Semiclassical analysis of the metastable driven and damped quantized anharmonic oscillator

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(Received 18 December 1997; revised manuscript received 9 March 1998)

Recent numerical work has demonstrated how one of the bistable states of a driven, damped, anharmonic classical oscillator is metastable in the corresponding quantum oscillator [D. Enzer and G. Gabrielse, Phys. Rev. Lett. **78**, 1211 (1997)]. Based on an analysis of the classical oscillator, we present a simple analytical treatment of the quantum oscillator. Calculated fluctuations in the quantum observables and the estimated decay rate of the metastable state are in an excellent agreement with the numerical results of Enzer and Gabrielse. [S1050-2947(98)01708-9]

PACS number(s): 42.50.Lc, 03.65.Sq, 42.65.Pc

I. INTRODUCTION

An electron in a Penning trap is an anharmonic oscillator, in that the relativistic mass increase causes the cyclotron frequency to decrease with the energy of the electron [1]. With anharmonicity comes the potential for bistability. Enzer and Gabrielse [2] argue that it will be possible to operate an externally driven, damped one-electron cyclotron oscillator in a regime in which the classical system would exhibit bistability, yet the electron should be treated quantum mechanically. They also demonstrate numerically how quantum fluctuations alone may cause transitions from one bistable state to the other [2].

In fact, the effects of fluctuations in nonlinear systems [3–6], be it thermal fluctuations, classical noise, or quantum fluctuations, are a much-studied subject, e.g., in optical bistability [7,8]. The charm of the quantum-mechanical, driven, damped, anharmonic oscillator [quantum nonlinear oscillator (QNLO)] is that it is just about as elementary theoretically as a bistable system can be. It is then no wonder that the QNLO has long served as a paradigm of optical bistability [9–11]. For instance, characteristic time scales of a QNLO were studied numerically in Ref. [10] using matrix continued fractions, and lifetimes of the bistable states are discussed analytically in Ref. [12].

The thrust of the present paper is that we match the straightforward numerical experiments of Ref. [2] with an equally straightforward analytical treatment of the QNLO, in the same limits that are anticipated to be experimentally the most relevant. We start with an explicit, detailed analysis of the corresponding classical, driven, damped, anharmonic oscillator (CNLO). The problem of the QNLO is next formulated in terms of a Fokker-Planck equation (FPE) for the Q quasiprobability distribution [13]. We then proceed to a discussion of a feature not shared by the CNLO, namely, fluctuations. These are studied in the standard manner by linearizing the FPE, with an emphasis on the close similarities between CNLO and QNLO. We demonstrate energy and phase fluctuations in complete agreement with Ref. [2].

In the second part of our development we address the lifetime of the metastable excited state of the QNLO. Here is where our pedagogical approach pays off. We combine a global picture of the Hamiltonian from the CNLO, local considerations of fluctuations from the QNLO, and the classical Kramer analysis [15,16] into a prediction of the lifetime of the metastable state. In this conceptually delicate but technically simple manner we arrive at a lifetime that exactly agrees with the numerical results of Ref. [2].

The paper is organized as follows. In Sec. II we study the CNLO. The emphasis is on fixed points in the phase space of the system, which are the potential multistable states. In Sec. III we begin our analysis of the QNLO in terms of the Q function. The focus is initially on the close analogies between the CNLO and the QNLO. In Sec. IV we analyze fluctuations of the QNLO by linearizing the FPE around the classical fixed points and in Sec. V we develop our estimate for the lifetime of the metastable state of the QNLO. A comparison with Ref. [12] is included. The brief remarks in Sec. VI conclude our paper.

II. CLASSICAL DRIVEN DAMPED ANHARMONIC OSCILLATOR

Although there are physical differences between an electron subject to cyclotron motion and a one-dimensional harmonic oscillator, for pedagogical reasons we first consider a classical, driven, damped oscillator that is nonlinear because of relativistic mass increase. The equations of motion for position and momentum X and P are

$$\dot{X} = P/m - aP^3, \tag{1a}$$

$$\dot{P} = -m\omega^2 X - \gamma P + F\cos(\nu t). \tag{1b}$$

The notation is self-explanatory, except for the coefficient of nonlinearity $a = 1/2m^3c^2$.

We convert Eqs. (1) to a frame rotating in the (X,P) plane at the frequency of the driving force ν and make the familiar rotating-wave approximation of quantum optics. This entails that we keep in the corotating frame only those terms in the equations of motion that vary slowly in comparison to oscillations at the frequency ν. We also adjust the

dimensions by defining $X \equiv \sqrt{\hbar/m\omega}x$ and $P \equiv \sqrt{\hbar m\omega}p$. We are naturally at liberty to use the constant $\hbar$ in the scaling, even if our approach is classical at this point. The variables x and p are thereby made dimensionless. In the rotating frame we have the equations of motion

$$\dot{x} = -\frac{\gamma}{2}x + \delta p - \frac{\chi}{2}p(p^2+x^2), \tag{2a}$$

$$\dot{p} = -\frac{\gamma}{2}p - \delta x + \frac{\chi}{2}x(p^2+x^2) + \frac{\Omega}{\sqrt{2}}. \tag{2b}$$

