, $\bar{n}$ is the particle density averaged over the cross section of the cloud [2,3], m is the atomic mass, and a_{sc} is the scattering length for atom-atom collisions. Nonlinear effects were also investigated in Ref. [3] and were found to be important for conditions of experimental relevance. The effects of dispersion were neglected in this study since the length scales of interest were much larger than the superfluid coherence length, which sets the scale on which dispersive effects become important. It is of interest to include the effects of dispersion and to investigate the possible existence of solitary waves in these systems.

Solitary waves have been studied in many different physical contexts [4,5]. We shall not attempt to review the extensive literature for the Gross-Pitaevskii equation, the nonlinear Schrödinger equation that describes motion of a Bose-Einstein condensate, apart from mentioning the pioneering work of Zakharov and Shabat [6] on solitary waves in bulk systems. There are a number of studies for clouds of Bose-Einstein condensed atoms in a trap. Morgan *et al.* [7] considered a class of solitary wave solutions that are, in effect, center-of-mass oscillations of the stationary states of the time-independent Gross-Pitaevskii equation.

It is possible experimentally to make very elongated clouds of Bose-Einstein condensed atoms in magnetic or optical traps, and in this paper we examine the possibility of solitary waves propagating along the axis of such a cloud. The solutions we seek are the analogs of the solitary waves first observed by Scott Russell on his historic horse ride along the canal [8]. However, an important difference between canals and Bose-Einstein condensed clouds is that, whereas it is often a good approximation to assume that the depth of water in a canal is constant and thus the sound speed is independent of position, this is a poor approximation in Bose-Einstein condensed clouds, where the density, and hence also the sound speed, vary significantly in directions transverse to the axis of the trap. Consequently, the

conditions under which one can find solitary wave solutions for Bose-Einstein condensed clouds is more restricted than for canals. We wish to consider in this paper situations where the variation of cloud properties along the axis of the trap may be neglected. In the directions perpendicular to the axis, we shall assume that the cloud is confined by a harmonic trapping potential. However, to set the stage and establish notation, let us first consider the case of a homogeneous medium and then generalize our discussion to atoms in traps that confine particles in two directions.

The condensate wave function Ψ satisfies the Gross-Pitaevskii equation,

$$i\hbar \frac{\partial \Psi}{\partial t} = \left(-\frac{\hbar^2 \nabla^2}{2m} + U_0 |\Psi|^2 + V \right) \Psi, \tag{1}$$

where V is the external potential, which may be taken to be zero for a homogeneous medium. We write the condensate wave function as $\Psi = n^{1/2} e^{i\phi}$, where n is the particle density, which is spatially uniform in equilibrium and in the absence of an external potential. To obtain equations resembling those of hydrodynamics, it is convenient to work in terms of a superfluid velocity, $\mathbf{v} = \hbar \nabla \phi / m$. We consider solutions that depend on a single coordinate, z. From the real and imaginary parts of the Gross-Pitaevskii equation one obtains two equations, the equation of continuity

$$\frac{\partial n}{\partial t} = -\frac{\partial (nv)}{\partial z}, \tag{2}$$

and the equation for the phase,

$$\hbar \frac{\partial \phi}{\partial t} = -nU_0 - \frac{1}{2}mv^2 + \frac{\hbar^2}{2mn^{1/2}} \frac{\partial^2 n^{1/2}}{\partial z^2}. \tag{3}$$

When differentiated with respect to z, the latter gives the generalized Euler equation,

$$m\frac{\partial v}{\partial t} = -\frac{\partial}{\partial z}\left(\mu(n) + \frac{1}{2}mv^2 - \frac{\hbar^2}{2mn^{1/2}} \frac{\partial^2 n^{1/2}}{\partial z^2} \right), \tag{4}$$

where we have introduced the chemical potential, μ. For a dilute Bose gas, $\mu(n) = nU_0$. We seek solutions to these equations for which the fluid velocity and particle density propagate at a uniform velocity, u, without change of form,

i.e., they depend on z and t only through the combination $z - ut$. Thus, one may write $\partial v/\partial t = -u(\partial v/\partial z)$ and $\partial n/\partial t = -u(\partial n/\partial z)$. Far away from the solitary wave, the condensate is at rest and has its equilibrium density, n_0. With these boundary conditions, the continuity equation, Eq. (2), gives

$$v = u\left(1 - \frac{n_0}{n}\right). \tag{5}$$

Also, Eq. (4) can be written as

$$-mu\frac{\partial v}{\partial z} = -\frac{\partial}{\partial z}\left(\mu(n) + \frac{1}{2}mv^2 - \frac{\hbar^2}{2mn^{1/2}}\frac{\partial^2 n^{1/2}}{\partial z^2}\right). \tag{6}$$

