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Exact wave function of a harmonic plus an inverse harmonic potential with time-dependent mass and frequency

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Using canonical and unitary transformations and the Lewis-Risenfeld invariant method, the exact Schrödinger wave function for a harmonic plus inverse harmonic potential with time-dependent mass and frequency is obtained analytically. [S1050-2947(98)05208-1]

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I. INTRODUCTION

For more than three decades, the study of the time-dependent harmonic oscillator has been one of the important problems in classical and quantum mechanics [1–5] because it can be treated as an exact solvable model and offers many applications in various fields of physics. The origin of this development for a time-dependent harmonic oscillator was based on the discovery of an exact invariant, the so-called time-dependent dynamical invariant, given by Lewis and Risenfeld [6]. After their work, many different derivations for the dynamical invariant have been presented: Lutzky [7] applied Noether's theorem, Ray and Reid [8] obtained the invariants from Ermakov's method, Leach and Günter [9] introduced the time-dependent canonical transformation, and Korsch [10] constructed the invariant via dynamical algebra.

Recently some authors have investigated the harmonic oscillator with time-dependent mass and frequency separately or together [11,12]. For constant mass, Dantas *et al.* [13] obtained the wave function that satisfies the Schrödinger equation, and Pedrosa [14] evaluated an exact wave function for the harmonic oscillator with time-dependent mass and frequency. One can also find quantum mechanical studies [15,16] of the harmonic plus inverse harmonic potential of the type

$$V(x,t)=a_1(t)x^2+\frac{a_2}{x^2}, \tag{1}$$

where a_1 and a_2 are the time-dependent and time-independent coefficients, respectively. It can be easily found that the reduced Schrödinger equation for a central harmonic potential is related to an effective potential of the type $V(x,t)$.

In several previous papers we have obtained the wave function, energy expectation values, uncertainty relations, and coherent states for the Calidrola-Kanai Hamiltonian via the path integral method [17] and utilizing various time-dependent quantum systems [18,19]. Moreover, we have developed the exact quantum theory of a time-dependent bound quadratic Hamiltonian system [12,20]. We have also obtained the exact wave function for the Caldirola-Kanai Hamiltonian through the Lewis-Risenfeld dynamical invariant method [21] and the relations of canonical and unitary transformations for a general time-dependent quadratic Hamiltonian system [22].

The main purpose of this paper is to evaluate an exact invariant and Schrödinger wave function for the harmonic oscillator plus inverse harmonic potential with time-dependent mass and frequency, where a_2 is not constant but a time-dependent coefficient in Eq. (1). In Sec. II we will treat a time-dependent harmonic plus inverse potential. The classical invariant and Schrödinger solution will be given in Sec. III. Finally, we give a summary in Sec. IV.

II. TIME-DEPENDENT HARMONIC PLUS INVERSE HARMONIC POTENTIAL

We first consider the harmonic plus inverse harmonic potential with time-dependent mass and frequency of the type

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$$H=\frac{p^2}{2M(t)}+\frac{M(t)\omega(t)^2q^2}{2}+\frac{1}{2M(t)q^2}, \tag{2}$$

where p and q are canonical conjugates and satisfy a commutation relation, i.e., $[q,p]=i\hbar$, and $M(t)$ and $\omega(t)$ are time-dependent mass and frequency. Taking $M(t)=me^{\gamma t}$ and $\omega(t)=\omega_0$, and neglecting the third term, the Caldirola-Kanai Hamiltonian [1,17] can be recovered:

$$H=e^{-\gamma t}\frac{p^2}{2m}+e^{\gamma t}\frac{m\omega_0^2q^2}{2}. \tag{3}$$

To take the time-dependent canonical transformation, we introduce the generating function

$$G(q,P,t)=qP[M(t)]^{1/2}-\frac{M(t)\gamma(t)q^2}{4}, \tag{4}$$

where $\gamma(t)$ is given by

$$\gamma(t)=\frac{d[\ln M(t)]}{dt}. \tag{5}$$

Making use of this generating function, we obtain the new canonical variables and Hamiltonian as follows:

$$Q=\frac{\partial G(q,P,t)}{\partial P}=q[M(t)]^{1/2}, \tag{6}$$

$$p=\frac{\partial G(q,P,t)}{\partial q}=P[M(t)]^{1/2}-\frac{[M(t)]^{1/2}\gamma(t)Q}{2}, \tag{7}$$

$$H_N=\frac{\partial G(q,P,t)}{\partial t}+H(t). \tag{8}$$

Through this transformation, we can obtain the new Hamiltonian H_N as

$$H_N=\frac{P^2}{2}+\frac{\Omega^2(t)Q^2}{2}+\frac{1}{2Q^2}, \tag{9}$$

$$\Omega^2(t)=\omega^2(t)-\frac{\gamma^2(t)}{4}-\frac{\dot{\gamma}(t)}{2}, \tag{10}$$

where $\Omega(t)$ is the new frequency. We notice that the commutation relation $[Q,P]=[q,p]$ holds for both coordinates.