Striving at quantum-optics-like notation, we have introduced the Rabi frequency $\Omega = F/\sqrt{2\hbar\omega m}$, a parameter of cubic nonlinearity $\chi = 3a\hbar m^2\omega^2/4$, and detuning

$$\delta = \omega - \nu. \tag{3}$$

In this paper we only consider the case when the frequency of the driving force ν is below the resonance frequency of the oscillator at zero energy ω. The notation is such that then the detuning δ is positive. If damping is neglected, the equations of motion (2) may be derived from the Hamiltonian

$$H = \frac{\delta}{2}(p^2+x^2) - \frac{\chi}{8}(p^2+x^2)^2 - \frac{\Omega}{\sqrt{2}}x. \tag{4}$$

The key feature of our CNLO is that, without an external drive, its oscillation frequency decreases with energy. In the rotating frame, the effective detuning decreases with the oscillator energy. A one-electron cyclotron oscillator in a Penning trap behaves in the same way because of the relativistic dependence of cyclotron frequency on energy. In a one-electron oscillator the damping is due to synchrotron radiation from the electron. From this point onward our model and the model of Ref. [2] are mathematically indistinguishable and we freely mix terms when speaking about the two models. There are only a few minor differences in the notation. In Ref. [2] no explicit symbol is introduced for the detuning, while we follow standard conventions in quantum optics and denote detuning by δ. In addition in Ref. [2] the variable δ was used to characterize the nonlinearity, while here we use χ for precisely that purpose. Finally, we denote the Rabi frequency by Ω not Ω_R.

Ignoring the driving force proportional to Ω for a moment, the Hamiltonian (4) exhibits rotational invariance in phase space. It is natural to investigate the system in terms of variables that reflect this symmetry. A possible choice is an energylike variable ϵ and an angle ϕ defined as

$$\epsilon = \frac{1}{2}(x^2+p^2), \tag{5a}$$

$$\tan\phi = x/p. \tag{5b}$$

The transformation $(x,p) \leftrightarrow (\phi,\epsilon)$ is canonical [17], making ϕ the generalized coordinate and ϵ the generalized momentum. The transformed Hamiltonian takes the form

$$H = \delta\epsilon - \frac{\chi}{2}\epsilon^2 - \sqrt{\epsilon}\Omega\sin\phi, \tag{6}$$

while the equations of motion, including the damping, are

$$\dot{\phi} = \delta - \chi\epsilon - \frac{\Omega}{2\sqrt{\epsilon}}\sin\phi, \tag{7a}$$

$$\dot{\epsilon} = -\gamma\epsilon + \sqrt{\epsilon}\Omega\cos\phi. \tag{7b}$$

Both sets of coordinates will be used for the rest of this paper. Locally, such as in linearized studies of fluctuations, the coordinate systems (x,p) and (ϕ,ϵ) work equally well, although the (ϕ,ϵ) representation yields the quantities of interest more easily. On the other hand, in the numerical results in Ref. [2] one sees crescent-shape distributions in coordinates analogous to x and p, which suggests that over large, global, scales the coordinates (ϕ,ϵ) might be more appropriate.

In the absence of damping, the CNLO will follow a constant-energy surface in phase space. Damping relaxes conservation of energy. In the case of a very weak damping one may think that the CNLO almost follows energy surfaces, except that it slowly drifts from one surface to the other. For the purposes of the present paper we make the plausible assumption that the time evolution eventually leads to a steady state, a fixed point found by equating the derivatives in either Eqs. (2a) and (2b) or Eqs. (7a) and (7b) to zero and solving for x and p, or ϕ and ϵ.

We use MATHEMATICA [18] to assist in the calculations, so we easily find exact expressions for the fixed points. However, here we rather utilize the scales of the parameters as given in Ref. [2] to extract simple approximate results. The relative magnitudes of the frequency parameters in the problem are such that

$$\gamma \ll \chi < \Omega < \delta, \quad \chi \ll \delta. \tag{8}$$

For that reason, all of our final results involve an expansion in the small parameter χ/δ or often actually in $\sqrt{\chi/\delta}$. Also, whenever it is helpful for the interpretation of the results, we take the limit when the damping coefficient γ is the smallest frequency parameter. We find three fixed points, which are

$$(x_e,p_e) = \left(-\sqrt{\frac{2\delta}{\chi}} - \frac{\Omega}{2\sqrt{2}\delta}, \frac{\sqrt{2}\gamma\delta}{\chi\Omega} + \frac{\gamma}{\sqrt{2\delta\chi}} - \frac{\gamma\Omega}{4\sqrt{2}\delta^2} \right), \tag{9a}$$

$$(x_r,p_r) = \left(\sqrt{\frac{2\delta}{\chi}} - \frac{\Omega}{2\sqrt{2}\delta}, \frac{\sqrt{2}\gamma\delta}{\chi\Omega} - \frac{\gamma}{\sqrt{2\delta\chi}} - \frac{\gamma\Omega}{4\sqrt{2}\delta^2} \right), \tag{9b}$$

$$(x_g,p_g) = \left(\frac{\Omega}{\sqrt{2}\delta}, \frac{\gamma\Omega}{2\sqrt{2}\delta^2} \right) \tag{9c}$$

in terms of Cartesian coordinates and

$$(\sin\phi_e,\epsilon_e) = \left(\frac{\gamma^2\delta}{2\chi\Omega^2} - 1, \frac{\delta}{\chi} + \frac{\Omega}{2\sqrt{\chi\delta}} - \frac{\Omega^2}{8\delta^2} \right), \tag{10a}$$

$$(\sin\phi_r,\epsilon_r) = \left(1 - \frac{\gamma^2\delta}{2\chi\Omega^2}, \frac{\delta}{\chi} - \frac{\Omega}{2\sqrt{\chi\delta}} - \frac{\Omega^2}{8\delta^2} \right), \tag{10b}$$

$$(\sin\phi_g, \epsilon_g) = \left(1 - \frac{7\chi^2\Omega^4}{32\delta^6}, \frac{\Omega^2}{4\delta^2} + \frac{\chi\Omega^4}{8\delta^5}\right) \tag{10c}$$

in terms of the energy-angle variables.