We integrate this equation to obtain

$$\frac{\hbar^2}{2mn^{1/2}}\frac{\partial^2 n^{1/2}}{\partial z^2} = (n-n_0)U_0 + \frac{m}{2}(v-u)^2 - \frac{m}{2}u^2. \tag{7}$$

Here, we have added the integration constant $-n_0U_0$ to impose the boundary condition $n\to n_0$ at infinity. Combining Eqs. (5) and (7), we obtain a differential equation for n. If we multiply this equation by $\partial n^{1/2}/\partial z$ and integrate with respect to z, we find

$$\frac{\hbar^2}{2m}\left(\frac{\partial n^{1/2}}{\partial z}\right)^2 = (nU_0 - mu^2)\frac{(n-n_0)^2}{2n}. \tag{8}$$

We note that the phase has the more general form $\phi(z,t) = \phi_1(z-vt) + \phi_2(t)$, which can be seen from the equation $\partial\phi/\partial z = mv/\hbar$. Thus, Eq. (3) can be written as

$$\hbar\frac{\partial\phi_2}{\partial t} = \frac{\hbar^2}{2mn^{1/2}}\frac{\partial^2 n^{1/2}}{\partial z^2} - nU_0 + \frac{m}{2}(v-u)^2 - \frac{m}{2}u^2. \tag{9}$$

Since the left side of this equation is a function of t and the right is a function of $z-ut$, each must be equal to a constant. This constant must equal $-n_0U_0$ in order to satisfy the boundary condition $n\to n_0$ at infinity. Thus, $\phi_2(t) = -n_0U_0t$ is a linear function of time.

We see from Eq. (8) that the condition $nU_0 - mu^2 \geq 0$ must be satisfied in order to obtain real solutions. In other words, the density must exceed the minimum value n_{min},

$$n_{min} = \frac{mu^2}{U_0}. \tag{10}$$

To obtain solutions that are localized in space, n must lie between n_{min} and n_0. This has a ready interpretation in terms of one-dimensional motion of a classical particle whose spatial coordinate is proportional to $n^{1/2}$ if z is regarded as the time variable. The classical potential is then proportional to $-(n-n_{min})(n-n_0)^2$, and the solitary wave solutions correspond to an oscillation of the "particle" from $n=n_0$ to $n = n_{min}$ and back again. Thus, solitary waves for this problem are depressions in the density. In contrast, solitary waves in canals correspond to elevations of the surface of the water. We also note that the velocity of the wave is equal to the sound speed at the minimum density, n_{min}. The sound speed in a uniform gas with density n_0 is given by $c^2 = n_0U_0/m$. It follows from Eq. (10) that
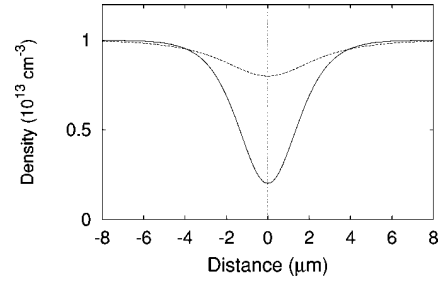

FIG. 1. Density profiles of solitary waves as a function of space, as given by Eq. (12). Here the background density of atoms is $n_0 = 10^{13}$ cm^{-3} and therefore for Na atoms $\xi(n_0) \approx 1.2$ μm. For the solid curve, the ratio $n_{min}/n_0 = 20\%$ and $\zeta \approx 1.9$ μm; for the dashed curve, $n_{min}/n_0 = 80\%$ and $\zeta \approx 3.8$ μm.