III. CLASSICAL INVARIANT AND THE EXACT SOLUTION FOR THE SCHRÖDINGER EQUATION

As mentioned earlier, for the construction of an exact invariant for the time-dependent classical dynamical system, several methods are developed. With the use of Ermakov's technique [8], we construct the invariant for the Hamiltonian [Eq. (9)]. To derive the invariant for this Hamiltonian, we consider the equation of motion from Eq. (9):

$$\ddot{Q}(t)+\Omega^2(t)Q(t)=\frac{1}{Q^3(t)}. \tag{11}$$

If $\rho(t)$ is some classical solution of Eq. (11), then this equation can be expressed as

$$\ddot{\rho}-\left(\frac{\ddot{Q}}{Q}-\frac{1}{Q^4}\right)\rho=\frac{1}{\rho^3} \tag{12}$$

or

$$Q\ddot{\rho}-\ddot{Q}\rho=\frac{Q}{\rho^3}-\frac{\rho}{Q^3}. \tag{13}$$

Multiplying Eq. (13) by $(Q\dot{\rho}-\dot{Q}\rho)$, we arrive at

$$(Q\dot{\rho}-\dot{Q}\rho)\frac{d}{dt}(Q\dot{\rho}-\dot{Q}\rho)=\left(\frac{Q}{\rho^3}-\frac{\rho}{Q^3}\right)(Q\dot{\rho}-\dot{Q}\rho) \tag{14}$$

or

$$\frac{1}{2}\frac{d}{dt}[(Q\dot{\rho}-\dot{Q}\rho)]^2=-\frac{1}{2}\frac{d}{dt}\left[\left(\frac{\rho}{Q}\right)^2+\left(\frac{Q}{\rho}\right)^2\right]. \tag{15}$$

Then the invariant can be written as

$$I(t)=\frac{1}{2}\left[(\rho\dot{Q}-\dot{\rho}Q)^2+\left(\frac{\rho}{Q}\right)^2+\left(\frac{Q}{\rho}\right)^2\right]. \tag{16}$$

Our invariant coincides with that of Ref. [14] that has been obtained by using the Lie algebra method.

Now we introduce the transformation

$$\rho(t)=x(t)M^{1/2}(t), \tag{17}$$

where $x(t)$ is a function of time to be determined. Substitution of Eqs. (7), (10), and (15) into Eq. (11) gives the equation

$$\ddot{q}+\gamma(t)\dot{q}+\omega^2(t)q=\frac{1}{M^2(t)q^3}. \tag{18}$$

If $x(t)$ is some classical solution of Eq. (18), the invariant, Eq. (16), becomes

$$I(t)=\frac{1}{2}\left[M^2(t)(q\dot{x}-x\dot{q})^2+\left(\frac{x}{q}\right)^2+\left(\frac{q}{x}\right)^2\right]. \tag{19}$$

Therefore Eq. (18) constitutes an Ermakov system or the Hamiltonian given as Eq. (2), that has an invariant in the form of Eq. (19). Taking $M(t)=1$, Eq. (19) is reduced to Eq. (15). In this case, the generating function, Eq. (4), corresponds to the identity transformation, and the structure of the Hamiltonian corresponding to Eq. (11) turns out to be

$$\overline{H}=\frac{1}{2}\left[\overline{p^2}+\Omega^2(t)\rho^2+\frac{1}{\rho^2}\right], \tag{20}$$

where $\overline{p}=\dot{\rho}$, and Eq. (20) is analogous to the form of Eq. (9). The invariant, Eq. (16), maps H into $\overline{H}$ and vice versa. Furthermore, when $M(t)$ is time-independent and the third term is neglected, one can recover the well-known time-dependent harmonic oscillator. The invariant $I(t)$ satisfies Hamilton's equation [2,6]

$$\frac{dI}{dt}=\frac{\partial I}{\partial t}+\frac{1}{i\hbar}[I,H]=0,\quad I^{\dagger}=I, \tag{21}$$

and the eigenstates $\phi_n(Q,t)$ of the invariant $I(t)$ are assumed to form a complete orthogonal set with time-dependent eigenvalue λ_n:

$$I\phi_n(Q,t)=\lambda_n\phi_n(Q,t). \tag{22}$$

We first consider the unitary transformation

$$\phi_n'(Q,t)=\exp\left[-\frac{i\dot{\rho}Q^2}{2\hbar\rho}\right]\phi_n(Q,t). \tag{23}$$

Performing this transformation, we can obtain new invariant $I'(t)[I'=UIU^{\dagger}]$:

$$I'\phi_n'(Q,t)=\lambda_n\phi_n'(Q,t), \tag{24}$$

$$I'=-\frac{\hbar^2}{2}\rho^2\frac{\partial^2}{\partial Q^2}+\frac{1}{2}\left(\frac{Q}{\rho}\right)^2+\frac{1}{2}\left(\frac{\rho}{Q}\right)^2. \tag{25}$$