The question remains whether these fixed points are attractors or repellors. We have already anticipated the answer in the subscripts given to the fixed points: e for attracting excited state, r for repelling state, and g for attracting ground state. These assignments are verified in the standard manner: By linearizing the equations of motion around the fixed points we find the equations of motion for small deviations from equilibrium $\tilde{x}_i = x - x_i$ and $\tilde{p}_i = p - p_i$, where the subscript i corresponds to one of the indices e, r, or g. Symbolically,

$$[\dot{\tilde{x}}_i, \dot{\tilde{p}}_i]^T = M_i^{xp}[\tilde{x}_i, \tilde{p}_i]^T, \tag{11}$$

where M_i^{xp} is a constant 2×2 matrix. If the real parts of both eigenvalues of M_i^{xp} are negative, the time evolution forces $\tilde{x}_i \to 0$ and $\tilde{p}_i \to 0$ and we have an attractor. On the other hand, if the real part of an eigenvalue is positive, the linearized equations of motion have an exponentially growing solution. In physics it always takes over and removes the CNLO from the neighborhood of the fixed point. This is the signature of a repellor. An equivalent calculation can be done for the (ϕ, ϵ) variables. The actual eigenvalues are

$$\lambda^{\pm}_{xp,e} = \lambda^{\pm}_{\phi\epsilon,e} = -\tfrac{1}{2}\gamma \pm i\,\delta^{1/4}\chi^{1/4}\Omega^{1/2}, \tag{12a}$$

$$\lambda^{\pm}_{xp,r} = \lambda^{\pm}_{\phi\epsilon,r} = -\tfrac{1}{2}\gamma \pm \delta^{1/4}\chi^{1/4}\Omega^{1/2}, \tag{12b}$$

$$\lambda^{\pm}_{xp,g} = \lambda^{\pm}_{\phi\epsilon,g} = -\tfrac{1}{2}\gamma \pm i\,\delta. \tag{12c}$$

Our initial labeling is seen to be correct.

Though neither attractors nor repellors, the undamped motion has three fixed points as well. These are obtained from Eqs. (9) or (10) by setting $\gamma = 0$. In the undamped case the fixed points are possible extrema of the Hamiltonian. We illustrate the global structure of the Hamiltonian in Fig. 1, which gives a contour plot of $H(x,p)$. The fixed points are identified with the same letters we use to designate the corresponding fixed points in the case of a weak damping. It turns out that the fixed point g is a local minimum, e a local maximum, and r a saddle point of the Hamiltonian. Two constant-energy curves cross at r, as appropriate for a hyperbolic fixed point. One of these curves wraps around the minimum g and the other around both extrema g and e. Outside of the latter curve the Hamiltonian drops steeply to $-\infty$ as either $x \to \infty$ or $p \to \infty$.

A weak damping shifts the fixed points slightly. For instance, e no longer resides at exactly the maximum of the Hamiltonian. In our qualitative discussions we often ignore such fine distinctions. It should be borne in mind, though, that in the calculations one should use the expressions for the fixed points that properly include the damping. Ignoring the damping prematurely may lead to quite puzzling wrong results.

To the leading order in χ, the fixed point g does not depend on the nonlinearity of the CNLO. In fact, it simply corresponds to the steady state of the driven harmonic oscillator, which reflects a balance between the drive strength and the detuning. The fixed point e corresponds in the laboratory frame to the situation in which the energy of the CNLO is just right, so that the nonlinearity has decreased the oscillator frequency sufficiently to render it on resonance with the driving force. Correspondingly, to the leading order in Ω this fixed point does not depend on the strength of the external drive.

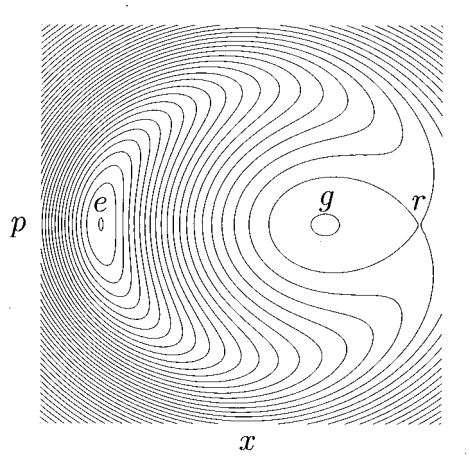

FIG. 1. Contour plot of the classical Hamiltonian $H(x,p)$ in the (x,p) plane for representative values of the parameters δ, χ, and Ω. The extrema and the saddle point of the Hamiltonian are labeled with the same symbols e, g, and r as in their corresponding fixed points. For clarity of the visualization, the contours are not equally spaced.

It may seem paradoxical that the CNLO is attracted to the *maximum* of the energy at e. After all, here the potential energy has a maximum too and in the neighborhood of the point e the force on the oscillator points down the potential hill. One way to resolve this paradox is to note that the effective mass $[\partial^2 H/\partial p^2]^{-1}$ is negative around e. The oscillator thus rolls opposite to the force, i.e., uphill.