$$\frac{n_{min}}{n_0} = \frac{u^2}{c^2}. \tag{11}$$

Integrating Eq. (8) we find that the profile of the wave is given by

$$n(z) = n_{min} + (n_0 - n_{min})\tanh^2(z/\zeta), \tag{12}$$

where $\zeta = 2^{1/2}\xi(n_0)[1 - (n_{min}/n_0)]^{-1/2}$ and $\xi(n_0)$ is the coherence length corresponding to the background density n_0, $\xi(n_0) = (8\pi n_0 a_{sc})^{-1/2}$. Therefore, ζ, which gives the spatial extent of the solitary wave, is on the order of the coherence length that corresponds to the background density. Figure 1 shows the profile of two solitary waves, $n(z)$, for Na atoms with a background density of $n_0 = 10^{13}$ cm^{-3}, and for two values of n_{min}/n_0, 20% (solid curve) and 80% (dashed curve).

For typical densities in the MIT experiment [1] ($n \approx 10^{14}$ cm^{-3} at the center of the cloud), ξ is $\approx 0.2-0.4$ μm with $a_{sc} = 27.5$ Å for Na atoms [9]. However, in order to be able to observe solitary waves, it is desirable to look at lower densities than these, so that the coherence length becomes sufficiently large that structures can be resolved by optical means. For nonzero n_{min} the solutions (12) correspond to what are termed "gray solitons" in Ref. [10]. Finally, we point out that, if $u=0$ ($n_{min}=0$), Eq. (12) gives the well-known kink solution $\Psi = n_0^{1/2}\tanh(z/\zeta)$ [11], which is sometimes referred to as a "dark soliton" [10,12].

We turn now to the more realistic problem of atoms in a trapping potential that is harmonic in the transverse direction and for which there is no restoring force along the axis of the trap, which we take to be the z axis. We assume that the transverse dimension of the cloud is so small that the time scale for adjustment of the transverse profile of the particle density to the equilibrium form appropriate for the instantaneous number of particles per unit length is small compared with the time for the pulse to pass a given point. Later, we shall investigate what this condition means quantitatively. The problem becomes one-dimensional, and the solitary pulse may be specified in terms of a local velocity, $v(z)$, and a local density of particles per unit length, $\sigma(z)$,

$$\sigma(z) = \int dx\,dy\,|\Psi(x,y,z)|^2. \tag{13}$$

Here, x and y are coordinates perpendicular to the axis of the trap. With this assumption, the wave function may be written in the form

$$\Psi(\mathbf{r},t) = f(z,t)g(x,y,\sigma), \qquad (14)$$

where g is the equilibrium wave function for the transverse motion; $g(x,y,\sigma)$ depends on time implicitly through the time dependence of σ. We choose g to be normalized so that $\int |g|^2 dx dy = 1$, and therefore from Eqs. (13) and (14), $|f|^2 = \sigma$.

It is convenient to derive the equations for f and g from a variational principle. The Ginzburg-Pitaevskii equation may be derived by requiring stationarity of the action

$$\mathcal{S} = -\int \frac{i\hbar}{2}\left(\Psi^* \frac{\partial \Psi}{\partial t} - \Psi \frac{\partial \Psi^*}{\partial t}\right) d\mathbf{r} dt$$
$$+ \int \left(\frac{\hbar^2}{2m}|\nabla\Psi|^2 + \frac{1}{2}U_0|\Psi|^4 + V|\Psi|^2\right) d\mathbf{r} dt. \quad (15)$$

Using Eq. (14) for Ψ, we can write Eq. (15) as

$$\mathcal{S} = -\int \frac{i\hbar}{2}\left((fg)^* \frac{\partial (fg)}{\partial t} - fg \frac{\partial (fg)^*}{\partial t}\right) d\mathbf{r} dt$$
$$+ \int \left[\frac{\hbar^2}{2m}\left(\left|g\frac{\partial f}{\partial z}\right|^2 + |f\nabla g|^2\right)\right] d\mathbf{r} dt \qquad (16)$$
$$+ \int \left[\frac{1}{2}U_0|fg|^4 + V|fg|^2\right] d\mathbf{r} dt.$$