Taking $\sigma=Q/\rho$, we can write Eq. (26) as

$$\left(-\frac{\hbar^2}{2}\frac{\partial^2}{\partial\sigma^2}+\frac{\sigma^2}{2}+\frac{1}{2\sigma^2}\right)\phi_n'(\sigma)=\lambda_n\phi_n'(\sigma). \tag{26}$$

Equation (26) represents an ordinary one-dimensional Schrödinger equation with potential $V(\sigma)=\sigma^2/2+1/2\sigma^2$, and the solution [4] is given as

$$\phi_n'=\left(\frac{4}{\hbar}\right)^{1/4}\left[\frac{\Gamma(n+1)}{\Gamma(n+a+1)}\right]^{1/2}\left(\frac{\sigma^2}{\hbar}\right)^{(2a+1)/4}\times\exp\left(-\frac{\sigma^2}{2\hbar}\right)L_n^a\left(\frac{\sigma^2}{\hbar}\right) \tag{27}$$

and

$$\lambda_n=\hbar(2n+a+1),\quad a=\frac{1}{2}\left(1+\frac{4}{\hbar^2}\right)^{1/2}, \tag{28}$$

where L_n^a are the associated Laguerre polynomials.

Now, we consider the time-dependent Schrödinger equation for the Hamiltonian, Eq. (9),

$$i\hbar\frac{\partial\psi_n}{\partial t}=H_N\psi_n, \tag{29}$$

$$H_N=-\frac{\hbar^2}{2}\frac{\partial^2}{\partial Q^2}+\frac{1}{2}\Omega^2(t)Q^2+\frac{1}{2Q^2}, \tag{30}$$

where we have used $P=-i\hbar\partial/\partial Q$. According to Lewis and Riesenfeld [2,6], the solution of this Schrödinger equation differs by only a time-dependent phase factor from the eigenstates of the invariant [6]; we may write the wave function $\psi_n(Q,t)$ as

$$\psi_n(Q,t)=\exp[i\alpha_n(t)]\phi_n(Q,t), \tag{31}$$

where $\alpha_n(t)$ satisfies the equation

$$\hbar\frac{d\alpha_n(t)}{dt}=\left\langle\phi_n\left|\left(i\hbar\frac{\partial}{\partial t}-H_N(t)\right)\right|\phi_n\right\rangle. \tag{32}$$

Using Eqs. (23), (24), and (27), we obtain the Schrödinger solution for Eq. (29):

$$\phi_n(\sigma,t)=\left(\frac{4}{\hbar}\right)^{1/4}\left[\frac{\Gamma(n+1)}{\Gamma(n+a+1)}\right]^{1/2}\left(\frac{\sigma^2}{\hbar}\right)^{(2a+1)/4}\times\exp\left[-\frac{\sigma^2}{2\hbar}+\frac{i\dot{\rho}\sigma^2}{2\hbar\rho}\right]L_n^a\left(\frac{\sigma^2}{\hbar}\right). \tag{33}$$

Applying the transformation Eq. (17) to Eq. (33), the eigenstates $\phi_n(q,t)$ of the invariant, Eq. (19), can be expressed as

$$\phi_n(q,t)=\left(\frac{4}{\hbar}\right)^{1/4}\left[\frac{\Gamma(n+1)}{\Gamma(n+a+1)}\right]^{1/2}\left(\frac{q^2}{\hbar x^2}\right)^{(2a+1)/4}\times\exp\left[\frac{iM(t)}{2\hbar}\left(\frac{\dot{x}}{x}+\frac{\gamma(t)}{2}+\frac{i}{M(t)x^2}\right)q^2\right]L_n^a\left(\frac{q^2}{\hbar x^2}\right). \tag{34}$$

Finally, we evaluate the phase factor $\alpha_n(t)$ that connects the solution for the original Schrödinger equation to that of the invariant. Performing the unitary transformation, Eq. (32) becomes

$$\hbar\frac{d\alpha_n(t)}{dt}=\left\langle\phi_n'\left|i\hbar\frac{\partial}{\partial t}-\frac{\hbar\dot{\rho}Q}{i\rho}\frac{\partial}{\partial Q}+i\hbar\frac{\dot{\rho}}{2\rho}-\frac{I'}{\rho^2}\right|\phi_n'\right\rangle. \tag{35}$$

With the use of the normalization of ϕ_n and Eq. (26), we get

$$\alpha_n(t)=-\left[2n+1+\frac{1}{2}\left(1+\frac{4}{\hbar^2}\right)^{1/2}\right]\int^t\frac{dt'}{M(t')x^2(t')}. \tag{36}$$

For the case of constant mass, i.e., $M(t)=m_0$, Eq. (36) is exactly reduced to our previous result [12].

IV. SUMMARY

In this paper, taking advantage of canonical and unitary transformations and the Lewis-Riesenfeld invariant method, we have derived the exact Schrödinger wave function for the harmonic plus inverse harmonic potential with time-dependent coefficient a_2, mass, and frequency. When $M(t)$ is constant, and $a_2=0$, our wave function is exactly reduced to those given in Refs. [12,14].

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