There are two apparent oddities in our formalism: energy is not bounded from below and the effective mass of the oscillator may be negative. We have expanded the relativistic mass increase to only the lowest nontrivial order in momentum, which unduly removes the lower limit from energy. In addition, we are analyzing the oscillator in the rotating frame, which both exacerbates the drop of energy toward $-\infty$ and leads to a volatile effective mass. The seemingly bizarre features of our model are not a sign of an inherent failure, but basically reflect the fact that our formulation is in the rotating frame.

The CNLO may be bistable. One of the operating points g is essentially the same stable state as for a driven harmonic oscillator, while the second operating point e is a nonlinear resonance. Our mission now is to find out what happens to the bistability in the presence of both thermal and quantum fluctuations.

III. CLASSICAL DESCRIPTION OF THE QUANTUM OSCILLATOR

We now quantize the nonlinear oscillator. The position and the momentum in Eqs. (1), X and P, become quantum operators $\hat{X}$ and $\hat{P}$ with the commutator $[\hat{X}, \hat{P}] = i\hbar$. The

quantum counterparts of the dimensionless position and momentum x and p therefore have the commutator $[\hat{x},\hat{p}]=i$. The rising and lowering operators a and $a^\dagger$ with the commutator $[a,a^\dagger]=1$ may be defined as

$$a=\frac{1}{\sqrt{2}}(\hat{x}+i\hat{p}),\quad a^\dagger=\frac{1}{\sqrt{2}}(\hat{x}-i\hat{p}). \tag{13}$$

Unfortunately, we cannot obtain the correct quantum Hamiltonian by simply substituting the operators $\hat{x}$ and $\hat{p}$ into the classical Hamiltonian. This comes about because the transformation $(X,P)\rightarrow(x,p)$ involves the extra factor $\hbar$ and is not canonical in the sense of classical mechanics. To obtain Heisenberg equations of motion for $\hat{x}$ and $\hat{p}$ that resemble the classical equations of motion as closely as possible, we insert $\hat{x}$ and $\hat{p}$ into the classical Hamiltonian (4) *and* multiply the Hamiltonian by $\hbar$. The result is written in terms of the operators a and $a^\dagger$ as

$$\frac{H}{\hbar}=\delta a^\dagger a-\frac{\chi}{2}a^\dagger a^\dagger a a-\frac{\Omega}{2}(a+a^\dagger). \tag{14}$$

We have dropped scalar constants from the Hamiltonian.

It should be noted that, as the Hamiltonian (4) contains products of what become the noncommuting operators $\hat{x}$ and $\hat{p}$, quantization is not unique. For instance, one particular ordering of the operator products would give $(a^\dagger a)^2$ instead of $a^\dagger a^\dagger a a$ in Eq. (14). Such a Hamiltonian could be cast into the form (14) as well, except that the oscillator frequency would then be $\delta-\chi/2$. We have no prescription for resolving ambiguities of operator order and simply choose to use the one in Eq. (14).

Damping, on the other hand, cannot be represented in Hamiltonian form. It must be described in terms of a master equation, which includes both the Hamiltonian evolution and the damping [13]. In the rotating frame the master equation reads

$$\begin{aligned}\dot{\rho}=&-i\left[\delta a^\dagger a-\frac{\chi}{2}a^\dagger a^\dagger a a-\frac{\Omega}{2}(a+a^\dagger),\rho\right]\\&+\frac{\gamma}{2}(2a\rho a^\dagger-a^\dagger a\rho-\rho a^\dagger a)\\&+\overline{N}\gamma(a^\dagger\rho a+a\rho a^\dagger-a^\dagger a\rho-\rho a a^\dagger).\end{aligned} \tag{15}$$

This equation is identical to the one used in Ref. [2], except for operator ordering. The parameter $\overline{N}$ is the number of thermal excitations at the oscillator frequency ω. $\overline{N}=0$ corresponds to zero temperature with only quantum noise remaining.

The Fock space representation of the density operator, as in Ref. [2], may be advantageous in direct computations. Alternatively, there exist well-known methods in quantum optics that allow us to represent ρ in terms of quasiprobability distributions in the phase space of the classical oscillator [13]. We may then draw from analogies between classical and quantum systems. We have chosen to work with the Q representation, in which one basically takes the expectation value of the density operator in coherent states $|\alpha\rangle$:

$$Q(\alpha)=\frac{1}{\pi}\langle\alpha|\rho|\alpha\rangle. \tag{16}$$

All quantum expectation values can be calculated from the Q function. The explicit rule is

$$\langle a^m(a^\dagger)^n\rangle_{\mathrm{QM}}=\int d^2\alpha\,\alpha^m(\alpha^*)^n Q(\alpha). \tag{17}$$

The subscript QM is a reminder of the fact that the result is the true quantum expectation value, even though it is obtained as something analogous to a phase-space integral.

In quantum optics one often introduces two quadratures x and p such that $\alpha=x+ip$. However, in order to make the connection to our CNLO explicit, we have found it more convenient to define the quadratures as

$$\alpha=\frac{1}{\sqrt{2}}(x+ip). \tag{18}$$

In this way, the Q function may be qualitatively interpreted as the distribution function of either the quantum variables $(\hat{x},\hat{p})$ or the classical-mechanics position and momentum (x,p), as things might be. It should be noted that the change of the variables $\alpha\leftrightarrow(x,p)$ involves a change in the integration measure $d\alpha=\frac{1}{2}\,dx\,dp$. Whenever we write $Q(x,p)$, we assume that Q has been properly normalized to make $\int dx\,dp\,Q(x,p)=1$. With this convention, Eq. (17) may be developed further as

$$\begin{aligned}&\langle a^m(a^\dagger)^n\rangle_{\mathrm{QM}}\\&\quad=\left(\frac{1}{2}\right)^{(m+n)/2}\int dx\,dp\,Q(x,p)(x+ip)^m(x-ip)^n.\end{aligned} \tag{19}$$