Terms containing $(\partial f/\partial z)(\partial g/\partial z)$ vanish because of the normalization condition on g. Minimizing Eq. (16) with respect to g^*, we find that g obeys the equation

$$-\frac{\hbar^2 \nabla^2}{2m}g + Vg + U_0|fg|^2 g + \frac{\hbar^2}{2m|f|^2}\left|\frac{\partial f}{\partial z}\right|^2 g = \mu(\sigma)g, \qquad (17)$$

where μ is the chemical potential. We neglect the last term on the left side of Eq. (17) since, as we show below, the characteristic length of pulses is sufficiently long that it is negligible in all cases of interest. For the same reason the term $f\nabla g$ in Eq. (16), which equals $f\nabla_\perp g + \hat{\mathbf{z}}f\partial g/\partial z$, is approximately equal to $f\nabla_\perp g$, where $\nabla_\perp = \hat{\mathbf{x}}\partial/\partial x + \hat{\mathbf{y}}\partial/\partial y$, with $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ being the unit vectors in the directions x, y, and z, respectively. Thus, Eq. (17) takes the final form

$$-\frac{\hbar^2 \nabla_\perp^2}{2m}g + Vg + U_0|fg|^2 g = \mu(\sigma)g. \qquad (18)$$

We now minimize Eq. (16) with respect to f^* and find

$$i\hbar \frac{\partial f}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 f}{\partial z^2} + \left(\frac{\hbar^2}{2m}\int |\nabla_\perp g|^2 dx dy\right) f$$
$$+ U_0\left(\int |g|^4 dx dy\right)|f|^2 f + \left(\int |g|^2 V dx dy\right)f. \qquad (19)$$

Following the same procedure as for the homogeneous medium, we assume that $f = \sigma^{1/2}e^{i\phi}$, with the velocity field associated with f being $v = (\hbar/m)\partial\phi/\partial z$. Again, we obtain hydrodynamic equations for σ and v, which are the same as Eqs. (2) and (4) but with n replaced by σ and μ given in the present case by

$$\mu(\sigma) = \left(\frac{\hbar^2}{2m}\int |\nabla_\perp g|^2 dx dy\right)$$
$$+ \left(\int |g|^2 V dx dy\right) + U_0\left(\int |g|^4 dx dy\right)\sigma. \quad (20)$$

The first term on the right side of Eq. (20) is the kinetic energy in the transverse direction, the second is the potential energy due to the confining potential, and the third is the energy due to interactions between atoms. Assuming that σ and v are functions of $z - ut$ only with u constant, we find that

$$\frac{\hbar^2}{2m}\left(\frac{\partial \sigma^{1/2}}{\partial z}\right)^2 = [\epsilon(\sigma) - \epsilon(\sigma_0)] - \mu(\sigma_0)(\sigma - \sigma_0)$$
$$- mu^2 \frac{(\sigma - \sigma_0)^2}{2\sigma}, \qquad (21)$$

where $\epsilon(\sigma) = \int_0^\sigma \mu(\sigma')d\sigma'$ is the energy per unit length. In Eq. (21), we have imposed the boundary conditions $\sigma = \sigma_0$ and $v = 0$ far away from the disturbance.

There is no simple expression for the energy per unit length for general values of σ. To obtain analytical results, we explore some limiting cases for the experimentally relevant situation where the confining potential in the transverse directions is harmonic and rotationally invariant, and given by $V = m\omega_\perp^2(x^2 + y^2)/2$. In the low-density regime, the interaction energy can be treated perturbatively and the problem reduces essentially to the one-dimensional case treated above. As we shall show below, in this limit $\epsilon(\sigma) \propto \sigma(1 + a_{sc}\sigma)$. In the high-density limit, the Thomas-Fermi approximation applies [13], and $\epsilon(\sigma)$ varies as $\sigma^{3/2}$.

Before examining the two cases separately, we estimate the density, σ_c, at which the crossover between the two limits occurs. This corresponds to the condition that the interaction energy per particle is on the order of the oscillator energy in the transverse direction. The interaction energy per particle is nU_0. If A is the cross section of the cloud and $R_\perp$ is the corresponding radius, $n = \sigma/A$. Therefore, the condition determining σ_c is $U_0\sigma_c/A \sim \hbar\omega_\perp$. Denoting the characteristic length scale for the ground state of a particle in the transverse confining potential by $a_\perp = (\hbar/m\omega_\perp)^{1/2}$ and assuming that $R_\perp = a_\perp$, the cross section of the cloud is $A = \pi a_\perp^2$; thus one finds $\sigma_c \sim a_{sc}^{-1}$, which gives $\sigma_c \approx 4 \times 10^6$ cm^{-1} for Na atoms. Alternatively, we can determine σ_c by equating the interaction energy per particle and the kinetic energy of the atoms due to their confinement in the transverse direction, $\sigma_c U_0/A \sim \hbar^2/(mA)$, which gives the same result for σ_c. This expression for σ_c is equivalent to the condition that the coherence length, ξ, be comparable to the transverse dimension of the cloud, $R_\perp$.

