

Relativistic Levinson theorem in two dimensions

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In the light of the generalized Sturm-Liouville theorem, the Levinson theorem for the Dirac equation in two dimensions is established as a relation between the total number n_j of the bound states and the sum of the phase shifts $\eta_j(\pm M)$ of the scattering states with the angular momentum j :

$$\eta_j(M) + \eta_j(-M) = \begin{cases} (n_j+1)\pi & \text{when a half bound state occurs at } E=M \text{ and } j=3/2 \text{ or } -1/2 \\ (n_j+1)\pi & \text{when a half bound state occurs at } E=-M \text{ and } j=1/2 \text{ or } -3/2 \\ n_j\pi & \text{the remaining cases.} \end{cases}$$

The critical case, where the Dirac equation has a finite zero-momentum solution, is analyzed in detail. A zero-momentum solution is called a half-bound state if its wave function is finite but does not decay fast enough at infinity to be square integrable. [S1050-2947(98)11109-5]

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I. INTRODUCTION

The Levinson theorem [1] is an important theorem in the quantum scattering theory, which sets up the relation between the number of bound states and the phase shift at zero momentum. It has been generalized [2–9] and applied to different fields in modern physics [10–16]. Recently, the Levinson theorem in two dimensions was studied both in experimental [17] and theoretical [18–20] aspects because of the wide interest in the lower dimensional field theories.

In this paper we will study the Levinson theorem for the Dirac equation in two dimensions:

$$\sum_{\mu=0}^2 i\gamma^\mu(\partial_\mu + ieA_\mu)\psi = M\psi, \quad (1)$$

where M is the mass of the particle, and

$$\gamma^0 = \sigma_3, \quad \gamma^1 = i\sigma_1, \quad \gamma^2 = i\sigma_2. \quad (2)$$

Throughout this paper the natural units $\hbar = c = 1$ are employed. Discuss the special case where only the zero component of A_μ is nonvanishing and cylindrically symmetric:

$$A_1 = A_2 = 0, \quad eA_0 = V(r). \quad (3)$$

The boundary condition at the origin for the potential $V(r)$ is necessary for the nice behavior of the wave function

$$\int_0^1 r|V(r)|dr < \infty. \quad (4)$$

For simplicity, we first discuss the case where the potential $V(r)$ is a cutoff one at a sufficiently large radius r_0 :

$$V(r) = 0 \quad \text{when } r \geq r_0. \quad (5)$$

The general case where the potential $V(r)$ has a tail at infinity will be discussed in Sec. V.

Introduce a parameter λ for the potential $V(r)$:

$$V(r, \lambda) = \lambda V(r). \quad (6)$$

As λ increases from zero to one, the potential $V(r, \lambda)$ changes from zero to the given potential $V(r)$. If λ changes its sign, the potential $V(r, \lambda)$ changes sign, too.

Letting

$$\psi_{jE}(t, \mathbf{r}, \lambda) = e^{-iEt} r^{-1/2} \begin{pmatrix} f_{jE}(r, \lambda) e^{i(j-1/2)\varphi} \\ g_{jE}(r, \lambda) e^{i(j+1/2)\varphi} \end{pmatrix}, \quad (7)$$

where j denotes the total angular momentum, $j = \pm 1/2, \pm 3/2, \dots$, we obtain the radial equations:

$$\frac{d}{dr} g_{jE}(r, \lambda) + \frac{j}{r} g_{jE}(r, \lambda) = [E - V(r, \lambda) - M] f_{jE}(r, \lambda), \quad (8)$$

$$-\frac{d}{dr} f_{jE}(r, \lambda) + \frac{j}{r} f_{jE}(r, \lambda) = [E - V(r, \lambda) + M] g_{jE}(r, \lambda).$$

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It is easy to see that the solutions with a negative j can be obtained from those with a positive j by interchanging $f_{jE}(r, \lambda) \leftrightarrow g_{-j-E}(r, -\lambda)$, so that in the following we only discuss the solutions with a positive j . The main results for the case with a negative j will be indicated in the text.

The physically admissible solutions are finite, continuous, vanishing at the origin, and square integrable:

$$f_{jE}(r, \lambda) = g_{jE}(r, \lambda) = 0 \quad \text{when } r \rightarrow 0, \quad (9)$$

$$\int_0^\infty dr \{ |f_{jE}(r, \lambda)|^2 + |g_{jE}(r, \lambda)|^2 \} < \infty. \quad (10)$$

The solutions for $|E| > M$ describe the scattering states, and those for $|E| \leq M$ describe the bound states. We will solve Eq. (8) in two regions, $0 \leq r < r_0$ and $r_0 < r < \infty$, and then match two solutions at r_0 by the match condition:

$$A_j(E, \lambda) \equiv \frac{f_{jE}(r, \lambda)}{g_{jE}(r, \lambda)} \Big|_{r=r_0-} = \frac{f_{jE}(r, \lambda)}{g_{jE}(r, \lambda)} \Big|_{r=r_0+}. \quad (11)$$

When r_0 is the zero point of $g_{jE}(r, \lambda)$, the match condition can be replaced by its inverse $g_{jE}(r, \lambda)/f_{jE}(r, \lambda)$ instead. The merit of using this match condition is that we need not care about the normalization factor in the solutions.

The establishment of the Levinson theorem for the Dirac equation is similar to that for the Schrödinger equation [20]. The main differences between them are that the ratio f/g of two radial functions in the Dirac problem plays the role of

the logarithmic derivative R'/R in the Schrödinger problem, and the energy E of bound states satisfies $|E| \leq M$ instead of $E \leq 0$. With the increment of the strength of the potential $V(r, \lambda)$, the scattering state may turn into a bound state at the energy M or $-M$ and the bound state may also turn into a scattering state at the energies.