The equation of motion for the Q function, a FPE in the variables α and α^* [13], may be derived from Eq. (15). However, since we are looking for an analogy with the CNLO, we are more interested in the FPE in terms of x and p,

$$\begin{aligned}\dot{Q}(x,p)=&\left\{\frac{\partial}{\partial x}\left[\frac{\gamma}{2}x-\delta p+\frac{\chi}{2}p(x^2+p^2)\right]\right.\\&+\frac{\partial}{\partial p}\left[\frac{\gamma}{2}p+\delta x-\frac{\chi}{2}x(x^2+p^2)-\frac{\Omega}{\sqrt{2}}\right]\\&+\frac{\partial}{\partial x}\left(\gamma\frac{1+\overline{N}}{2}+\frac{\chi}{2}px\right)\frac{\partial}{\partial x}\\&+\frac{\partial}{\partial x}\frac{\chi}{4}(p^2-x^2)\frac{\partial}{\partial p}+\frac{\partial}{\partial p}\left(\gamma\frac{1+\overline{N}}{2}-\frac{\chi}{2}px\right)\frac{\partial}{\partial p}\\&\left.+\frac{\partial}{\partial p}\frac{\chi}{4}(p^2-x^2)\frac{\partial}{\partial x}\right\}Q(x,p).\end{aligned} \tag{20}$$

In the process the detuning has been renormalized, $\delta+\chi\rightarrow\delta$. Of course, in the relevant limit of the parameters (8), this change is of little consequence.

The two pairs of square brackets in the drift terms in Eq. (20) enclose the negatives of the right-hand sides of the equations of motion (2a) and (2b) for the CNLO. In fact, suppose that the symmetric 2×2 matrix D of the coefficients D_{xx}, D_{xp}, ..., in the diffusion terms

$$\frac{\partial}{\partial x}D_{xx}(x,p)\frac{\partial}{\partial x}Q+\frac{\partial}{\partial x}D_{xp}(x,p)\frac{\partial}{\partial p}Q+\cdots$$

is positive-definite everywhere. Then the function $Q(x,p)$ may be interpreted as the phase-space density for a diffusing CNLO that evolves according to the classical equations of motion, plus some added noise [13]:

$$dx=-\frac{\gamma}{2}x\,dt+\delta p\,dt-\frac{\chi}{2}p(x^2+p^2)dt+d\eta_x, \quad (21a)$$

$$dp=-\frac{\gamma}{2}p\,dt-\delta x\,dt+\frac{\chi}{2}x(x^2+p^2)dt+\frac{\Omega}{\sqrt{2}}dt+d\eta_p. \quad (21b)$$

Here $d\eta_x$ and $d\eta_p$ are random increments with an appropriate statistics. Our diffusion matrix actually is not positive-definite, but we disregard this mathematical inconvenience and make use of the physical picture that the classical interpretation conveys to us.

The effect of noise (diffusion) is that an oscillator that would otherwise travel on a deterministic trajectory in phase space will now make random excursions around the classical path. Otherwise, our analysis of the CNLO may be taken over directly. Fixed points and their stability properties are the same as before. Nonetheless, the possibility opens up that fluctuations make the QNLO hop between the fixed points. This is the topic of Sec. V below.

The analogy between CNLO and QNLO is not limited to the (x,p) representation. Simply by making the transformation of variables (5), we find a $Q(\phi,\epsilon)$ and its equation of motion

$$\dot{Q}(\phi,\epsilon)=\left[\frac{\partial}{\partial\phi}\left(\chi\epsilon-\delta+\frac{\Omega\sin\phi}{2\sqrt{\epsilon}}\right)+\frac{\partial}{\partial\epsilon}(\gamma\epsilon-\sqrt{\epsilon}\Omega\cos\phi)+\frac{\partial}{\partial\epsilon}\epsilon\gamma(1+\overline{N})\frac{\partial}{\partial\epsilon}+\frac{\partial}{\partial\epsilon}\frac{\chi\epsilon}{2}\frac{\partial}{\partial\phi}+\frac{\partial}{\partial\phi}\frac{\chi\epsilon}{2}\frac{\partial}{\partial\epsilon}+\frac{\partial}{\partial\phi}\frac{\gamma(1+\overline{N})}{4\epsilon}\frac{\partial}{\partial\phi}\right]Q(\phi,\epsilon). \quad (22)$$

A canonical transformation of coordinates and momenta does not change positive-definiteness, or lack thereof, of the diffusion matrix. With precisely the same caveats as before, we thus interpret the FPE (22) as the equation of motion of a diffusing CNLO,

$$d\phi=\delta\,dt-\chi\epsilon\,dt-\frac{\Omega\,\sin\phi}{2\sqrt{\epsilon}}dt+d\eta_\phi, \quad (23a)$$

$$d\epsilon=-\gamma\epsilon\,dt+\sqrt{\epsilon}\Omega\,dt\,\cos\phi+d\eta_\epsilon, \quad (23b)$$

where $d\eta_\epsilon$ and $d\eta_\phi$ are appropriate random noises.