We now examine the low-density limit, $\sigma \ll \sigma_c$, in which the interaction energy can be treated perturbatively. To find

the differential equation that g satisfies in this limit, we neglect the interaction-energy term (the third term on the left side) in Eq. (18). The last term on the left side is $\sim \hbar^2/(2m\zeta^2)(\delta f/f)^2$, where δf is the deviation of f, which is much less than $\hbar^2/(2ma_\perp^2)$ because of the condition $\zeta \gg a_\perp$. Therefore, this term is negligible, and g thus satisfies the Schrödinger equation in a harmonic potential,

$$-\frac{\hbar^2 \nabla_\perp^2}{2m} g + Vg = \mu g, \qquad (22)$$

which means that $|g| \propto \exp[-(x^2+y^2)/2a_\perp^2]$. From Eq. (20), we find that $\mu(\sigma) = \hbar \omega_\perp (1 + 2a_{sc}\sigma)$. The first term in μ is the ground-state energy of the harmonic oscillator and the second is the interaction energy. Thus, $\epsilon(\sigma) = \hbar \omega_\perp \sigma(1 + a_{sc}\sigma)$ in the low-density limit, and Eq. (21) takes the form

$$\frac{\hbar^2}{2m}\left(\frac{\partial \sigma^{1/2}}{\partial z}\right)^2 = \left(\frac{\sigma}{2A}U_0 - mu^2\right)\frac{(\sigma - \sigma_0)^2}{2\sigma}, \qquad (23)$$

by analogy with Eq. (8). The results for the profile and the width of the cloud are analogous to the one-dimensional case, $u^2 = \sigma_{min}U_0/(2mA)$, where σ_{min} is the minimum of σ associated with the solitary wave and

$$\sigma(z) \approx \sigma_{min} + (\sigma_0 - \sigma_{min})\tanh^2(z/\zeta). \qquad (24)$$

For this problem,

$$\zeta = 2\xi(n_0)[1 - (\sigma_{min}/\sigma_0)]^{-1/2} \qquad (25)$$

or $\zeta = 2^{1/2}\xi(n_0/2)[1 - (\sigma_{min}/\sigma_0)]^{-1/2}$ with $n_0 = \sigma_0/A$. From Eq. (23), we see that the density that determines u^2 is $\sigma_{min}/2A$ and the one that determines ζ is $\sigma_0/2A$. The factor of 1/2 in these results compared with the analogous results for the homogeneous case is due to the average of the equilibrium density of atoms across the trap, $n(x,y)$, given by $\int n^2(x,y)dxdy/\int n(x,y)dxdy$. Its origin is thus completely different from that of the factor of 1/2 that occurs in the Thomas-Fermi limit [3]. The latter is $\int n(x,y)dxdy/[n(0,0)\int dxdy]$. It is amusing that the sound speed expressed in terms of the density of particles on the axis of the trap is given by precisely the same expression, $c = [n(0,0)U_0/(2m)]^{1/2}$, in both the high- and low-density limits for harmonic transverse trapping potentials.

We argued above that the motion would be quasi-one-dimensional if the time scale for adjustments in the transverse structure of the cloud is short compared with the time for passage of the pulse, and we now check the consistency of this assumption. In the low-density regime, $\sigma \ll \sigma_c$, the characteristic time for adjustment of the profile is on the order of $\omega_\perp^{-1}$, while the time scale for passage of the pulse is at least of order ξ/c. The ratio of these scales is $nU_0/\hbar\omega_\perp$, which is much less than unity in this regime, and therefore our assumption is consistent.