The key point for the proof of the Levinson theorem is that the ratio $f_{jE}(r, \lambda)/g_{jE}(r, \lambda)$ is monotonic with respect to the energy E , which is called the generalized Sturm-Liouville theorem [21] and will be proved in Sec. II. Based on the generalized Sturm-Liouville theorem, in Sec. III the number of bound states will be related with the variance of the ratio at r_0- as the potential changes. In Sec. IV, we further prove that this variance of the ratio also determines the sum of the phase shifts at the energies $\pm M$. In the course of the proof, it can be seen evidently that as the potential changes, the phase shift at the energy M jumps by π while a scattering state of a positive energy becomes a bound state, and the phase shift at the energy $-M$ jumps by $-\pi$ while a bound state becomes a scattering state of a negative energy, or vice versa. The critical case, where the Dirac equation has a finite zero-momentum solution, will be studied in Sec. IV in detail. A zero-momentum solution is called a half-bound state if its wave function is finite but does not decay fast enough at infinity to be square integrable. Thus, the Levinson theorem relates the number n_j of bound states with angular momentum j to the sum of phase shifts $\eta_j(\pm M)$ with j at the energies $\pm M$:

$$\eta_j(M) + \eta_j(-M) = \begin{cases} (n_j + 1)\pi & \text{when a half bound state occurs at } E = M \text{ and } j = 3/2 \text{ or } -1/2, \\ (n_j + 1)\pi & \text{when a half bound state occurs at } E = -M \text{ and } j = 1/2 \text{ or } -3/2, \\ n_j\pi & \text{the remaining cases.} \end{cases} \quad (12)$$

The problem that the potential has a tail at infinity will be discussed in Sec. V.

II. GENERALIZED STURM-LIOUVILLE THEOREM

Suppose that f, g and f_1, g_1 are two solutions of Eq. (8) with the energies E and E_1 , respectively. From Eq. (8) we have

$$\frac{d}{dr}(f_1g - g_1f) = -(E_1 - E)(f_1f + g_1g). \quad (13)$$

From the boundary condition that both solutions vanish at the origin, we integrate Eq. (13) in the region $0 \leq r \leq r_0$ and obtain

$$(f_1g - g_1f)|_{r=r_0-} = -(E_1 - E) \int_0^{r_0} (f_1f + g_1g) dr.$$

Taking the limit as E_1 tends to E , we have

$$\begin{aligned} \lim_{E_1 \rightarrow E} \frac{f_1g - g_1f}{E_1 - E} \Big|_{r=r_0-} &= \{g_{jE}(r_0, \lambda)\}^2 \frac{\partial}{\partial E} A_j(E, \lambda) \\ &= - \int_0^{r_0} \{f_{jE}^2(r, \lambda) + g_{jE}^2(r, \lambda)\} dr < 0, \end{aligned} \quad (14)$$

where we denote the solution f and g by $f_{jE}(r_0, \lambda)$ and $g_{jE}(r_0, \lambda)$. Thus, when $|E| \geq M$ we have

$$A_j(E, \lambda) = A_j(M, \lambda) - c_1^2 k^2 + \dots$$

when $E > M$ and $E \sim M$,

$$A_j(E, \lambda) = A_j(-M, \lambda) + c_2^2 k^2 + \dots$$

when $E < -M$ and $E \sim -M$, (15)

where c_1^2 and c_2^2 are non-negative numbers, and the momentum k is defined as follows:

$$k = (E^2 - M^2)^{1/2}. \quad (16)$$

Similarly, from the boundary condition that the radial functions $f_{jE}(r, \lambda)$ and $g_{jE}(r, \lambda)$ for $|E| \leq M$ tend to zero at infinity, we obtain by integrating Eq. (13) in the region $r_0 \leq r < \infty$

$$\begin{aligned} & \{g_{jE}(r_0, \lambda)\}^2 \frac{\partial}{\partial E} \left(\frac{f_{jE}(r, \lambda)}{g_{jE}(r, \lambda)} \right) \Big|_{r=r_0+} \\ &= \int_{r_0}^{\infty} [f_{jE}^2(r, \lambda) + g_{jE}^2(r, \lambda)] dr > 0. \end{aligned} \quad (17)$$

Thus, as the energy E increases, the ratio $f_{jE}(r, \lambda)/g_{jE}(r, \lambda)$ at r_0- , i.e., $A_j(E, \lambda)$, decreases monotonically, but the ratio $f_{jE}(r, \lambda)/g_{jE}(r, \lambda)$ at r_0+ when $|E| \leq M$ increases monotonically. This is called the generalized Sturm-Liouville theorem [21].

III. NUMBER OF BOUND STATES

Now, we solve Eq. (8) for the energy $|E| \leq M$. In the region $0 \leq r < r_0$, when $\lambda = 0$, we have

$$\begin{aligned} f_{jE}(r, 0) &= e^{-i(j-1/2)\pi/2} \{(M+E)\pi\kappa r/2\}^{1/2} J_{j-1/2}(i\kappa r), \\ g_{jE}(r, 0) &= e^{-i(j-3/2)\pi/2} \{(M-E)\pi\kappa r/2\}^{1/2} J_{j+1/2}(i\kappa r), \end{aligned} \quad (18)$$

where $J_m(x)$ is the Bessel function, and

$$\kappa = (M^2 - E^2)^{1/2}. \quad (19)$$

The ratio at $r = r_0-$ when $\lambda = 0$ is

$$\begin{aligned} A_j(E, 0) &= -i \left(\frac{M+E}{M-E} \right)^{1/2} \frac{J_{j-1/2}(i\kappa r_0)}{J_{j+1/2}(i\kappa r_0)} \\ &= \begin{cases} -\frac{2M(2j+1)}{\kappa^2 r_0} \sim -\infty & \text{when } E \sim M, \\ -\frac{2j+1}{2Mr_0} & \text{when } E \sim -M. \end{cases} \end{aligned} \quad (20)$$