IV. CORRELATIONS OF QUANTUM OBSERVABLES

According to Eq. (17), the shape of the Q function is needed for a calculation of the correlation functions of a and $a^\dagger$, e.g., for the moments of x and p, or ϕ and ϵ. However, if we are only interested in (steady-state equal-time) correlation functions up to order 2, we may resort to a simple technique that allows us to analyze these quantities without solving for Q globally. We assume that the Q function is tightly localized in phase space around one of the fixed points. We may then put the diffusion matrix equal to its value at the fixed point and expand the drift terms to first order in x and p (or ϕ and ϵ) around the fixed point. Finally, we multiply the linearized FPE with second-order products of $\tilde{x}=x-x_i$ and $\tilde{p}=p-p_i$ (or $\tilde{\phi}=\phi-\phi_i$ and $\tilde{\epsilon}=\epsilon-\epsilon_i$), where i denotes one of the three fixed points, and integrate over the phase space using partial integration to eliminate the derivatives from the equations.

For instance, for the moment of $\tilde{x}^2$ we find

$$\begin{aligned}\frac{d}{dt}\langle\tilde{x}^2\rangle&=\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dp\,\tilde{x}^2\dot{Q}(x,p)\\&\simeq\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dp([\chi x_i p_i+\gamma(1+\overline{N})]\\&\quad-2\tilde{x}\{[(\tfrac{1}{2}\gamma+\chi p_i x_i)\tilde{x}\\&\quad+(-\delta+\tfrac{3}{2}\chi p_i^2+\tfrac{1}{2}\chi x_i^2)\tilde{p}]\})Q(x,p)\\&=\gamma(1+\overline{N})+\chi p_i x_i-(\gamma+2\chi p_i x_i)\langle\tilde{x}^2\rangle\\&\quad+(2\delta-3\chi p_i^2-\chi x_i^2)\langle\tilde{x}\tilde{p}\rangle. \quad (24)\end{aligned}$$

The first-order moments cancel after the proper fixed point including the damping (9) is inserted and the corresponding terms have been omitted in Eq. (24). The other moment equations have a similar structure, so that the second-order moment equations form a closed set. Since we are interested in steady-state moments, we equate the time derivatives of the moments to zero and solve the resulting set of linear equations. To leading order in χ/δ we find

$$\langle\tilde{x}^2\rangle_e=\langle\tilde{x}^2\rangle_r=\frac{3+2\overline{N}}{4}, \quad (25a)$$

$$\langle\tilde{p}^2\rangle_e=-\langle\tilde{p}^2\rangle_r=\sqrt{\frac{\delta^3}{\chi\Omega^2}}(1+2\overline{N}), \quad (25b)$$

$$\langle\tilde{x}^2\rangle_g=\langle\tilde{p}^2\rangle_g=1+\overline{N}, \quad (25c)$$

$$\langle\tilde{\epsilon}^2\rangle_e=\langle\tilde{\epsilon}^2\rangle_r=\frac{\delta}{\chi}\frac{3+2\overline{N}}{2}, \quad (25d)$$

$$\langle\tilde{\phi}^2\rangle_e=-\langle\tilde{\phi}^2\rangle_r=\sqrt{\frac{\delta\chi}{\Omega^2}}\frac{1+2\overline{N}}{2}, \quad (25e)$$

$$\langle \tilde{\epsilon}^2 \rangle_g = \frac{\Omega^2}{\delta^2} \frac{1+\bar{N}}{2}, \tag{25f}$$

$$\langle \tilde{\phi}^2 \rangle_g = \frac{2\delta^2}{\Omega^2}(1+\bar{N}). \tag{25g}$$

All cross correlations such as $\langle \tilde{x}\tilde{p} \rangle$ are of the order γ and thus negligible. The negative moments encountered at the fixed point r indicate that, as expected, there is no stationary solution for the FPE localized around the repellor r.

Similar results were obtained for the excited state e in the numerical investigation in Ref. [2]. In order to see how well our results actually agree one should note that

$$\begin{aligned} \bar{n} \equiv \langle n \rangle_{\mathrm{QM}} &= \langle a^\dagger a \rangle_{\mathrm{QM}} = \int d^2\alpha (|\alpha|^2 - 1) Q(\alpha) \\ &= \int dx\, dp \left(\frac{1}{2}(x^2+p^2) - 1 \right) Q(x,p) \\ &= \frac{1}{2}\langle x^2+p^2 \rangle - 1 = \int d\epsilon \int d\phi [\epsilon - 1] Q(\phi, \epsilon) \\ &= \langle \epsilon \rangle - 1, \end{aligned} \tag{26}$$

$$\begin{aligned} (\Delta n)^2 \equiv \langle (\Delta n)^2 \rangle_{\mathrm{QM}} &= \langle\langle (a^\dagger a - \langle a^\dagger a \rangle)^2 \rangle\rangle_{\mathrm{QM}} \\ &= \int d^2\alpha [(|\alpha|^2 - \langle |\alpha|^2 \rangle)^2 - |\alpha|^2] Q(\alpha) \\ &= \frac{1}{4}\langle\langle (x^2+p^2 - \langle x^2+p^2 \rangle)^2 \rangle\rangle - \frac{1}{2}\langle x^2+p^2 \rangle \\ &= \langle \tilde{\epsilon}^2 \rangle - \langle \epsilon \rangle. \end{aligned} \tag{27}$$

Again, the subscript QM stands to denote the true quantum-mechanical expectation value, while the expectation values without this subscript are computed as if Q was a classical probability distribution for its variables. By virtue of Eq. (10a), at the excited state we have $\langle \epsilon \rangle = \epsilon_e \simeq \delta/\chi$. Combining this observation with Eqs. (25d) and (27) gives

$$\Delta n \simeq \sqrt{\frac{1+2\bar{N}}{2}} \sqrt{\bar{n}}. \tag{28}$$

In comparison, according to Ref. [2], Δn is proportional to $\sqrt{\bar{n}}$, with the proportionality constant that is 0.7 for $\bar{N}=0$ and increases by 20% when $\bar{N}=0.2$. Our result is in perfect agreement with this description.