Another case that can be solved analytically is that of small-amplitude solitary waves for arbitrary densities. We expand $\epsilon(\sigma)$ to order $(\sigma - \sigma_0)^3$ and write Eq. (21) as

$$\frac{\hbar^2}{2m}\left(\frac{\partial \sigma^{1/2}}{\partial z}\right)^2 \approx \frac{1}{2}\epsilon''(\sigma_0)(\sigma - \sigma_0)^2 + \frac{1}{6}\epsilon'''(\sigma_0)(\sigma - \sigma_0)^3$$
$$-mu^2\frac{(\sigma - \sigma_0)^2}{2\sigma}, \qquad (26)$$

where the prime denotes differentiation with respect to σ. Clearly, two roots of the right side of the above equation are equal to σ_0. The remaining root gives an expression for the velocity u as a function of σ_0 and σ_{min},

$$u^2 \approx \frac{\sigma_{min}}{m}\left(\epsilon''(\sigma_0) + \frac{\epsilon'''(\sigma_0)}{3}(\sigma_{min} - \sigma_0)\right). \qquad (27)$$

Because $\epsilon''(\sigma)$ depends on σ, the velocity of propagation of the wave is not generally equal to the bulk velocity at the minimum density, except in the low-density limit, when $\epsilon''(\sigma)$ is independent of σ. Integrating Eq. (26), we can determine the profile of the solitary wave, $\sigma(z)$. For small values of $\sigma - \sigma_0$, Eq. (26) takes the simple form

$$\frac{\partial \sigma^{1/2}}{\partial z} \approx \frac{(\sigma - \sigma_{min})^{1/2}}{l}\left(1 - \frac{\sigma}{\sigma_0}\right), \qquad (28)$$

where the length l is given by $l^2 = \hbar^2/\{m\sigma_0[\epsilon''(\sigma_0) + \sigma_0\epsilon'''(\sigma_0)/3]\}$. Again assuming that σ is close to σ_0, we find that

$$\sigma(z) \approx \sigma_{min} + (\sigma_0 - \sigma_{min})\tanh^2(z/\zeta) \qquad (29)$$

with $\zeta = l[1 - (\sigma_{min}/\sigma_0)]^{-1/2}$.

We now consider the calculation of $\epsilon(\sigma)$ in the high-density limit where the kinetic-energy term is negligible [13]. We also neglect the last term on the left of Eq. (18). The consistency of this assumption will be checked below. With these approximations, we write the Thomas-Fermi equation for g,

$$Vg + U_0|fg|^2g = \mu g. \qquad (30)$$

Thus, $|g|^2$ is a parabola, $|g|^2 \propto [1 - (x^2+y^2)/R_\perp^2]$. This form for $|g|$ implies that the potential energy is equal to $m\omega_\perp^2 R_\perp^2/6$ and that the interaction energy is equal to $4U_0\sigma/(3\pi R_\perp^2)$. From Eq. (20) we see that $\mu(\sigma)$ is the sum of these two terms in this limit. To find the explicit dependence of μ on σ, we calculate $R_\perp$ [3] now. The density of atoms, $n(x,y,z)$, has the functional form of $|g|^2$,

$$n(x,y,z) = n(0,0,z)\left(1 - \frac{x^2+y^2}{R_\perp^2}\right), \qquad (31)$$

where $n(0,0,z)$ is the density on the axis of the trap. Thus, the number of particles per unit length is given by

$$\sigma(z) = \int n(x,y,z)dxdy = \frac{1}{2}n(0,0,z)\pi R_\perp^2. \qquad (32)$$

In the Thomas-Fermi approximation where the kinetic energy is negligible, the sum of the chemical potential $\mu = n(x,y,z)U_0$ plus the potential energy V is a constant, and therefore the density on the axis of the trap is given by

$$n(0,0,z)U_0 = \tfrac{1}{2} m \omega_\perp^2 R_\perp^2 . \tag{33}$$

Thus, we find that $R_\perp^2 = 4a_\perp^2 (\sigma a_{\rm sc})^{1/2}$. Equation (20) then implies that $\mu(\sigma) = 2\hbar \omega_\perp (\sigma a_{\rm sc})^{1/2}$ and therefore $\epsilon(\sigma) = (4/3)\hbar \omega_\perp (a_{\rm sc}\sigma^3)^{1/2}$. The width, ζ, of the solitary waves is therefore given by $\zeta = (12/5)^{1/2}\xi(n_0)[1 - (\sigma_{\rm min}/\sigma_0)]^{-1/2}$, where $n_0 = \sigma_0/(\pi R_\perp^2)$.