In the region $r_0 < r < \infty$, due to the cutoff potential we have $V(r) = 0$ and

$$\begin{aligned} f_{jE}(r, \lambda) &= e^{i(j+1/2)\pi/2} \{(M+E)\pi\kappa r/2\}^{1/2} H_{j-1/2}^{(1)}(i\kappa r), \\ g_{jE}(r, \lambda) &= e^{i(j+3/2)\pi/2} \{(M-E)\pi\kappa r/2\}^{1/2} H_{j+1/2}^{(1)}(i\kappa r), \end{aligned} \quad (21)$$

where $H_m^{(1)}(x)$ is the Hankel function of the first kind. The ratio at $r = r_0+$ does not depend on λ and is given as follows:

$$\frac{f_{jE}(r, \lambda)}{g_{jE}(r, \lambda)} \Big|_{r=r_0+} = -i \left(\frac{M+E}{M-E} \right)^{1/2} \frac{H_{j-1/2}^{(1)}(i\kappa r_0)}{H_{j+1/2}^{(1)}(i\kappa r_0)} = \begin{cases} \frac{2Mr_0}{2j-1} & \text{when } E \sim M \text{ and } j \geq 3/2, \\ -2Mr_0 \ln(\kappa r_0) \sim \infty & \text{when } E \sim M \text{ and } j = 1/2, \\ \frac{\kappa^2 r_0}{2M(2j-1)} \sim 0 & \text{when } E \sim -M \text{ and } j \geq 3/2, \\ -\frac{\kappa^2 r_0 \ln(\kappa r_0)}{2M} \sim 0 & \text{when } E \sim -M \text{ and } j = 1/2. \end{cases} \quad (22)$$

It is evident from Eqs. (20) and (22) that as the energy E increases from $-M$ to M , there is no overlap between two variant ranges of the ratio at two sides of r_0 when $\lambda = 0$ (no potential) except for $j = 1/2$ where there is a half-bound state at $E = M$. The half-bound state will be discussed in the next section.

As λ increases from zero to one, the potential $V(r, \lambda)$ changes from zero to the given potential $V(r)$, and $A_j(E, \lambda)$ changes, too. If $A_j(M, \lambda)$ decreases across the value $2Mr_0/(2j-1)$ as λ increases, an overlap between the variant ranges of the ratios at two sides of r_0 appears. Since the ratio $A_m(E, \lambda)$ of two radial functions at r_0- decreases monotonically as the energy E increases, and the ratio at r_0+ increases monotonically, the overlap means that there must be one and only one energy where the matching condition (11) is satisfied, namely a bound state appears.

As λ increases, $A_j(M, \lambda)$ may decrease to $-\infty$, jump to ∞ , and then decrease again across the value $2Mr_0/(2j-1)$,

so that another bound state appears. Note that when r_0 is a zero point of the wave function $g_{jE}(r, \lambda)$, $A_j(E, \lambda)$ goes to infinity. It is not a singularity.

On the other hand, as λ increases, if $A_j(-M, \lambda)$ decreases across zero, an overlap between the variant ranges of the ratios at two sides of r_0 disappears so that a bound state disappears.

Therefore, each time $A_j(M, \lambda)$ decreases across the value $2Mr_0/(2j-1)$ as λ increases, a new overlap between the variant ranges of the ratios at two sides of r_0 appears such that a scattering state of a positive energy becomes a bound state. On the other hand, each time $A_j(-M, \lambda)$ decreases across zero, an overlap between the variant ranges of the ratio at two sides of r_0 disappears such that a bound state becomes a scattering state of a negative energy. Conversely, each time $A_j(M, \lambda)$ increases across the value $2Mr_0/(2j-1)$, an overlap between the variant ranges disappears such that a bound state becomes a scattering state of a positive

energy, and each time $A_j(-M, \lambda)$ increases across zero, a new overlap between the variant ranges appears such that a scattering state of a negative energy becomes a bound state.

Now, the number n_j of bound states with the angular momentum j is equal to the sum (or subtraction) of four times as λ increases from zero to one: the times that $A_j(M, \lambda)$ decreases across the value $2Mr_0/(2j-1)$, minus the times that $A_j(M, \lambda)$ increases across the value $2Mr_0/(2j-1)$, minus the times that $A_j(-M, \lambda)$ decreases across zero, plus the times that $A_j(-M, \lambda)$ increases across zero.

When $j=1/2$, the value $2Mr_0/(2j-1)$ becomes infinity. We may check the times that $A_j(M, \lambda)^{-1}$ increases (or decreases) across zero to replace the times that $A_j(M, \lambda)$ decreases (or increases) across infinity.