Reference [2] also gives a phase width $\Delta\phi$ for the excited state e. The question of the phase conjugate to the number of quanta is somewhat delicate, but the phase width quoted in Ref. [2] is much larger than the minimum-uncertainty limit one would surmise for phase and number of excitations. We therefore simply ignore possible quantum corrections and use the phase spread from Eq. (25e) as the square of the width. We find

$$\Delta\phi \simeq \Delta n \sqrt{\frac{\chi}{\sqrt{\bar{n}}\Omega}}. \tag{29}$$

This too agrees with the numerical results of Ref. [2], up to a ''proportionality constant approximately equal to unity.'' Since the definition of the phase width is not given unambiguously in Ref. [2], we cannot assess the accuracy of the numerical constant.

V. DECAY RATE OF THE METASTABLE STATE

The final issue we address is the lifetime of the metastable state. Usually, the lifetime would be estimated by an application of Kramers analysis of a particle diffusing over a potential barrier [15,16]. However, in the present case we have a highly nontrivial situation since our FPE does not have a potential solution [16]. We therefore have to invoke a few quite subtle arguments and deviate slightly from a direct approach in terms of Kramers reasoning. Nonetheless, in the end we will again reach a complete agreement with numerical experiments.

We begin with a reflection on the meaning of the Q function. It is well known that in a coherent state $|\alpha_0\rangle$ the operator quadratures $\hat{x}$ and $\hat{p}$ both (separately) have a Gaussian probability distribution with the spreads $(\Delta\hat{x})^2 = (\Delta\hat{p})^2 = \frac{1}{2}$. The corresponding Q function, though, is

$$Q(\alpha) = \frac{1}{\pi}|\langle \alpha | \alpha_0 \rangle|^2 = \frac{1}{\pi} e^{-|\alpha-\alpha_0|^2}, \tag{30a}$$

$$Q(x,p) = \frac{1}{2\pi} e^{-[(x-x_0)^2 + (p-p_0)^2]/2}. \tag{30b}$$

If we were to calculate the spreads of the variables x and p from this Q function, we would find $(\Delta x)^2 = (\Delta p)^2 = 1$. Here the function Q behaves as if it were the distribution for the underlying ''physical'' variables x_p and p_p smeared by a convolution with a Gaussian of the form

$$K(x,p) = \frac{1}{\pi} e^{-(x^2+p^2)}. \tag{31}$$

In general, the Q function is derived from the density operator ρ as in Eq. (16). We thus posit that the function $Q(x,p)$ is not the ''true'' distribution $W(x_p, p_p)$ for the physical observables x_p and p_p, but rather the distribution W blurred by the size of the coherent state. In precise terms, we view $Q(x,p)$ as the result after the true distribution $W(x_p, p_p)$ has been blurred by a convolution with the Gaussian (31), whose spreads are $(\Delta x)^2 = (\Delta p)^2 = \frac{1}{2}$. Similarly, consider the variables ϵ and ϕ, with $\epsilon \simeq \epsilon_0$ large enough that over the region we are considering the version of the mapping $(x,p) \leftrightarrow (\phi, \epsilon)$ linearized about a point (ϕ_0, ϵ_0) may be used to transfer between the coordinate systems. In this region the function $Q(\phi, \epsilon)$ represents the true distribution of ϕ_p and ϵ_p as viewed through a Gaussian blur with the spreads $(\Delta\epsilon)^2 \simeq \epsilon_0$ and $(\Delta\phi)^2 \simeq 1/4\epsilon_0$.

In fact, we have seen an example of the latter already. A direct calculation with the Q function in the neighborhood of

the fixed point e gives the spread $(\Delta\epsilon)^2 \simeq \epsilon(3+2\overline{N})/2$. However, this is contaminated by ϵ worth of blur, so that the physical energy variable would have the spread $(\Delta\epsilon_p)^2 \simeq \epsilon(1+2\overline{N})/2$. The physical energy should be related to the excitation number by $n=\epsilon_p$, so that the spread of the excitation number is also $(\Delta n)^2=\overline{n}(1+2\overline{N})/2$. This agrees with Eq. (28).

In the second step we assume that the Q function is concentrated strongly in the neighborhood of the fixed point e. We therefore expand the Hamiltonian to second order in the variables to obtain the local Hamiltonian $H_e(\phi,\epsilon)$. We derive the drift terms in the FPE from this linearized Hamiltonian and simply use the fixed-point values of ϵ and ϕ for the diffusion coefficients. The ensuing linearized FPE is exactly solvable and gives a Gaussian distribution. However, according to our argument above, this Gaussian is not the true distribution of the variables ϵ and ϕ. We have to deconvolute the blur. In the limit $\gamma\to 0$ this gives a probability distribution of the form

$$W(\phi_p,\epsilon_p)\propto\exp\left[-\frac{H_e(\phi_p,\epsilon_p)}{T}\right] \tag{32}$$

for the physical variables. In the leading order in χ/δ the temperaturelike parameter T is given by

$$T=-\frac{\delta}{\chi}\frac{1+2\overline{N}}{2}. \tag{33}$$

What exactly is the meaning of the parameter T and its relation to the actual temperature (if there is any) is a thorny issue, which we will not attempt to address. Nonetheless, formally T looks exactly like temperature in classical statistical mechanics and henceforth we treat it as such. The temperature is negative, but this is just a fluke of our system, in that a tightly bound distribution is set up in the neighborhood of the maximum of the Hamiltonian.