In considering the limits of validity of our calculation in the high-density regime, we observe that the time scale for adjustment of the profile of the pulse is $\sim R_\perp/c$ and that the time for passage of the pulse is $\sim \zeta/c$. If the motion is to be essentially one-dimensional, $R_\perp$ must be much smaller than ζ. In the Thomas-Fermi approximation, $R_\perp$ is larger than $a_\perp$ due to the repulsive interactions between the atoms. This can be seen from the formula for the radius $R_\perp = 2a_\perp (\sigma a_{\rm sc})^{1/4}$ derived above. To satisfy the condition $\zeta \gg R_\perp$, the quantity $\sigma a_{\rm sc}$ must be large compared with $(3/10)^2 (\sigma_0 a_{\rm sc})^{-1}[1 - (\sigma_{\rm min}/\sigma_0)]^{-2}$. Thus, if $\delta\sigma = \sigma_0 - \sigma_{\rm min}$ is the amplitude of the disturbance, we see that $\delta\sigma/\sigma_0 \ll 3/(10\sigma_0 a_{\rm sc})$. This indicates that the amplitude of the solitary wave, $\delta\sigma$, must be extremely small if the one-dimensional approximation is to be valid when $\sigma \gg \sigma_c$, and therefore our small-amplitude treatment given above is applicable. For the experimental conditions of Ref. [1], the number of particles in the trap is $N \sim 5 \times 10^6$, and the length of the trap in the axial direction L is ~ 450 μm, implying that $\sigma_0 \sim 10^8$ cm^{-1}. Validity of the one-dimensional approximation therefore requires that $\delta\sigma/\sigma_0 \ll 0.01$. Finally, we check the consistency of the assumption made in deriving the Thomas-Fermi equation for g. The last term on the left of Eq. (18) has an upper bound of $\hbar^2/(2m\zeta^2)$. This term is approximately equal to $\hbar^2/(2m\xi^2)(\delta\sigma/\sigma_0) \ll \hbar^2/(2m\xi^2) \sim nU_0$. Therefore, this term is indeed negligible.

Our calculations indicate that the most favorable conditions for observing solitary waves in trapped Bose-condensed gases occur when the density is sufficiently low to permit a perturbative treatment of particle interactions. In this case, one-dimensional behavior persists even for large-amplitude solitary waves. Low-density systems have the further advantage that the coherence length is correspondingly large. Since the coherence length determines the size of solitary waves, this simplifies resolution of these structures. However, a lower density of particles makes the detection of these effects more difficult because of the lower signal.

We now wish to estimate the experimental conditions required for observation of solitary waves in the low-density regime. The spatial resolution of current experiments using direct imaging methods is ~ 4 μm [14]. As a theoretical estimate for the width of the solitary wave, we take the full width of the dip in the density profile at half the maximum depression, that is, the distance between points where $\sigma = (\sigma_0 + \sigma_{\rm min})/2$. This is given by $2\zeta \tanh^{-1}(1/\sqrt{2}) \approx 3.5\xi(n_0)[1 - (\sigma_{\rm min}/\sigma_0)]^{-1/2}$, which must exceed the experimental resolution if solitary waves are to be observable. This leads to the condition

$$\xi(n_0) \gtrsim [1 - (\sigma_{\rm min}/\sigma_0)]^{1/2} \ \mu\text{m}, \tag{34}$$

or $\xi(n_0) \gtrsim 1$ μm for solitary waves with $\sigma_{\rm min}/\sigma_0 \ll 1$. For the coherence length to exceed 1 μm, the density per unit volume must be less than $n_{\rm obs} \approx 10^{13}$ cm^{-3}. The correspond-