IV. RELATIVISTIC LEVINSON THEOREM

We turn to a discussion of the phase shifts of the scattering states. Solving Eq. (8) in the region $r_0 < r < \infty$ for the energy $|E| > M$, we have

$$f_{jE}(r, \lambda) = B(E) \left(\frac{\pi kr}{2} \right)^{1/2} \{ \cos \eta_j(E, \lambda) J_{j-1/2}(kr) - \sin \eta_j(E, \lambda) N_{j-1/2}(kr) \},$$

$$- \sin \eta_j(E, \lambda) N_{j-1/2}(kr) \}, \tag{23}$$

$$g_{jE}(r, \lambda) = \left(\frac{\pi kr}{2} \right)^{1/2} \{ \cos \eta_j(E, \lambda) J_{j+1/2}(kr) - \sin \eta_j(E, \lambda) N_{j+1/2}(kr) \},$$

where $N_m(x)$ denotes the Neumann function, the momentum k is given in Eqs.(16), and $B(E)$ is defined as

$$B(E) = \begin{cases} \left(\frac{E+M}{E-M} \right)^{1/2} & \text{when } E > M, \\ - \left(\frac{|E|-M}{|E|+M} \right)^{1/2} & \text{when } E < -M. \end{cases} \tag{24}$$

The asymptotic form of the solution (23) at $r \rightarrow \infty$ is

$$f_{jE}(r, \lambda) \sim B(E) \cos[kr - j\pi/2 + \eta_j(E, \lambda)], \tag{25}$$

$$g_{jE}(r, \lambda) \sim \sin[kr - j\pi/2 + \eta_j(E, \lambda)].$$

Substituting Eq. (23) into the match condition (11), we obtain the formula for the phase shift $\eta_j(E, \lambda)$:

$$\begin{aligned} \tan \eta_j(E, \lambda) &= \frac{J_{j+1/2}(kr_0)}{N_{j+1/2}(kr_0)} \frac{A_j(E, \lambda) - B(E) J_{j-1/2}(kr_0) / J_{j+1/2}(kr_0)}{A_j(E, \lambda) - B(E) N_{j-1/2}(kr_0) / N_{j+1/2}(kr_0)} \\ &= \frac{J_{j-1/2}(kr_0)}{N_{j-1/2}(kr_0)} \frac{\{A_j(E, \lambda)\}^{-1} - B(E)^{-1} J_{j+1/2}(kr_0) / J_{j-1/2}(kr_0)}{\{A_j(E, \lambda)\}^{-1} - B(E)^{-1} N_{j+1/2}(kr_0) / N_{j-1/2}(kr_0)}. \end{aligned} \tag{26}$$

The phase shift $\eta_j(E, \lambda)$ is determined up to a multiple of π due to the period of the tangent function. We use the convention that the phase shifts for the free particles [$V(r)=0$] are vanishing:

$$\eta_j(E, 0) = 0. \tag{27}$$

Under this convention, the phase shifts $\eta_j(E)$ are determined completely as λ increases from zero to one:

$$\eta_j(E) \equiv \eta_j(E, 1). \tag{28}$$

The phase shifts $\eta_j(\pm M, \lambda)$ are the limits of the phase shifts $\eta_j(E, \lambda)$ as E tends to $\pm M$. At the sufficiently small k , $k \ll 1/r_0$, we have

$$\tan \eta_j(E, \lambda) \sim \begin{cases} - \frac{\pi (kr_0/2)^{2j+1}}{(j+1/2)!(j-1/2)!} \frac{A_j(M, \lambda) - 2M(2j+1)/(k^2 r_0)}{A_j(M, \lambda) - c_1^2 k^2 - \frac{2Mr_0}{2j-1} \left(1 + \frac{(kr_0)^2}{(2j-1)(2j-3)} \right)} & \text{when } j > 3/2, \\ - \frac{\pi}{2} \left(\frac{kr_0}{2} \right)^4 \frac{A_j(M, \lambda) - 8M/(k^2 r_0)}{A_j(M, \lambda) - c_1^2 k^2 - Mr_0 \left(1 - \frac{(kr_0)^2}{2} \ln(kr_0) \right)} & \text{when } j = 3/2, \\ \frac{\pi}{2 \ln(kr_0)} \frac{\{A_j(M, \lambda)\}^{-1} + c_1^2 k^2 - k^2 r_0 / (4M)}{\{A_j(M, \lambda)\}^{-1} + c_1^2 k^2 + \{2Mr_0 \ln(kr_0)\}^{-1}} & \text{when } j = 1/2 \end{cases} \tag{29}$$

for $E > M$, and

$$\tan \eta_j(E, \lambda) \sim \begin{cases} -\frac{\pi(kr_0/2)^{2j+1}}{(j+1/2)!(j-1/2)!} \frac{A_j(-M, \lambda) + (2j+1)/(2Mr_0)}{A_j(-M, \lambda) + c_2^2 k^2 + \frac{k^2 r_0}{2M(2j-1)}} & \text{when } j \geq 3/2, \\ -\pi \left(\frac{kr_0}{2}\right)^2 \frac{A_j(-M, \lambda) + 1/(Mr_0)}{A_j(-M, \lambda) + c_2^2 k^2 - k^2 r_0 \ln(kr_0)/(2M)} & \text{when } j = 1/2, \end{cases} \quad (30)$$

for $E < -M$. The asymptotic forms (15) have been used in driving Eqs. (29) and (30). In addition to the leading terms, we include in Eqs. (29) and (30) some next leading terms, which are useful only for the critical case where the leading terms are canceled.

First, from Eqs. (29) and (30) we see that $\tan \eta_j(E, \lambda)$ tends to zero as E goes to $\pm M$, namely, $\eta_j(\pm M, \lambda)$ are always equal to the multiple of π . In other words, if the phase shift $\eta_j(E, \lambda)$ for a sufficiently small k is expressed as a positive or negative acute angle plus $n\pi$, its limit $\eta_j(M, \lambda)$ [or $\eta_j(-M, \lambda)$] is equal to $n\pi$. It means that $\eta_j(M, \lambda)$ [or $\eta_j(-M, \lambda)$] changes discontinuously when $\eta_j(E, \lambda)$ changes through the value $(n+1/2)\pi$, where n is an integer.