In the last step of our argument we reiterate the observation that energy is not conserved as a result of the damping proportional to γ. Thus the system explores the energy space at the innate time scale γ^{-1}. We assume that the oscillator switches from the fixed point (ϕ_e,ϵ_e) to the fixed point (ϕ_g,ϵ_g) once such an excursion has brought it past the energy surface of the unstable fixed point $H(\phi,\epsilon)=H(\phi_r,\epsilon_r)$; cf. Fig. 1 and its discussion. The rate for switching is therefore estimated as

$$R_e\simeq\gamma\exp\left[-\frac{H(\phi_r,\epsilon_r)-H(\phi_e,\epsilon_e)}{T}\right], \tag{34}$$

where the exponential is nothing but the Boltzmann factor. Inserting the known coordinates of the fixed points, we find to the leading order in χ/δ

$$R_e\simeq\gamma\exp\left[-\frac{4\Omega}{\sqrt{\overline{n}}\chi(1+2\overline{N})}\right]. \tag{35}$$

This is exactly the result found numerically in Ref. [2], albeit the $\overline{N}$ dependence was not clearly resolved in the numerical experiments.

The same argument may also be carried out to analyze the stability of the ground state g. This gives the rate for the hops $g\to e$ as

$$R_g\simeq\gamma\exp\left[-\frac{\delta}{2\chi(1+\overline{N})}\right]. \tag{36}$$

The escape rate from the ground state seems to be much lower than that from the metastable excited state. This is not due to a large temperature difference. In fact, for $\overline{N}=0$ the temperature in the ground state is positive and twice the magnitude of the temperature in the excited state. Instead, the explanation is to be found in the depth of the ground state well. As soon as the oscillator reaches the stable ground state, the fluctuations are too small to kick the oscillator out of the well again.

An astute reader may have noticed that our true distribution $W(x_p,p_p)$ is nothing but the Wigner function [13]. We have used the Q function mainly because it obeys a FPE. Moreover, the Q function is non-negative everywhere, thus featuring an interpretation in terms of probabilities. The Wigner function, on the other hand, does not obey a FPE, nor does it need to be non-negative. The deconvolution of Q is therefore not guaranteed to produce a valid classical distribution for x_p and p_p. In our cases it did, though, and we crassly interpret the results as classical position-momentum distributions.

On a more fundamental level, we have quietly made a number of implicit assumptions. For instance, after the nonlinear canonical transformation $(x,p)\to(\phi,\epsilon)$, the precise quantum mechanical significance of the function $Q(\phi,\epsilon)$ is obscure. Nonetheless, we silently continue to assume that there are some quantum observables corresponding to the variables ϕ and ϵ ($\epsilon\cong a^\dagger a$) and that the function Q gives us a slightly hazy glimpse of these observables. As another example, the diffusion tensor is not positive-definite in our FPE. We avoid the issue by only studying explicitly local solutions in the neighborhoods of the fixed points. One should not expect that one may treat quantum systems essentially classically without having to resort to this type of heuristic reasoning at some point.

Dykman and Smelyanskii address the same problem of the lifetime of the bistable states analytically using more sophisticated techniques [12]. They adopt the position representation for the density operator instead of the Q representation and resort to a quasiclassical approximation, essentially the density matrix analog of the WKB method. The quasiclassical density operator may be found by solving an auxiliary two-particle problem in complex phase space using the methods of classical mechanics, which they proceed to do. For the metastable state, in the same limits we are considering, their result is precisely the same as our Eq. (35). On the other hand, for the lifetime of the ground state their results are qualitatively different from our Eq. (36). These authors also point out that their predicted rate may be much smaller than the rate of genuine quantum tunneling. Quantum tunneling is fully incorporated in the Q function, but it

can only be treated correctly if the failure of positive-definiteness of the diffusion tensor is accounted for. Obviously, in the absence of numerical corroboration, our Eq. (36) should be viewed with caution.

VI. CONCLUSION

At least in the absence of thermal fluctuations $\overline{N}=0$, the finite spreads of the energy and the angle and the finite lifetime of the nonlinear resonance *e* of the driven, damped, anharmonic oscillator are purely quantum mechanical phenomena. Quantum mechanics gives us certain diffusion terms in the FPE to work with. From there on, though, we have used essentially classical arguments to reproduce the results obtained numerically in Ref. [2].

The estimates of the energy and angle widths of the metastable state were derived by linearizing the FPE. This is a standard approach [14] and should be accurate as long as the metastable state is sufficiently well localized in phase space.

Our analysis of the lifetime of the metastable state involves both local and global arguments in the phase space of the oscillator. While studies of the lifetime would probably have been quite demanding if ordinary techniques for the FPE had been applied [10,16] and *are* demanding within the quasiclassical framework [12], we were able to obtain an accurate estimate in a simple fashion. The complete agreement with the numerical experiments may be fortuitous, but we find it remarkable nonetheless. The questions of why and how our simple argument worked so well and whether it will work in other similar situations should make interesting research topics in their own right.

ACKNOWLEDGMENTS

Gerald Gabrielse has repeatedly suggested the problem addressed in this paper to J.J. over many years and Mark Dykman gave us the bibliographic data of Ref. [12]. B.V. wishes to acknowledge the hospitality of the University of Connecticut, where this work was performed. This work is supported in part by the National Science Foundation, Grant No. PHY-9421116.

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