ing number of particles per unit length σ is $n_{\rm obs}A \approx n_{\rm obs}\pi a_\perp^2$. For a transverse trapping frequency of 240 Hz, which is a typical value of the MIT trap [1,15], the oscillator length, $a_\perp$, in this direction is ~ 1.4 μm, implying that the number of particles per unit length must be less than 8×10^5 cm^{-1} for structures to be observable. This value corresponds to $\sigma_c/5$, and therefore the coupling would be weak. Let us now calculate the number of particles that should be used in the MIT trap for σ to have the value 8×10^5 cm^{-1}. The kinetic energy along the axial direction is typically of order $\hbar^2/2mL^2$, and the interaction energy is $\sim \sigma U_0/(\pi a_\perp^2)$. The ratio of these quantities is $\sim (a_\perp/L)^2 (8\sigma a_{\rm sc})^{-1}$. Thus for $L \gg a_\perp/(8\sigma a_{\rm sc})^{1/2}$, the kinetic energy along the z direction is negligible, and the Thomas-Fermi approximation [13] may be used to determine the structure along the axis of the trap. We shall assume this to be the case and subsequently will check the consistency of this assumption. Under these conditions, the Thomas-Fermi approximation may be used to calculate the z dependence of σ, even though it may not be used to calculate the structure of the cloud in the transverse directions. In the presence of a confining potential in the z direction, the Thomas-Fermi condition is that the sum of the chemical potential and the z-dependent part of the trapping potential should be constant. Since the chemical potential is $\mu = \hbar \omega_\perp (1 + 2\sigma a_{\rm sc})$ in the low-density limit, this condition is

$$\hbar \omega_\perp [1 + 2\sigma(z)a_{\rm sc}] + \tfrac{1}{2} m \omega_z^2 z^2 = \text{const}, \tag{35}$$

where ω_z is the frequency of the trapping potential along the z axis. We equate the value of the left side of Eq. (35) at $z = 0$ to the value at $z = Z$, where Z is the distance from the center of the cloud to its edge, $Z = L/2$. Since σ vanishes at the edges of the cloud, we get

$$2\hbar \omega_\perp \sigma(z=0)a_{\rm sc} = \tfrac{1}{2} m \omega_z^2 Z^2, \tag{36}$$

where $\sigma(z=0)$ is the number of particles per unit length at the center of the cloud. Solving the above equation for Z, we find

$$Z = 2\frac{a_z^2}{a_\perp}[\sigma(z=0)a_{\rm sc}]^{1/2} = 2\sqrt{\pi}(n_{\rm obs}a_{\rm sc})^{1/2}a_z^2, \tag{37}$$

where $a_z = (\hbar/m\omega_z)^{1/2}$ is the oscillator length in the axial direction. From Eqs. (35) and (37) we see that the number of particles per unit length can be written as

$$\sigma(z) = \sigma(z=0)\left(1 - \frac{z^2}{Z^2}\right). \tag{38}$$

The total number of particles is $N = \int_{-Z}^{Z} \sigma(z)dz$; thus from Eqs. (37) and (38) we find

$$N = \frac{4}{3}\sigma(z=0)Z = \frac{8\pi^{3/2}}{3}n_{\rm obs}^{3/2}(a_z a_\perp)^2 a_{\rm sc}^{1/2}. \tag{39}$$

For the MIT trap, the frequency in the axial direction is ~ 7.9 Hz [15] and therefore $a_z \sim 7.5$ μm; the total number of particles, N, should be equal to $\sim 2 \times 10^3$ if $\sigma(z=0)$ is to be

8×10^5 cm^{-1}. The total width of the trap in the z direction would then be $L\approx38$ μm. Returning to our assumption, we see that for $\sigma=\sigma_c/5$ (the highest value of σ for which these structures are observable), the condition $L\gg a_\perp/(8\sigma a_{sc})^{1/2}$ implies that L must be much larger than $0.8a_\perp$. This is indeed true and the Thomas-Fermi approximation is valid along the z axis of the cloud.

The development of traps with stronger transverse confinement, such as the optical dipole trap of the MIT group [16], promises to facilitate experiments on solitary waves since, for a given value of the dimensionless coupling σa_{sc}, the corresponding densities will be higher. However, the disadvantage is that the higher density will result in shorter length scales for structures.

In this paper we have discussed the propagation of solitary waves in elongated traps and have estimated characteristic sizes of these structures. Clearly, many questions remain. These include the stability of these structures, the role they play in other phenomena (such as dissipation), and the question of how pulses propagate when the motion is not quasi-one-dimensional.

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