Second, from Eq. (26) we have

$$\left. \frac{\partial \eta_j(E, \lambda)}{\partial A_j(E, \lambda)} \right|_E = -\left(\frac{E+M}{E-M}\right)^{1/2} \frac{2\{\cos \eta_j(E, \lambda)\}^2}{\pi k r_0 \{N_{j+1/2}(kr_0)A_j(E, \lambda) - B(E)N_{j-1/2}(kr_0)\}^2} \leq 0, \quad E > M, \quad (31)$$

$$\left. \frac{\partial \eta_j(E, \lambda)}{\partial A_j(E, \lambda)} \right|_E = \left(\frac{|E|-M}{|E|+M}\right)^{1/2} \frac{2\{\cos \eta_j(E, \lambda)\}^2}{\pi k r_0 \{N_{j+1/2}(kr_0)A_j(E, \lambda) - B(E)N_{j-1/2}(kr_0)\}^2} \geq 0, \quad E < -M.$$

Namely, as the ratio $A_j(E, \lambda)$ decreases, the phase shift $\eta_j(E, \lambda)$ for $E > M$ increases monotonically, but $\eta_j(E, \lambda)$ for $E < -M$ decreases monotonically. In terms of the monotonic properties we are able to determine the jump of the phase shifts $\eta_j(\pm M, \lambda)$.

We first consider the scattering states of a positive energy with a sufficiently small momentum k . As $A_j(E, \lambda)$ decreases, if $\tan \eta_j(E, \lambda)$ changes sign from positive to negative, the phase shift $\eta_j(M, \lambda)$ jumps by π . Note that in this case if $\tan \eta_j(E, \lambda)$ changes sign from negative to positive, the phase shift $\eta_j(M, \lambda)$ stays invariant. Conversely, as $A_j(E, \lambda)$ increases, if $\tan \eta_j(E, \lambda)$ changes sign from negative to positive, the phase shift $\eta_j(M, \lambda)$ jumps by $-\pi$. Therefore, as λ increases from zero to one, each time the $A_j(M, \lambda)$ decreases from near and larger than the value $2Mr_0/(2j-1)$ to smaller than that value, the denominator in Eq. (29) changes sign from positive to negative and the rest factor stays positive, so that the phase shift $\eta_j(M, \lambda)$ jumps by π . It has been shown in the preceding section that each time the $A_j(M, \lambda)$ decreases across the value $2Mr_0/(2j-1)$, a scattering state of a positive energy becomes a bound state. Conversely, each time the $A_j(M, \lambda)$ increases across that value, the phase shift $\eta_j(M, \lambda)$ jumps by $-\pi$, and a bound state becomes a scattering state of a positive energy.

Then, we consider the scattering states of a negative energy with a sufficiently small k . As $A_j(E, \lambda)$ decreases, if $\tan \eta_j(E, \lambda)$ changes sign from negative to positive, the phase shift $\eta_j(-M, \lambda)$ jumps by $-\pi$. However, in this case if $\tan \eta_j(E, \lambda)$ changes sign from positive to negative, the phase shift $\eta_j(-M, \lambda)$ stays invariant. Conversely, as $A_j(E, \lambda)$ increases, if $\tan \eta_j(E, \lambda)$ changes sign from positive to negative, the phase shift $\eta_j(-M, \lambda)$ jumps by π . Therefore, as λ increases from zero to one, each time the

$A_j(-M, \lambda)$ decreases from a small and positive number to a negative one, the denominator in Eq. (29) changes sign from positive to negative and the rest factor stays negative, so that the phase shift $\eta_j(-M, \lambda)$ jumps by $-\pi$. In the preceding section it was shown that each time the $A_j(-M, \lambda)$ decreases across zero, a bound state becomes a scattering state of a negative energy. Conversely, each time the $A_j(-M, \lambda)$ increases across zero, the phase shift $\eta_j(-M, \lambda)$ jumps by π , and a scattering state of a negative energy becomes a bound state. Therefore, we obtain the Levinson theorem for the Dirac equation in two dimensions for noncritical cases:

$$\eta_j(M) + \eta_j(-M) = n_j \pi. \quad (32)$$

It is obvious that the Levinson theorem (32) holds for both positive and negative j in the noncritical cases.

For the case of $j=1/2$ and $E \sim M$, where the value $2Mr_0/(2j-1)$ is infinity, since $\{A_j(E, \lambda)\}^{-1}$ increases as $A_j(E, \lambda)$ decreases, we can study the variance of $\{A_j(E, \lambda)\}^{-1}$ in this case instead. For the energy $E > M$ where the momentum k is sufficiently small, when $\{A_j(M, \lambda)\}^{-1}$ increases from negative to positive as λ increases, both the numerator and denominator in Eq. (29) change signs, but not simultaneously. The numerator changes sign first, and then the denominator changes. The front factor in Eq. (29) is negative so that $\tan \eta_j(E, \lambda)$ first changes from negative to positive when the numerator changes sign, and then changes from positive to negative when the denominator changes sign. It is in the second step that the phase shift $\eta_j(M, \lambda)$ jumps by π . Similarly, each time $\{A_j(M, \lambda)\}^{-1}$ decreases across zero as λ increases, $\eta_j(M, \lambda)$ jumps by $-\pi$.

For $\lambda=0$ and $j=1/2$, the numerator in Eq. (29) is equal to zero, and the phase shift $\eta_j(M, 0)$ is defined to be zero. For this case there is a half-bound state at $E=M$ [see Eq. (33)].

If $\{A_j(M,\lambda)\}^{-1}$ increases [$A_j(M,\lambda)$ decreases] as λ increases from zero, the front factor in Eq. (29) is negative, the numerator becomes positive first, and then the denominator changes sign from negative to positive, such that the phase shift $\eta_j(M,\lambda)$ jumps by π and simultaneously the half-bound state becomes a bound state with $E < M$.

Now, we turn to study the critical cases. First, we study the critical case for $E \sim M$, where the ratio $A_j(M,1)$ is equal to the value $2Mr_0/(2j-1)$. It is easy to obtain the following solution of $E = M$ in the region $r_0 < r < \infty$, satisfying the radial equations (8) and the match condition (11) at r_0 :

$$f_{jM}(r,1) = 2Mr^{-j+1}, \quad g_{jM}(r,1) = (2j-1)r^{-j}. \quad (33)$$

It is a bound state when $j > 3/2$, but called a half-bound state when $j = 3/2$ or $j = 1/2$. A half-bound state is not a bound state, because its wave function is finite but not square integrable.

For definiteness, we assume that in the critical case, as λ increases from a number near and less than one and finally reaches one, $A_j(M,\lambda)$ decreases and finally reaches, but not across, the value $2Mr_0/(2j-1)$. In this case, when $\lambda = 1$ a new bound state of $E = M$ appears for $j > 3/2$, but does not appear for $j = 3/2$ or $j = 1/2$. We should check whether or not the phase shift $\eta_j(M,1)$ increases by an additional π as λ increases and reaches one.

It is evident from the next leading terms in the denominator of Eq. (29) that the denominator for $j \geq 3/2$ has changed sign from positive to negative as $A_j(M,\lambda)$ decreases and finally reaches the value $2Mr_0/(2j-1)$, namely, the phase shift $\eta_j(M,\lambda)$ jumps by an additional π at $\lambda = 1$. Simultaneously, a new bound state of $E = M$ appears for $j > 3/2$, but only a half-bound state appears for $j = 3/2$, so that the Levinson theorem (32) holds for the critical case with $j > 3/2$, but it has to be modified for the critical case with $j = 3/2$:

$$\eta_j(M) + \eta_j(-M) = (n_j + 1)\pi \quad (34)$$

when a half bound state occurs at $E = M$ and $j = 3/2$.

For $j = 1/2$ the next leading term with $\ln(kr_0)$ in the denominator of Eq. (29) dominates so that the denominator stays negative (does not change sign) as $\{A_j(M,\lambda)\}^{-1}$ increases and finally reaches zero, namely, the phase shift $\eta_j(M,\lambda)$ does not jump, no matter whether the rest part in Eq. (28) stays positive or has changed to negative. Simultaneously, only a new half-bound state of $E = M$ for $j = 1/2$ appears, so that the Levinson theorem (32) holds for the critical case with $j = 1/2$.

This conclusion holds for the critical case where $A_j(M,\lambda)$ increases and finally reaches, but not across, the value $2Mr_0/(2j-1)$.

Second, we study the critical case for $E = -M$, where the ratio $A_j(-M,1)$ is equal to zero. It is easy to obtain the following solution of $E = -M$ in the region $r_0 < r < \infty$, satisfying the radial equations (8) and the match condition (11) at r_0 :

$$f_{jM}(r,\lambda) = 0, \quad g_{jM}(r,\lambda) = r^{-j}. \quad (35)$$

It is a bound state when $j \geq 3/2$, but a half-bound state when $j = 1/2$.

For definiteness, we again assume that in the critical case, as λ increases from a number near and less than one and finally reaches one, $A_j(-M,\lambda)$ decreases and finally reaches zero, so that when $\lambda = 1$ the energy of a bound state decreases to $E = -M$ for $j \geq 3/2$, but a bound state becomes a half-bound state for $j = 1/2$. We should check whether or not the phase shift $\eta_j(-M,1)$ decreases by π as λ increases and reaches one.

For the energy $E < -M$ where the momentum k is sufficiently small, one can see from the next leading terms in the denominator of Eq. (30) that the denominator does not change sign as $A_j(-M,\lambda)$ decreases and finally reaches zero, namely, the phase shift $\eta_j(-M,\lambda)$ does not jump by an additional $-\pi$ at $\lambda = 1$. Simultaneously, the energy of a bound state decreases to $E = -M$ for $j \geq 3/2$, but a bound state becomes a half-bound state for $j = 1/2$, so that the Levinson theorem (32) holds for the critical case with $j \geq 3/2$, but it has to be modified for the critical case with $j = 1/2$:

$$\eta_j(M) + \eta_j(-M) = (n_j + 1)\pi \quad (36)$$

when a half-bound state occurs at $E = -M$ and $j = 1/2$.

Combining Eqs. (32), (34), (36) and their corresponding forms for the negative j , we obtain the relativistic Levinson theorem (12) in two dimensions.

V. DISCUSSION

Now, we discuss the general case where the potential $V(r)$ has a tail at $r \geq r_0$. Let r_0 be so large that only the leading term in $V(r)$ is concerned:

$$V(r) \sim br^{-n}, \quad r \geq r_0, \quad (37)$$

where b is a nonvanishing constant and n is a positive constant, not necessary to be an integer. Substituting it into Eq. (8) and changing the variable r to ξ :

$$\xi = \begin{cases} kr = r\sqrt{E^2 - M^2} & \text{when } |E| > M, \\ \kappa r = r\sqrt{M^2 - E^2} & \text{when } |E| \leq M, \end{cases} \quad (38)$$

we obtain the radial equations in the region $r_0 \leq r < \infty$:

$$\begin{aligned} \frac{d}{d\xi} g_{jE}(\xi) + \frac{j}{\xi} g_{jE}(\xi) &= \left(\frac{E}{|E|} \sqrt{\frac{E-M}{E+M}} - \frac{b}{\xi^n} \kappa^{n-1} \right) f_{jE}(\xi), \\ -\frac{d}{dr} f_{jE}(\xi) + \frac{j}{r} f_{jE}(\xi) &= \left(\frac{E}{|E|} \sqrt{\frac{E+M}{E-M}} - \frac{b}{\xi^n} \kappa^{n-1} \right) g_{jE}(\xi), \end{aligned} \quad (39)$$

for $|E| > M$, and

$$\begin{aligned} \frac{d}{d\xi} g_{jE}(\xi) + \frac{j}{\xi} g_{jE}(\xi) &= \left(-\sqrt{\frac{M-E}{M+E}} - \frac{b}{\xi^n} \kappa^{n-1} \right) f_{jE}(\xi), \\ -\frac{d}{dr} f_{jE}(\xi) + \frac{j}{r} f_{jE}(\xi) &= \left(\sqrt{\frac{M+E}{M-E}} - \frac{b}{\xi^n} \kappa^{n-1} \right) g_{jE}(\xi), \end{aligned} \quad (40)$$

for $|E| \leq M$. As far as the Levinson theorem is concerned, we are only interested in the solutions with the sufficiently small k and κ . If $n \geq 3$, in comparison with the first term on the right hand side of Eq. (39) or Eq. (40), the potential term with a factor k^{n-1} (or κ^{n-1}) is too small to affect the phase shift at the sufficiently small k and the variant range of the ratio $f_{jE}(r, \lambda)/g_{jE}(r, \lambda)$ at r_0+ . Therefore, the proof given in the preceding sections is effective for those potentials with a tail so that the Levinson theorem (12) holds.

When $n=2$ and $b \neq 0$, we will only keep the leading terms for the small parameter k (or κ) in solving Eq. (39) [or Eq. (40)]. First, we calculate the solutions with the energy $E \sim M$. Let

$$\alpha = (j^2 - j + 2Mb + 1/4)^{1/2} \neq j - 1/2. \tag{41}$$

If $\alpha^2 < 0$, there is an infinite number of bound states. We will not discuss this case nor the case with $\alpha=0$ here. When $\alpha^2 > 0$, we take $\alpha > 0$ for convenience. Some formulas given in the preceding sections will be changed.

When $E \leq M$ we have

$$\begin{aligned} f_{jE}(r, \lambda) &= e^{i(\alpha+1)\pi/2} 2M(\pi\kappa r/2)^{1/2} H_\alpha^{(1)}(i\kappa r), \\ g_{jE}(r, \lambda) &= e^{i(\alpha+1)\pi/2} \kappa(\pi\kappa r/2)^{1/2} \\ &\times \left\{ -\frac{d}{d(\kappa r)} H_\alpha^{(1)}(i\kappa r) + \frac{j-1/2}{\kappa r} H_\alpha^{(1)}(i\kappa r) \right\}. \end{aligned} \tag{42}$$

Hence, the ratio at $r=r_0+$ for $E=M$ is

$$\left. \frac{f_{jE}(r, \lambda)}{g_{jE}(r, \lambda)} \right|_{r=r_0+} = \frac{2Mr_0}{j + \alpha - 1/2}, \quad E=M. \tag{43}$$

When $E > M$ we have

$$\begin{aligned} f_{jE}(r, \lambda) &= 2M(\pi\kappa r/2)^{1/2} \\ &\times \{ \cos \delta_\alpha(E, \lambda) J_\alpha(\kappa r) - \sin \delta_\alpha(E, \lambda) N_\alpha(\kappa r) \}, \\ g_{jE}(r, \lambda) &= k(\pi\kappa r/2)^{1/2} \left\{ \cos \delta_\alpha(E, \lambda) \left(-\frac{d}{d(\kappa r)} J_\alpha(\kappa r) \right. \right. \\ &\left. \left. + \frac{j-1/2}{\kappa r} J_\alpha(\kappa r) \right) - \sin \delta_\alpha(E, \lambda) \right. \\ &\left. \times \left(-\frac{d}{d(\kappa r)} N_\alpha(\kappa r) + \frac{j-1/2}{\kappa r} N_\alpha(\kappa r) \right) \right\}. \end{aligned} \tag{44}$$

When κr tends to infinity, the asymptotic form of the solution is

$$\begin{aligned} f_{jE}(r, \lambda) &\sim 2M \cos[\kappa r - \alpha\pi/2 - \pi/4 + \delta_\alpha(E, \lambda)], \\ g_{jE}(r, \lambda) &\sim k \sin[\kappa r - \alpha\pi/2 - \pi/4 + \delta_\alpha(E, \lambda)]. \end{aligned}$$

In comparison with the solution (25) we obtain the phase shift $\eta_j(E, \lambda)$ for $E > M$:

$$\eta_j(E, \lambda) = \delta_\alpha(E, \lambda) + (j - \alpha - 1/2)\pi/2, \quad E > M. \tag{45}$$

From the match condition (11), for the sufficiently small k we obtain

$$\begin{aligned} \tan \delta_\alpha(E, \lambda) &\sim \frac{-\pi(\kappa r_0/2)^{2\alpha} (j - \alpha - 1/2)}{\Gamma(\alpha + 1)\Gamma(\alpha) (j + \alpha - 1/2)} \\ &\times \frac{A_j(M, \lambda) - 2Mr_0/(j - \alpha - 1/2)}{A_j(M, \lambda) - 2Mr_0/(j + \alpha - 1/2)}. \end{aligned} \tag{46}$$

Therefore, as λ increases from zero to one, each time the $A_j(M, \lambda)$ decreases from near and larger than the value $2Mr_0/(j + \alpha - 1/2)$ to smaller than that value, the denominator in Eq. (46) changes sign from positive to negative and the rest factor stays positive, so that $\delta_\alpha(M, \lambda)$ jumps by π . Simultaneously, from Eq. (43) a new overlap between the variant ranges of the ratio at two sides of r_0 appears such that a scattering state of a positive energy becomes a bound state. Conversely, each time the $A_j(M, \lambda)$ increases across that value, $\delta_\alpha(M, \lambda)$ jumps by $-\pi$, and a bound state becomes a scattering state of a positive energy.

Second, we calculate the solutions with the energy $E \sim -M$. Let

$$\beta = (j^2 + j - 2Mb + 1/4)^{1/2} \neq j + 1/2. \tag{47}$$

Similarly, we only discuss the cases with $\beta^2 > 0$, and take $\beta > 0$.

When $E \geq -M$ we have

$$\begin{aligned} f_{jE}(r, \lambda) &= -e^{i(\beta+1)\pi/2} \kappa(\pi\kappa r/2)^{1/2} \\ &\times \left\{ \frac{d}{d(\kappa r)} H_\beta^{(1)}(i\kappa r) + \frac{j+1/2}{\kappa r} H_\beta^{(1)}(i\kappa r) \right\}, \\ g_{jE}(r, \lambda) &= e^{i(\beta+1)\pi/2} 2M(\pi\kappa r/2)^{1/2} H_\beta^{(1)}(i\kappa r). \end{aligned} \tag{48}$$

Hence, the ratio at $r=r_0+$ for $E=-M$ is

$$\left. \frac{f_{jE}(r, \lambda)}{g_{jE}(r, \lambda)} \right|_{r=r_0+} = -\frac{j - \beta + 1/2}{2Mr_0}, \quad E=-M. \tag{49}$$

When $E < -M$ we have

$$\begin{aligned} f_{jE}(r, \lambda) &= -k(\pi\kappa r/2)^{1/2} \left\{ \cos \delta_\beta(E, \lambda) \left(\frac{d}{d(\kappa r)} J_\beta(\kappa r) \right. \right. \\ &\left. \left. + \frac{j+1/2}{\kappa r} J_\beta(\kappa r) \right) - \sin \delta_\beta(E, \lambda) \left(\frac{d}{d(\kappa r)} N_\beta(\kappa r) \right. \right. \\ &\left. \left. + \frac{j+1/2}{\kappa r} N_\beta(\kappa r) \right) \right\}, \\ g_{jE}(r, \lambda) &= 2M(\pi\kappa r/2)^{1/2} \{ \cos \delta_\beta(E, \lambda) J_\beta(\kappa r) \\ &\quad - \sin \delta_\beta(E, \lambda) N_\beta(\kappa r) \}, \end{aligned} \tag{50}$$

When κr tends to infinity, the asymptotic form for the solution is

$$\begin{aligned} f_{jE}(r, \lambda) &\sim k \sin[\kappa r - \beta\pi/2 - \pi/4 + \delta_\beta(E, \lambda)], \\ g_{jE}(r, \lambda) &\sim 2M \cos[\kappa r - \beta\pi/2 - \pi/4 + \delta_\beta(E, \lambda)]. \end{aligned}$$

In comparison with the solution (25) we obtain the phase shift $\eta_j(E, \lambda)$ for $E < -M$:

$$\eta_j(E, \lambda) = \delta_\beta(E, \lambda) + (j - \beta + 1/2)\pi/2, \quad E < -M. \quad (51)$$

From the match condition (11), for the sufficiently small k we obtain

$$\tan \delta_\alpha(E, \lambda) \sim \frac{-\pi(kr_0/2)^{2\beta}}{\Gamma(\beta+1)\Gamma(\beta)} \times \frac{A_j(-M, \lambda) + (j + \beta + 1/2)/(2Mr_0)}{A_j(-M, \lambda) + (j - \beta + 1/2)/(2Mr_0)}. \quad (52)$$

Therefore, as λ increases from zero to one, each time the $A_j(-M, \lambda)$ decreases from near and larger than the value $-(j - \beta + 1/2)/(2Mr_0)$ to smaller than that value, the denominator in Eq. (52) changes sign from positive to negative and the rest factor stays negative, so that $\delta_\beta(-M, \lambda)$ jumps by $-\pi$. Simultaneously, from Eq. (49) an overlap between the variant ranges of the ratio at two sides of r_0 disappears such that a bound state becomes a scattering state of a negative energy. Conversely, each time the $A_j(-M, \lambda)$ increases across that value, $\delta_\beta(-M, \lambda)$ jumps by π , and a scattering state of a negative energy becomes a bound state.

In summary, we obtain the modified relativistic Levinson theorem for noncritical cases when the potential has a tail (37) with $n=2$ at infinity:

$$\eta_j(M) + \eta_j(-M) = n_j\pi + (2j - \alpha - \beta)\pi/2. \quad (53)$$

We will not discuss the critical cases in detail. In fact, the modified relativistic Levinson theorem (53) holds for the critical cases of $\alpha > 1$ and $\beta > 1$. When $0 < \alpha < 1$ or $0 < \beta < 1$, $\delta_\alpha(M, 1)$ or $\delta_\beta(-M, 1)$ in the critical case will not be multiple of π , respectively, so that Eq. (53) is violated for those critical cases.

Furthermore, for the potential (37) with a tail at infinity, when $n > 2$, even if it contains a logarithm factor, for any arbitrarily small positive ϵ , one can always find a sufficiently large r_0 such that $|V(r)| < \epsilon/r^2$ in the region $r_0 < r < \infty$. Thus, from Eqs. (41) and (47) we have for the sufficiently small ϵ

$$\alpha = (j^2 - j \pm 2M\epsilon + 1/4)^{1/2} \sim j - 1/2,$$

$$\beta = (j^2 + j \mp 2M\epsilon + 1/4)^{1/2} \sim j + 1/2.$$

Hence, Eq. (53) coincides with Eq. (32). In this case the Levinson theorem (32) holds for the noncritical case.

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